New York Journal of Mathematics

New York J. Math. **8** (2002) 145–159.

Inductive Limit Algebras from Periodic Weighted Shifts on Fock Space

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Abstract. Noncommutative multivariable versions of weighted shift operators arise naturally as 'weighted' left creation operators acting on the Fock space Hilbert space. We identify a natural notion of periodicity for these N-tuples, and then find a family of inductive limit algebras determined by the periodic weighted shifts which can be regarded as noncommutative multivariable generalizations of the Bunce-Deddens C*-algebras. We establish this by proving that the C*-algebras generated by shifts of a given period are isomorphic to full matrix algebras over Cuntz-Toeplitz algebras. This leads to an isomorphism theorem which parallels the Bunce-Deddens and UHF classification scheme.

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The primary goal of this paper is to initiate the study of *noncommutative multivariable weighted shifts*. Almost three decades ago, Bunce and Deddens [3, 4] introduced a family of inductive limit C*-algebras generated by periodic unilateral weighted shift operators. On the other hand, we now know that noncommutative multivariable versions of unilateral shifts arise in theoretical physics and free probability theory as the so-called left creation operators acting on the full Fock space Hilbert space. There is now an extensive body of research for these operators and the algebras they generate (see [1, 9, 11, 10, 21, 23, 24] for example).

Received July 2, 2002.

Mathematics Subject Classification. 46L05, 46L35, 47B37, 47L40.

Key words and phrases. Hilbert space, operator, weighted shift, noncommutative multivariable operator theory, Fock space, creation operators, Cuntz-Toeplitz C*-algebras, K-theory.

Partially supported by a Canadian NSERC Post-doctoral Fellowship.

ISSN 1076-9803/02

In this paper, we introduce a family of C*-algebras which can be regarded as noncommutative multivariable generalizations of the Bunce-Deddens algebras. In accomplishing this, based on the creation operators, we introduce the concept of a noncommutative multivariable weighted shift and discover a satisfying notion of periodicity based on the structure of Fock space. We characterize these algebras in terms of inductive limits of full matrix algebras over the Cuntz-Toeplitz and Cuntz algebras. This leads to a classification theorem which parallels the classification of UHF algebras by Glimm [16], and the Bunce-Deddens algebras classification [3, 4], by supernatural numbers.

In the opening section we recall the formulation of Fock space and the creation operators. We also quickly review the basics of the Cuntz and Cuntz-Toeplitz algebras. In the second section we introduce noncommutative weighted shifts and investigate their basic structure. The third section describes a pictorial method for thinking of these shifts, by using the Fock space 'tree' structure. This leads to a natural notion of periodicity, and then we define the C*-algebras we study in the rest of the paper. The final two sections consist of an in-depth analysis of these algebras. Most importantly, we prove they are isomorphic to inductive limits of full matrix algebras of distinguished sizes over Cuntz and Cuntz-Toeplitz algebras. Using this characterization we establish a classification theorem based on K-theory for the Cuntz algebras.

1. Introduction

We begin by recalling the formulation of the full Fock space Hilbert space and its associated creation operators. For $N \geq 2$, let $\mathbb{F}_N^+$ be the unital free semigroup on N *noncommuting* letters $\{1, 2, \ldots, N\}$. We denote the unit in $\mathbb{F}_N^+$ by e. One way to realize N-variable Fock space is as $\mathcal{H}_N = \ell^2(\mathbb{F}_N^+)$. From this point of view, the vectors $\{\xi_w : w \in \mathbb{F}_N^+\}$ form an orthonormal basis for $\mathcal{H}_N$ which can be thought of as a generalized Fourier basis. The left creation operators (also known as the Cuntz-Toeplitz isometries we will see below) $L = (L_1, \ldots, L_N)$ are defined on $\mathcal{H}_N$ by their actions on basis vectors,

$$L_i \xi_w = \xi_{iw}, \quad \text{for} \quad 1 \leq i \leq N \quad \text{and} \quad w \in \mathbb{F}_N^+.$$

The L_i are isometries with pairwise orthogonal ranges for which the sum of the range projections satisfies $\sum_{i=1}^N L_i L_i^* = I - P_e$, where $P_e = \xi_e \xi_e^*$ is the rank one projection onto the span of the vacuum vector ξ_e. We will discuss a helpful pictorial method for thinking of the actions of these operators in Section 3.

Most importantly for our purposes, this N-tuple forms the noncommutative multivariable version of a unilateral shift. This claim is well supported by a number of facts. For instance, the unilateral shift *is* obtained for $N = 1$, and otherwise each of the L_i is unitarily equivalent to a shift of infinite multiplicity. Further, the study of the L_i in operator theory and operator algebras was at least partly initiated by the dilation theorem of Frazho [15], Bunce [2] and Popescu [22], which provided the noncommutative multivariable version of Sz.-Nagy's classical minimal isometric dilation of a contraction [14]. Namely, every row contraction of operators on Hilbert space has a minimal joint dilation to isometries, acting on a larger space, with pairwise orthogonal ranges. The classical Wold decomposition shows that every isometry breaks up into an orthogonal direct sum of a unitary together

with copies of the shift. Analogously, Popescu's version [22] shows that every N-tuple of isometries with pairwise orthogonal ranges decomposes into an orthogonal direct sum of isometries which form a representation of the Cuntz C*-algebra $\mathcal{O}_N$ (see below), together with copies of $L = (L_1, \ldots, L_N)$.

In addition, the WOT-closed nonselfadjoint algebras generated by the L_i have been shown by Davidson, Pitts, Arias, Popescu and others to be the appropriate *noncommutative analytic Toeplitz algebras* (see [1, 11, 10, 21, 24]). We also mention that the WOT-closed nonselfadjoint algebras generated by the weighted shifts discussed here have been investigated in [19], where a number of results from the single variable setting have been generalized, at the same time exposing new noncommutative phenomena. Finally, we note that compressing the creation operators to symmetric Fock space yields the *commutative* multivariable shift. The C*-algebras generated by weighted versions of which were studied in [6] for instance.

