New York Journal of Mathematics

New York J. Math. **8** (2002) 235–240.

Cosemisimple Hopf Algebras with Antipode of Arbitrary Finite Order

Julien Bichon

Abstract. Let $m \geq 1$ be a positive integer. We show that, over an algebraically closed field of characteristic zero, there exist cosemisimple Hopf algebras having an antipode of order $2m$. We also discuss the Schur indicator for such Hopf algebras.

Contents

1. Introduction 235
2. Proof of Theorem 1 236
3. The Schur indicator for cosemisimple Hopf algebras 237
References 239

1. Introduction

The order of the antipode of a Hopf algebra has been an important subject of study since the beginning of the theory in the 60's. Let us first recall some of the highlights.

- First, there is the following fact: the antipode of a commutative or cocommutative Hopf algebra is involutive.
- In 1971, Taft [13] constructed finite-dimensional Hopf algebras with antipode of arbitrary even order. The Taft algebras are not cosemisimple.
- In 1975, Kaplansky [6] conjectured that the antipode of a finite-dimensional cosemisimple Hopf algebra is involutive.
- Radford ([12], 1976) proved that the order of the antipode of a finite-dimensional Hopf algebra is finite.
- Kaplansky's conjecture was proved in characteristic zero by Larson-Radford ([9, 10], 1987). The conjecture remains opened in general, although Etingof-Gelaki ([4], 1998) proved it in positive characteristic, under the additional assumption of semisimplicity.

Received May 21, 2002.
Mathematics Subject Classification. 16W30.
Key words and phrases. Hopf algebra, antipode, Schur indicator.

ISSN 1076-9803/02

It seems that the only known examples of cosemisimple Hopf algebras have either involutive antipode or antipode of infinite order (the antipode of a cosemisimple Hopf algebra is always bijective). In this note, we present, under the assumption that the base field is algebraically closed of characteristic zero, examples of cosemisimple Hopf algebras with antipode of arbitrary even order:

Theorem 1. *Let $m \geq 1$ be a positive integer and let k be an algebraically closed field of characteristic zero. There exists a cosemisimple Hopf algebra over k having an antipode of order $2m$.*

In fact the Hopf algebras of the Theorem were introduced by Dubois-Violette and Launer [3]. Their properties are straightforward consequences of the results of [1]. Using these Hopf algebras, we also have a partial negative answer to a question raised in [5] (see Remark 5).

In Section 3, we study the Schur indicator for cosemisimple Hopf algebras. We get a generalization of the Frobenius-Schur theorem of Linchenko-Montgomery [11] to the case of cosemisimple Hopf algebras with involutive antipode. Finally we use the Hopf algebras of the second section to illustrate the difficulty for proving a wider generalization, even when the order of the antipode is finite.

We work over k, a fixed algebraically closed field of characteristic zero.

2. Proof of Theorem 1

We consider the universal Hopf algebra associated to a nondegenerate bilinear form, introduced by Dubois-Violette and Launer [3]. Let $n \in \mathbb{N}^*$, and let $E \in GL_n(k)$. We consider the following algebra $\mathcal{B}(E)$: it is the universal algebra with generators $(a_{ij})_{1\leq i,j\leq n}$ and satisfying the relations

$$E^{-1t}aEa = I = aE^{-1t}aE,$$

where a is the matrix $(a_{ij})_{1\leq i,j\leq n}$ and I is the identity matrix. The algebra $\mathcal{B}(E)$ admits a Hopf algebra structure, with comultiplication Δ defined by $\Delta(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj}$, $1 \leq i,j \leq n$, with counit ε defined by $\varepsilon(a_{ij}) = \delta_{ij}$, $1 \leq i,j \leq n$, and with antipode S defined on the matrix $a = (a_{ij})$ by $S(a) = E^{-1t}aE$.

For $\alpha_1, \ldots, \alpha_n \in k$, the corresponding anti-diagonal matrix is denoted by $\mathrm{AD}(\alpha_1, \ldots, \alpha_n)$ (the coefficient α_n is located at the top-right of the matrix). The corresponding diagonal matrix is denoted by $\mathrm{D}(\alpha_1, \ldots, \alpha_n)$.

Now fix $m \geq 1$ and $\xi \in k^*$ a primitive m-th root of unity. We consider the matrices

$$E = \mathrm{AD}(\xi, 1, 1, 1, 1, 1) \quad \text{and} \quad F = E^{-1t}E = \mathrm{D}(\xi^{-1}, 1, 1, 1, 1, \xi).$$

Proposition 2. *The Hopf algebra $\mathcal{B}(E)$ is cosemisimple, with antipode of order $2m$.*

Proof. We first prove that $\mathcal{B}(E)$ is cosemisimple. Let $q \in k^*$ be such that $q^2 + \mathrm{tr}(F)q + 1 = 0$. Then by [1], Theorem 1.1, there exists an equivalence of monoidal categories

$$\mathrm{Comod}(\mathcal{B}(E)) \cong^{\otimes} \mathrm{Comod}(\mathcal{O}(SL_q(2))).$$

It is well-known that $\mathcal{O}(SL_q(2))$ is cosemisimple if and only if $q = \pm 1$ or if q is not a root of unity (see e.g., [7]). For the value of q chosen here, it is easily seen that

either $q = -1$ or q is not a root of unity (e.g., embedding $\mathbb{Q}(\xi)$ into $\mathbb{C}$). Hence $\mathcal{B}(E)$ is cosemisimple.

Since $S^2(a) = FaF^{-1}$, we have $S^{2m}(a) = a$