The C*-algebras determined by the isometries $L = (L_1, \ldots, L_N)$ have also been studied extensively. The C*-algebra generated by $L_1, \ldots, L_N$ is called the *Cuntz-Toeplitz algebra* and is denoted $\mathcal{E}_N$. The ideal generated by the rank one projection $I - \sum_{i=1}^N L_i L_i^*$ in $\mathcal{E}_N$ yields a copy of the compact operators. When this ideal is factored out, the C*-algebra obtained is the *Cuntz algebra* $\mathcal{O}_N$. It is the universal C*-algebra generated by the relations

$$s_i^* s_j = \delta_{ij} \mathbf{1} \quad \text{for} \quad 1 \leq i, j \leq N \quad \text{and} \quad \sum_{i=1}^N s_i s_i^* = \mathbf{1}.$$

Up to isomorphism, $\mathcal{O}_N$ is the C*-algebra generated by *any* N isometries $S = (S_1, \ldots, S_N)$ which satisfy these relations, since it is simple.

The K-theory for a C*-algebra consists of a series of invariants which hold information on equivalence classes of projections in the matrix algebras over the algebra. The K-theory for $\mathcal{O}_N$ was worked out by Cuntz [5]. In particular, its K_0 group is the finite abelian group $K_0(\mathcal{O}_N) = \mathbb{Z}/(N-1)\mathbb{Z}$. In connection with classification results for inductive limits of Cuntz algebras we mention work of Rørdam [25]. We also note that our isomorphism theorem has overlap with work of Evans [13].

2. Noncommutative weighted shifts

From the discussion in the previous section, we are led to the following definition for noncommutative multivariable weighted shifts. We shall drop the multivariable reference for succinctness. We mention that the idea for considering these weighted shifts came during the author's preparation of [20], where a related class of N-tuples was used in the analysis there.

Definition 2.1. We say that an N-tuple of operators $S = (S_1, \ldots, S_N)$ acting on a Hilbert space $\mathcal{H}$ forms a *noncommutative weighted shift* if there is a unitary $U : \mathcal{H}_N \to \mathcal{H}$, operators $T = (T_1, \ldots, T_N)$ on $\mathcal{H}_N$, and scalars $\{\lambda_{i,w} : 1 \leq i \leq N \quad \text{and} \quad w \in \mathbb{F}_N^+\}$ such that $S_i = U T_i U^*$ for $1 \leq i \leq N$ and

$$T_i \xi_w = \lambda_{i,w} \xi_{iw} \quad \text{for} \quad 1 \leq i \leq N \quad \text{and} \quad w \in \mathbb{F}_N^+.$$

Note 2.2. For brevity, we assume that the weighted shifts $T = (T_1, \ldots, T_N)$ we consider actually act on Fock space $\mathcal{H}_N = \ell^2(\mathbb{F}_N^+)$. Further, the proposition below

will allow us to make the following simplifying assumption on weights throughout the paper:

$$\textit{Assumption: } \lambda_{i,w} \geq 0 \quad \text{for} \quad 1 \leq i \leq N \quad \text{and} \quad w \in \mathbb{F}_N^+.$$

Indeed, every shift is jointly unitarily equivalent to a shift with nonnegative weights:

Proposition 2.3. *Suppose $T = (T_1, \ldots, T_N)$ is a weighted shift with weights $\{\lambda_{i,w}\}$. Then there is a unitary $U \in \mathcal{B}(\mathcal{H}_N)$, which is diagonal with respect to the standard basis for $\mathcal{H}_N$, such that the weighted shift*

$$(UT_1U^*, \ldots, UT_NU^*)$$

has weights $\{|\lambda_{i,w}|\}$.

Proof. We build the unitary by inductively choosing scalars μ_w and defining $U\xi_w = \mu_w \xi_w$. Put $\mu_e = 1$. Let $k \geq 1$ and assume the scalars $\{\mu_w : |w| = k-1\}$ corresponding to words of length $k-1$ in $\mathbb{F}_N^+$ (as the empty word, the unit e is taken to have length zero) have been chosen. The scalars $\{\mu_w : |w| = k\}$ are obtained in the following manner. For $iw \in \mathbb{F}_N^+$ with $|w| = k-1$ and $1 \leq i \leq N$, choose $\mu_{iw} \in \mathbb{C}$ of modulus one such that

$$(\overline{\mu_w}\, \lambda_{i,w})\, \mu_{iw} \geq 0.$$

Now if $1 \leq i \leq N$ and $w \in \mathbb{F}_N^+$ are arbitrary, we have

$$\begin{aligned} \left(UT_iU^*\right) \xi_w = \overline{\mu_w}\, UT_i\xi_w &= \overline{\mu_w}\lambda_{i,w} U\xi_{iw} \\ &= \left(\overline{\mu_w}\lambda_{i,w}\mu_{iw}\right) \xi_{iw}. \end{aligned}$$

This yields the desired conclusion. □

We next present a direct generalization of the factorization of weighted shift operators into products of the unilateral shift and diagonal weight operators.

Proposition 2.4. *Let $T = (T_1, \ldots, T_N)$ be a weighted shift. Then each T_i factors as $T_i = L_iW_i$, where W_i is a positive operator which is diagonal with respect to the standard basis for $\mathcal{H}_N$. It follows that the norms of the T_i and the row matrix T are given by:*

(i) $\|T_i\| = \sup\{\lambda_{i,w} : w \in \mathbb{F}_N^+\}$ *for* $1 \leq i \leq N$.
(ii) $\|T\| = \sup_{1 \leq i \leq N} \|T_i\| = \sup\{\lambda_{i,w} : w \in \mathbb{F}_N^+ \text{ and } 1 \leq i \leq N\}$.

Proof. For $1 \leq i \leq N$, the operators W_i are given by the equation

$$W_i\xi_w = (T_i\xi_w, \xi_{iw})\xi_w = \lambda_{i,w}\xi_w.$$

Since $W_i \geq 0$, we have $T_i^*T_i = W_i^*L_i^*L_iW_i = W_i^2$, which is diagonal. Hence, $W_i = (T_i^*T_i)^{1/2}$ and $T_i = L_iW_i$. Further, this shows that

$$\begin{aligned} \|T_i\|^2 = \|T_i^*T_i\| = \|W_i^2\| &= \sup\{\|W_i^2\xi_w\| : w \in \mathbb{F}_N^+\} \\ &= \sup\{\lambda_{i,w}^2 : w \in \mathbb{F}_N^+\}. \end{aligned}$$

On the other hand, the entries of the $N \times N$ matrix T^*T consist of $T_i^*T_i$'s down the diagonal and zero off the diagonal, since the ranges of the T_i are pairwise orthogonal. Hence, from the above computation

$$\|T\| = \|T^*T\|^{1/2} = \sup_{1 \leq i \leq N} \|T_i\| = \sup_{1 \leq i \leq N} \{\lambda_{i,w} : w \in \mathbb{F}_N^+\},$$

which completes the proof. □

We finish this section by observing that the C*-algebra generated by a noncommutative weighted shift, which is bounded below in an appropriate sense, contains the Cuntz-Toeplitz algebra.

Definition 2.5. Let $T = (T_1, \ldots, T_N)$ be a weighted shift. If each W_i is bounded away from zero, in other words if

$$\inf\{\lambda_{i,w} : 1 \leq i \leq N \text{ and } w \in \mathbb{F}_N^+\} > 0,$$

we say that T is *bounded below.*

Corollary 2.6. *The* C*-*algebra* $\mathrm{C}^*(T_1, \ldots, T_N)$ *generated by the operators* $\{T_1, \ldots, T_N\}$ *from a weighted shift* $T = (T_1, \ldots, T_N)$ *contains* $\mathcal{E}_N = \mathrm{C}^*(L_1, \ldots, L_N)$ *when* T *is bounded below.*

Proof. From the proof of the previous proposition, we see that W_i is invertible precisely when $\inf\{\lambda_{i,w} : w \in \mathbb{F}_N^+\} > 0$. Thus, T being bounded below implies that each W_i is invertible. However, $W_i = (T_i^* T_i)^{1/2}$ belongs to $\mathrm{C}^*(T_i)$, and hence to $\mathrm{C}^*(T_1, \ldots, T_N)$, thus so does $L_i = T_i W_i^{-1}$ for $1 \leq i \leq N$. □

Note 2.7. We mention that the C*-algebras $\mathrm{C}^*(T_1, \ldots, T_N)$ generated by the T_i from a single weighted shift $T = (T_1, \ldots, T_N)$ are the focus of analysis in [7].

3. Fock space trees and periodicity

In this section we aim to convey to the reader a helpful pictorial method for thinking of noncommutative weighted shifts. In doing so, we introduce what seems to be a natural notion of periodicity for these operators. We also define the operator algebras which will be studied in the rest of the paper.

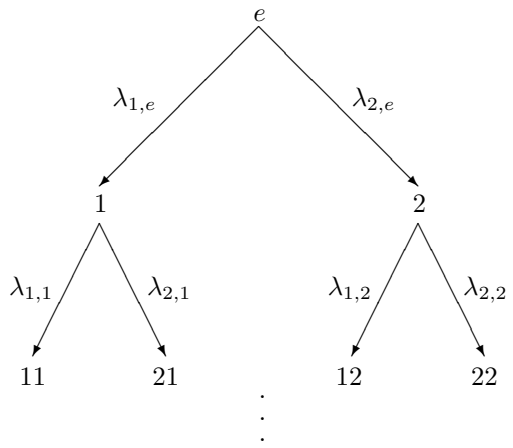
Recall that N-variable Fock space $\mathcal{H}_N = \ell^2(\mathbb{F}_N^+)$ has the orthonormal basis $\{\xi_w : w \in \mathbb{F}_N^+\}$. This basis yields a natural tree structure for Fock space which is traced out by the creation operators, and more generally by weighted shifts.

Definition 3.1. Let $T = (T_1, \ldots, T_N)$ be a weighted shift. Let $\mathcal{F}_T$ be the set of vertices $\{w : w \in \mathbb{F}_N^+\}$, together with the 'weighted' directed edges which correspond to the directions

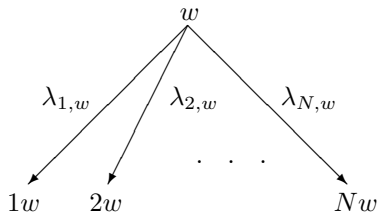
$$\{\lambda_{i,w} := w \mapsto iw \;\big|\; \text{for } 1 \leq i \leq N \text{ and } w \in \mathbb{F}_N^+\}.$$

We regard an edge $\lambda_{i,w}$ as lying to the left of another edge $\lambda_{j,w}$ precisely when $i < j$. We call $\mathcal{F}_T$ the *weighted Fock space tree* generated by T.

Pictorially, with $N = 2$ as an example, a typical weighted Fock space tree is given by the following diagram:



Notice that this structure is really determined by the operators $T = (T_1, \ldots, T_N)$. Indeed, given a basis vector ξ_w, the directed edge $\lambda_{i,w}$ corresponds to the action of T_i on ξ_w, namely mapping it to $\lambda_{i,w}\xi_{iw}$. Thus, more generally, we have the following picture for weighted edges leaving a typical vertex in the tree:



There are a number of conceptual benefits obtained by identifying these trees with weighted shifts. For instance, this point of view leads to the following notion of periodicity.

Definition 3.2. Let $k \geq 1$ be a positive integer. We say that a weighted shift $T = (T_1, \ldots, T_N)$ is of *period* k if

$$T_i\xi_w = \lambda_{i,u}\xi_{iw} \quad \text{for} \quad w \in \mathbb{F}_N^+,$$

where $w = uv$ is the *unique* decomposition of w with $0 \leq |u| < k$ and $|v| \equiv 0 \pmod{k}$.

Note 3.3. Observe that this says the scalars $\{\lambda_{i,u} : 0 \leq |u| < k\}$ *completely* determine the shift. They can be thought of as a 'remainder tree top'. For $N = 1$ the standard notion of periodicity is recovered, since the tree collapses to a single infinite stalk. For $N \geq 2$, it is most satisfying to think of this notion of periodicity in terms of the tree structure: If $T = (T_1, \ldots, T_N)$ is period k, then the remainder tree top, that is the finite top of the tree determined by vertices $\{w : |w| < k\}$ and edges $\{\lambda_{i,w} : |w| < k\}$, is 'repeated' throughout the entire weighted tree.

In fact, this finite tree top is repeated with a certain exponential growth. For instance, at the level of the tree corresponding to words $w \in \mathbb{F}_N^+$ of length nk for some positive integer $n \geq 1$, the top of this finite tree is repeated N^{nk} times, once for every word of length nk.

We mention that related tree top constructions play a key role in the paper [18].

Finally, we introduce the operator algebras which we are interested in studying.

Definition 3.4. For positive integers $N \geq 2$ and $k \geq 1$, let $\mathrm{C}_N^*(\text{per } k)$ be the C*-algebra (contained in $\mathcal{B}(\mathcal{H}_N)$) generated by the T_i from all weighted shifts $T = (T_1, \ldots, T_N)$ of period k.

It is clear from the picture given by the Fock space trees that if $n_1 | n_2$, then $\mathrm{C}_N^*(\text{per } n_1)$ is contained in $\mathrm{C}_N^*(\text{per } n_2)$. Thus, given an increasing sequence of positive integers $\{n_k\}_{k\geq 1}$ with $n_k | n_{k+1}$ for $k \geq 1$, we may consider the inductive limit algebra

$$\mathfrak{A}(n_k) = \overline{\bigcup_{k\geq 1} \mathrm{C}_N^*(\text{per } n_k)}$$

determined by this sequence. Let q be the quotient map of $\mathcal{B}(\mathcal{H}_N)$ onto the Calkin algebra. We are also interested in describing the inductive limit algebras $q(\mathfrak{A}(n_k))$.

Note 3.5. The reader may find it helpful to know that $\mathrm{C}_N^*(\text{per } k)$ is generated by the T_i from a single weighted shift. This is proved in the next section, using the matrix decompositions obtained there.

4. Main Theorem

The C*-algebra $\mathrm{C}_N^*(\text{per } k)$ generated by the k-periodic weighted shifts can be described in terms of a full matrix algebra with entries in a Cuntz-Toeplitz algebra. From the discussion in Section 1, recall the Cuntz-Toeplitz algebra $\mathcal{E}_{N^k}$ is the C*-algebra generated by the creation operators $L = (L_1, \ldots, L_{N^k})$ acting on N^k-variable Fock space $\mathcal{H}_{N^k}$.

Theorem 4.1. *For positive integers $N \geq 2$ and $k \geq 1$, let $d_{N,k}$ be the total number of words in $\mathbb{F}_N^+$ of length strictly less than k; that is, $d_{N,k} = 1 + N + \cdots + N^{k-1}$. Then the algebra $\mathrm{C}_N^*(\text{per } k)$ of k-periodic weighted shifts is unitarily equivalent to the algebra $\mathcal{M}_{d_{N,k}}(\mathcal{E}_{N^k})$ of $d_{N,k} \times d_{N,k}$ matrices with entries in $\mathcal{E}_{N^k}$. Further, this algebra is generated by the T_i from a single shift $T = (T_1, \ldots, T_N)$.*

Remark 4.2. At first glance the N^k appearing in the theorem may seem somewhat peculiar to the reader. We shall see that it arises from the exponential nature of periodicity here. We mention that the special case $N = 2$ and $k = 2$ of the theorem is expanded on in Example 4.7.

We shall prove the theorem in several stages. Throughout, $N \geq 2$ and $k \geq 1$ will be fixed positive integers. The first step is to decompose Fock space in a manner which will lead to simple matrix representations of the periodic weighted shifts.

Lemma 4.3. *For $w \in \mathbb{F}_N^+$ with $|w| < k$, the subspaces $\mathcal{K}_w$ of N-variable Fock space $\mathcal{H}_N$ given by*

$$\mathcal{K}_w = \operatorname{span}\{\xi_{wv} : |v| = km,\ m \geq 0\},$$

are pairwise orthogonal and

$$\mathcal{H}_N = \sum_{|w|<k} \oplus \mathcal{K}_w.$$

Further, for $|w| < k$, the operators $U_w : \mathcal{K}_e \to \mathcal{K}_w$ defined by $U_w \xi_v = \xi_{wv}$, for $|v| = km$ with $m \geq 0$, are unitary. Hence

$$U := \sum_{|w|<k} \oplus U_w \ : \ \mathcal{K}_e^{(d_{N,k})} \longrightarrow \mathcal{H}_N$$

is a unitary operator.

Proof. The subspaces $\mathcal{K}_w$ for $|w| < k$ clearly span $\mathcal{H}_N = \ell^2(\mathbb{F}_N^+)$ since any word $u \in \mathbb{F}_N^+$ can be written, in fact uniquely, as $u = u_1u_2$, where $|u_1| < k$ and k divides $|u_2|$. To see orthogonality, let w_1, w_2 be words with $|w_i| < k$, and consider typical basis vectors $\xi_{w_iv_i}$ for $\mathcal{K}_{w_i}$, where $|v_i| = km_i$ and $m_i \geq 0$. The only way the inner product $(\xi_{w_1v_1}, \xi_{w_2v_2})$ can be nonzero, is if $w_1v_1 = w_2v_2$, and hence by uniqueness of factorization we have $w_1 = w_2$ and $v_1 = v_2$.

The operators U_w as defined are unitary since they send one orthonormal basis to another. Spatially, these unitaries can be thought of as the restrictions of the isometries L_w, where $L_w := L_{i_1} \dots L_{i_s}$ when w is the word $w = i_1 \dots i_s$, to a distinguished subspace $\mathcal{K}_e$ of Fock space. Alternatively, the action of the adjoint U_w^* on $\mathcal{K}_w$ is described by restricting L_w^* to $\mathcal{K}_w$. The last statement of the lemma is immediate from the spatial decomposition of $\mathcal{H}_N$. □

We will distinguish between coordinate spaces of $\mathcal{K}_e^{(d_{N,k})}$ in the following manner: For $w \in \mathbb{F}_N^+$ with $|w| < k$, let

$$\{\, \xi_u^w \,:\, u \in \mathbb{F}_N^+ \quad \text{with} \quad |u| = km \quad \text{for} \quad m \geq 0\}$$

be the standard basis for the *wth coordinate space* of $\mathcal{K}_e^{(d_{N,k})}$, which is given by $U^*\mathcal{K}_w = U_w^*\mathcal{K}_w$. Notice that for $|v|, |w| < k$ and $|u| = km$, the vectors ξ_u^w and ξ_u^v really correspond to the *same* vector ξ_u in $\mathcal{K}_e$. Further, the action of U is described by

$$U\xi_u^w = \xi_{wu}, \tag{1}$$

for $w, u \in \mathbb{F}_N^+$ with $|w| < k$ and $|u| = km$.

The next step is to point out a relationship between particular Fock space trees.

Definition 4.4. We define a natural bijective correspondence between; the N^k words of length k in $\mathbb{F}_N^+$ on the one hand, and the N^k letters which generate $\mathbb{F}_{N^k}^+$ on the other, through the function

$$\varphi : \{w \in \mathbb{F}_N^+ : |w| = k\} \longrightarrow \{w \in \mathbb{F}_{N^k}^+ : |w| = 1\}$$

given by

$$\varphi(i_1i_2 \dots i_k) = (i_1 - 1)N^{k-1} + \dots + (i_{k-1} - 1)N + i_k,$$

for $1 \leq i_j \leq N$ and $1 \leq j \leq k$. This correspondence is also characterized by associating the words $\{iw \in \mathbb{F}_N^+ : |w| = k - 1\}$ with the 'ith block' of N^{k-1} letters in the listing $\{1, 2, \dots, N^k\}$. Notice with this ordering that the operators $\{L_{\varphi(w)} : w \in \mathbb{F}_N^+, |w| = k\}$ are the N^k creation operators associated with N^k-variable Fock space $\mathcal{H}_{N^k}$.

The map φ extends in a natural way to a bijective identification φ_m of the set $\{w \in \mathbb{F}_N^+ : |w| = km\}$ with $\{w \in \mathbb{F}_{N^k}^+ : |w| = m\}$ for $m \geq 0$. Given $w_1, \dots, w_m \in \mathbb{F}_N^+$ with $|w_i| = k$, the extensions are given by

$$\varphi_m(w_1 \dots w_m) = \varphi(w_1) \dots \varphi(w_m),$$

The units in $\mathbb{F}_N^+$ and $\mathbb{F}_{N^k}^+$ are identified with each other. We will use the notation φ for the extended map as well. This ordering leads to the following spatial equivalence.

Lemma 4.5. *The map from* $\mathcal{K}_e = \text{span}\{\xi_w : w \in \mathbb{F}_N^+, |w| = km, m \geq 0\}$ *to* N^k*-variable Fock space* $\mathcal{H}_{N^k} = \ell^2(\mathbb{F}_{N^k}^+)$ *which sends a basis vector* $\xi_{w_1 \ldots w_m} \in \mathcal{K}_e$*, where each* $|w_i| = k$*, to the basis vector* $\xi_{\varphi(w_1)\ldots\varphi(w_m)} \in \mathcal{H}_{N^k}$*, is unitary.*

Proof. This follows directly from the definitions of the space $\mathcal{K}_e$ and the map φ. □

This lemma gives us a tight spatial equivalence between the orthogonal direct sums $\mathcal{K}_e^{(d_{N,k})}$ and $\mathcal{H}_{N^k}^{(d_{N,k})}$, which carries through for the weighted shifts. We wish to preserve the correspondence discussed after Lemma 4.3. In particular, for $w \in \mathbb{F}_N^+$ with $|w| < k$, we let

$$\{\, \xi^w_{\varphi(u)} \,:\, u \in \mathbb{F}_N^+ \quad \text{with} \quad |u| = km \quad \text{for} \quad m \geq 0\}$$

be the standard basis for the wth coordinate space of $\mathcal{H}_{N^k}^{(d_{N,k})}$. Once again, with this identification, for $|v|, |w| < k$ and $|u| = km$, the vectors $\xi^w_{\varphi(u)}$ and $\xi^v_{\varphi(u)}$ correspond to the same vector $\xi_{\varphi(u)}$ in $\mathcal{H}_{N^k}$. Finally, we let $V : \mathcal{H}_{N^k}^{(d_{N,k})} \to \mathcal{K}_e^{(d_{N,k})}$ be the unitary operator which encodes this correspondence and the action of the map from the previous lemma. For $w, u \in \mathbb{F}_N^+$ with $|w| < k$ and $|u| = km$, the action of V is given by

$$V\xi^w_{\varphi(u)} = \xi^w_u. \tag{2}$$

With these Fock space decompositions in hand, we are now ready to focus on the particular actions of weighted shifts.

Lemma 4.6. *For* $w \in \mathbb{F}_N^+$ *with* $|w| < k$*, let* P_w *be the orthogonal projection of* $\mathcal{H}_{N^k}^{(d_{N,k})}$ *onto the* w*th coordinate space of* $\mathcal{H}_{N^k}^{(d_{N,k})}$*, so that* $I = \sum_{|w|<k} \oplus P_w$*. Let* $T = (T_1, \ldots, T_N)$ *be a* k*-periodic weighted shift acting on* $\mathcal{H}_N$*. Then the operators* $\text{Ad}_{UV}(T_i) = V^*U^*T_iUV$ *act on* $\mathcal{H}_{N^k}^{(d_{N,k})}$ *and have the following block matrix decompositions:*

(i) *For* $|w| < k-1$ *and* $|v| < k$*,*

$$P_v\big(\text{Ad}_{UV}(T_i)\big)P_w = \begin{cases} \lambda_{i,w} I_{\mathcal{H}_{N^k}} & \text{if } v = iw \\ 0 & \text{if } v \neq iw. \end{cases}$$

(ii) *For* $|w| = k-1$ *and* $|v| < k$*,*

$$P_v\big(\text{Ad}_{UV}(T_i)\big)P_w = \begin{cases} \lambda_{i,w} L_{\varphi(iw)} & \text{if } v = e \\ 0 & \text{if } v \neq e. \end{cases}$$

Proof. We first prove case (i). From the preceding discussion, the vectors $\xi^w_{\varphi(u)}$, where $u \in \mathbb{F}_N^+$ with $|u| = km$, form an orthonormal basis for the range of P_w. Further, from equations (1) and (2) we have

$$UV\xi^w_{\varphi(u)} = U\xi^w_u = \xi_{wu}.$$

Lastly, as $T = (T_1, \ldots, T_N)$ is k-periodic, we have $T_i\xi_{wu} = \lambda_{i,w}\xi_{iwu}$. Since $|w| < k-1$, these facts lead us to the following computation:

$$\begin{aligned} P_v\big(\text{Ad}_{UV}(T_i)\big)\xi^w_{\varphi(u)} &= P_vV^*U^*T_i\xi_{wu} \\ &= \lambda_{i,w}P_vV^*U^*\xi_{(iw)u} \\ &= \lambda_{i,w}P_v\xi^{iw}_{\varphi(u)} = \lambda_{i,w}\delta_{v,iw}\,\xi^{iw}_{\varphi(u)}, \end{aligned}$$

where $\delta_{v,iw}$ is equal to 1 if $v = iw$, and is 0 otherwise. But recall that the vectors $\xi^w_{\varphi(u)}$ and $\xi^{iw}_{\varphi(u)}$ both correspond to the same vector $\xi_{\varphi(u)}$ in $\mathcal{H}_{N^k}$. Thus, case (i) is established.

Now suppose that $|w| = k-1$. Again, k-periodicity gives us $T_i\xi_{wu} = \lambda_{i,w}\xi_{iwu}$. In this case $|iw| = k$, hence the definition of U and V from (1) and (2) yields:

$$\begin{aligned} P_v\big(\mathrm{Ad}_{UV}(T_i)\big)\xi^w_{\varphi(u)} &= P_vV^*U^*T_i\xi_{wu} \\ &= \lambda_{i,w}P_vV^*U^*\xi_{(iw)u} \\ &= \lambda_{i,w}P_vV^*\xi^e_{iwu} \\ &= \lambda_{i,w}P_v\xi^e_{\varphi(iwu)} = \lambda_{i,w}\delta_{v,e}\,\xi^e_{\varphi(iwu)}. \end{aligned}$$

However, from the definition of φ, we have $\varphi(iwu) = \varphi(iw)\varphi(u)$, since $|iw| = k$. Therefore,

$$P_v\big(\mathrm{Ad}_{UV}(T_i)\big)\xi^w_{\varphi(u)} = \lambda_{i,w}\delta_{v,e}L_{\varphi(iw)}\xi^e_{\varphi(u)}.$$

Once again, the vectors $\xi^w_{\varphi(u)}$ and $\xi^e_{\varphi(u)}$ both correspond to $\xi_{\varphi(u)}$ in $\mathcal{H}_{N^k}$. This establishes case (ii), and completes the proof. □

Proof of Theorem 4.1. We define an injective homomorphism of C*-algebras $\pi : \mathrm{C}^*_N(\mathrm{per}\ k) \to \mathcal{M}_{d_{N,k}}(\mathcal{E}_{N^k})$ by $\pi(T_i) = \mathrm{Ad}_{UV}(T_i)$, for every k-periodic weighted shift $T = (T_1, \ldots, T_N)$. The map π is clearly an injective homomorphism since it is a unitary equivalence. Further, it follows from case (i) in Lemma 4.6, that all the matrix units in $\mathcal{M}_{d_{N,k}}(\mathcal{E}_{N^k})$ can be obtained in the image of π, by judicious choice of scalars $\lambda_{i,w}$'s and appropriate matrix multiplication. From case (ii) in that lemma, we see that the N^k creation operators which generate $\mathcal{E}_{N^k}$ can be obtained in certain matrix entries. Since all the matrix units are present in the image, these creation operators can be moved around to every entry. Therefore, it follows that π is also surjective, and hence defines a $*$-isomorphism.

Lastly, it is not hard to see from the matrix decompositions of Lemma 4.6 that the algebra $\mathrm{C}^*_N(\mathrm{per}\ k)$ is generated by the T_i from a single shift $T = (T_1, \ldots, T_N)$. For instance, from work in [7] it follows that any shift will do for which the N^k numeric k-tuples corresponding to the weights on each path of length k in the associated tree are different. □

Before continuing, we discuss a special case of the theorem which may help to clarify some of the technical issues.

Example 4.7. Consider the case when $N = 2$ and $k = 2$. Then $\mathrm{C}^*_2(\mathrm{per}\,2)$ is the C*-algebra generated by the T_i from all 2-periodic shifts $T = (T_1, T_2)$. The theorem shows that this algebra is unitarily equivalent to the matrix algebra $\mathcal{M}_3(\mathcal{E}_4)$. Let us expand on this point.

Such 2-tuples act on the Fock space $\mathcal{H}_2$, which has orthonormal basis $\{\xi_w : w \in \mathbb{F}_2^+\}$. As in the previous section, the remainder tree top which determines the weighted Fock space tree for a given 2-periodic shift $T = (T_1, T_2)$ is generated by six scalars $\{a, b, c, d, e, f\}$ as follows:

$$T_1\xi_e = a\,\xi_1,\ \ T_1\xi_1 = c\,\xi_{1^2},\ \ T_1\xi_2 = e\,\xi_{12}.$$

and

$$T_2\xi_e = b\,\xi_2,\ \ T_2\xi_1 = d\,\xi_{21},\ \ T_2\xi_2 = f\,\xi_{2^2}.$$

Thus, for example, the action of T_1 on basis vectors is given by

$$T_1\xi_w = \begin{cases} a\,\xi_{1w} & \text{if } |w| \text{ is even} \\ c\,\xi_{1w} & \text{if } w = 1v \text{ and } |v| \text{ is even} \\ e\,\xi_{1w} & \text{if } w = 2v \text{ and } |v| \text{ is even.} \end{cases}$$

In the proof of the theorem for this case, 2-variable Fock space $\mathcal{H}_2 = \ell^2(\mathbb{F}_2^+)$ decomposes into a direct sum $\mathcal{H}_2 = \mathcal{K}_e \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$ of $d_{2,2} = 1+2 = 3$ subspaces, each of which may be naturally identified with $(2^2 =)4$-variable Fock space $\mathcal{H}_4 = \ell^2(\mathbb{F}_4^+)$. Take $\mathcal{K}_e$ for example. It is given by

$$\mathcal{K}_e = \operatorname{span}\Big\{\xi_e, \{\xi_{1^2}, \xi_{12}, \xi_{21}, \xi_{2^2}\}, \{\xi_w : w \in \mathbb{F}_2^+, |w| = 4\}, \dots\Big\}.$$

The unitary equivalence produced by this spatial identification yields the following block matrix form for our given 2-periodic shift $T = (T_1, T_2)$, with respect to the decomposition $\mathcal{H}_2 = \mathcal{K}_e \oplus \mathcal{K}_1 \oplus \mathcal{K}_2 \simeq \mathcal{H}_4 \oplus \mathcal{H}_4 \oplus \mathcal{H}_4$,

$$T_1 \simeq \begin{bmatrix} 0 & cL_1 & eL_3 \\ aI & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad T_2 \simeq \begin{bmatrix} 0 & dL_2 & fL_4 \\ 0 & 0 & 0 \\ bI & 0 & 0 \end{bmatrix},$$

where $\{L_1, L_2, L_3, L_4\}$ are the standard creation operators on $\mathcal{H}_4$.

Since we have complete freedom in $\mathrm{C}_2^*(\mathrm{per}\,2)$ on the choices of scalars a, b, c, d, e, f, it is now easy to see why it is unitarily equivalent to the matrix algebra $\mathcal{M}_3(\mathcal{E}_4)$. Further, it follows from these matrix decompositions that $\mathrm{C}_2^*(\mathrm{per}\,2)$ is generated, for instance, by $\{T_1, T_2\}$ with $a = b = 1$, $c = 1/2$, $d = 1/4$, $e = 1/8$, $f = 1/16$.

From Theorem 4.1 it follows that when we factor out the ideal of compact operators from $\mathrm{C}_N^*(\mathrm{per}\ k)$, simple C*-algebras are obtained. The key point being that the Cuntz-Toeplitz algebra is the extension of the compacts by the Cuntz algebra.

Corollary 4.8. *Let q be the quotient map of $\mathcal{B}(\mathcal{H}_N)$ onto the Calkin algebra. Then for $N \geq 2$ and $k \geq 1$, the algebra $q(\mathrm{C}_N^*(\mathrm{per}\ k))$ is $*$-isomorphic to the matrix algebra $\mathcal{M}_{d_{N,k}}(\mathcal{O}_{N^k})$. In particular, it is a simple* C*-*algebra.*

It follows that the inductive limit algebras $q(\mathfrak{A}(n_k))$ defined in the previous section are simple and have real rank zero.

Corollary 4.9. *Let $N \geq 2$ and let $\{n_k\}_{k\geq 1}$ be an increasing sequence of positive integers such that n_k divides n_{k+1} for $k \geq 1$. Then the inductive limit algebra $q(\mathfrak{A}(n_k))$ is simple and has real rank zero.*

Proof. Every ideal in $q(\mathfrak{A}(n_k))$ is the closed union of ideals of the subalgebras $q(\mathrm{C}_N^*(\mathrm{per}\ n_k))$. Thus, simplicity follows immediately from the previous corollary. These algebras have real rank zero because, as observed in [25], the class of C*-algebras of real rank zero is closed under tensoring with finite dimensional C*-algebras, forming direct sums, and forming inductive limits. The Cuntz algebras $\mathcal{O}_{N^{n_k}}$ have real rank zero since they are purely infinite. □

We finish this section by pointing out the connection between our results and the single variable setting results of Bunce-Deddens.

Remark 4.10. The focus of this paper is on the noncommutative multivariable setting; however, we remark that the proof of Theorem 4.1 goes through as presented for $N = 1$. Namely, the C*-algebra generated by all unilateral weighted

shift operators of period k is isomorphic to $\mathcal{M}_k(\mathcal{E}_1)$, where $\mathcal{E}_1$ is the C*-algebra generated by the unilateral shift, also realized as the algebra of Toeplitz operators with continuous symbol [3, 4]. The proof presented here recaptures this result for $N = 1$, although from a different perspective. In particular, the Bunce-Deddens proof heavily relies on the associated function theory which is omnipresent in the single variable case. Conceptually speaking the proof here is more spatially oriented.

While more effort is required to prove Theorem 4.1 for $N \geq 2$, the simplicity in Corollary 4.6 is more easily obtained as compared to the single variable case. The basic point is that $\mathcal{O}_N$ is simple for $N \geq 2$, while for $N = 1$ it is the C*-algebra generated by the bilateral shift operator, the algebra of continuous functions on the unit circle, which is not simple. Nonetheless, the inductive limit algebras $q(\mathfrak{A}(n_k))$ turn out to be simple for $N = 1$, and thus our results on these algebras for $N \geq 2$ can be regarded as a noncommutative multivariable generalization of their result.

5. Classification

In this section, we establish an isomorphism theorem for the limit algebras discussed in the previous two sections. Let $\{n_k\}_{k\geq 1}$ be an increasing sequence of positive integers with n_k dividing n_{k+1} for $k \geq 1$. Then for each prime p, there is a unique α_p in $\mathbb{N} \cup \{\infty\}$ which is the supremum of the exponents of p which divide n_k as $k \to \infty$. The *supernatural number* determined by the sequence $\{n_k\}_{k\geq 1}$ is the formal product $\delta(n_k) = \prod_{p \text{ prime}} p^{\alpha_p}$. Given two such sequences $\{n_k\}_{k\geq 1}$ and $\{m_j\}_{j\geq 1}$, it follows that $\delta(n_k) = \delta(m_j)$ precisely when: for all $k \geq 1$, there is a $j \geq 1$ with $n_k | m_j$; and for all $j \geq 1$, there is a $k \geq 1$ with $m_j | n_k$.

Supernatural numbers have been used to classify UHF algebras [16], and Bunce-Deddens algebras [3, 4]. They also distinguish between the inductive limit algebras of the current paper, as do the associated K_0 groups.

Theorem 5.1. *Let $N \geq 2$ be a positive integer. Let $\{n_k\}_{k\geq 1}$ and $\{m_j\}_{j\geq 1}$ be increasing sequences of positive integers for which $n_k | n_{k+1}$ and $m_j | m_{j+1}$ for $j, k \geq 1$. Then the following are equivalent:*

(i) *The supernatural numbers $\delta(n_k)$ and $\delta(m_j)$ are the same.*
(ii) *The algebras $\mathfrak{A}(n_k)$ and $\mathfrak{A}(m_j)$ are equal.*
(iii) *The algebras $q(\mathfrak{A}(n_k))$ and $q(\mathfrak{A}(m_j))$ are equal.*
(iv) *If $\mathfrak{B}(n_k)$ is an inductive limit of Cuntz algebras determined by a sequence $B_{n_1} \to B_{n_2} \to \ldots$ such that $B_{n_k} \cong \mathcal{M}_{d_{N,n_k}}(\mathcal{O}_{N^{n_k}})$, and $\mathfrak{B}(m_j)$ is similarly defined, then $\mathfrak{B}(n_k)$ and $\mathfrak{B}(m_j)$ are $*$-isomorphic.*
(v) *The groups $K_0(\mathfrak{B}(n_k))$ and $K_0(\mathfrak{B}(m_j))$ are isomorphic.*

Proof. To see (i) $\Rightarrow$ (ii), observe the division property associated with (i) shows that each n_k divides some m_j. Thus,

$$\mathrm{C}_N^*(\text{per } n_k) \subseteq \mathrm{C}_N^*(\text{per } m_j) \subseteq \mathfrak{A}(m_j)$$

for $k \geq 1$. Whence, $\mathfrak{A}(n_k) \subseteq \mathfrak{A}(m_j)$. The converse inclusion follows by symmetry. The implications (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are obvious. Since $\mathfrak{B}(n_k) \cong q(\mathfrak{A}(n_k))$ and $\mathfrak{B}(m_j) \cong q(\mathfrak{A}(m_j))$ by Corollary 4.8, we have (iii) $\Rightarrow$ (iv).

It remains to establish the implication (v) ⇒ (i). Recall the K_0 group of $\mathcal{O}_N$ is the finite abelian group $K_0(\mathcal{O}_N) = \mathbb{Z}/(N-1)\mathbb{Z}$ of order $N-1$. Hence,

$$K_0(\mathcal{M}_{d_{N,n_k}}(\mathcal{O}_{N^{n_k}})) = K_0(\mathcal{O}_{N^{n_k}}) = \mathbb{Z}/(N^{n_k}-1)\mathbb{Z}.$$

Since $\mathfrak{B}(n_k)$ is the inductive limit of the algebras B_{n_k}, we have

$$K_0(\mathfrak{B}(n_k)) = \varinjlim K_0(B_{n_k}).$$

There are similar facts for $K_0(\mathfrak{B}(m_j)) = \varinjlim K_0(B_{m_j})$.

For $k \geq 1$, let $g_k \in K_0(\mathfrak{B}(n_k))$ be an element of order $N^{n_k}-1$. Let $\Gamma : K_0(\mathfrak{B}(n_k)) \to K_0(\mathfrak{B}(m_j))$ be a group isomorphism. Then $\Gamma(g_k) \in K_0(\mathfrak{B}(m_j))$, and it follows that the order of $\Gamma(g_k)$ must divide $N^{m_j}-1$ for some $j \geq 1$. (Every element of $K_0(\mathfrak{B}(m_j))$ has this property.)

However, for positive integers $N \geq 2$ and $k \geq 1$, recall that

$$d_{N,k} = 1 + N + \cdots + N^{k-1} = \frac{N^k - 1}{N-1}.$$

Thus we have just observed that d_{N,n_k} divides d_{N,m_j}. But this implies that n_k divides m_j. Indeed, suppose $d_{N,m_j} = c\, d_{N,n_k}$ for some positive integer c. Consider the base N^{n_k} expansion of c given by $c = c_0 + c_1 N^{n_k} + \cdots + c_l (N^{n_k})^l$, where $0 \leq c_i < N^{n_k}$ for $0 \leq i \leq l$. By comparing coefficients in $d_{N,m_j} = c\, d_{N,n_k}$, we get each $c_i = 1$ and $m_j - 1 = l n_k + n_k - 1$. Whence, $m_j = (l+1)n_k$, and n_k divides m_j.

By symmetry, every m_j divides some n_k. Hence by the remarks preceding the theorem, this shows that the supernatural numbers $\delta(n_k)$ and $\delta(m_j)$ are identical. □

Remark 5.2. After preparing this article, the author became aware of related work of Evans [13] on Cuntz-Krieger algebras. The most notable overlap between our papers is that the equivalence of conditions (i) and (v) in the previous theorem follows from the Cuntz algebra case of Theorem 4.3 from [13]. We also point out that a related notion of periodicity is used to define certain inductive limits of Cuntz-Krieger algebras in [13]. In the Cuntz algebra case, we see it is a more restrictive version; requiring scalars $\{c_k : k \geq 0\}$ such that $\lambda_{i,w} = c_k$ for $1 \leq i \leq N$ and all $|w| = k$. Thus the periodicity introduced here is new, as is our main result Theorem 4.1.

We finish by pointing out that, not surprisingly, as for the Bunce-Deddens algebras, the algebras here are *not* almost finite dimensional. We need the following easy generalization of a theorem of Halmos.

Lemma 5.3. *For $1 \leq i \leq N$, the operator L_i is not quasitriangular.*

Proof. In [17], Halmos proves that the unilateral shift is not quasitriangular. However, as he points out before proving this result, the proof really only depends on the operator of concern being an isometry, with adjoint having nontrivial kernel. The L_i clearly have this property since L_i^* annihilates the vacuum vector. □

We can follow the lines of the Bunce-Deddens proof to show these algebras are not AF.

Theorem 5.4. *The algebras $\mathfrak{A}(n_k)$ and $q(\mathfrak{A}(n_k))$ are not approximately finite dimensional.*

Proof. Suppose there are finite dimensional C*-algebras $\{\mathfrak{B}_l\}_{l\geq 1}$ for which $\mathfrak{B}_l \subseteq \mathfrak{B}_{l+1}$ and $q(\mathfrak{A}(n_k)) = \overline{\bigcup_{l\geq 1} \mathfrak{B}_l}$. Then given $\varepsilon > 0$, there is a $B_i \in \bigcup_{l\geq 1} \mathfrak{B}_l$ with $\|q(L_i) - B_i\| < \varepsilon$. Choose $A_i \in \mathfrak{A}(n_k)$ such that $q(A_i) = B_i$. Then $\|q(L_i - A_i)\| < \varepsilon$, so there is a compact operator C_i with $\|L_i - A_i - C_i\| < \varepsilon$. But B_i belongs to a finite dimensional C*-algebra, hence there is a nontrivial polynomial p with $p(B_i) = p(q(A_i)) = 0$. Thus A_i is polynomially compact, and as such, it is quasitriangular. (This was proved initially in [12].) In particular, $A_i + C_i$ is quasitriangular, so that L_i belongs to the norm closure of the quasitriangular operators, and is itself quasitriangular [17]. This contradicts the previous lemma. Thus $q(\mathfrak{A}(n_k))$ is not approximately finite dimensional. The proof that $\mathfrak{A}(n_k)$ is not AF is easier. □

Acknowledgements. The author is grateful to Raul Curto, Ken Davidson, Palle Jorgensen and Paul Muhly for enlightening conversations. Thanks also to the Department of Mathematics at the University of Iowa for kind hospitality during the preparation of this article.

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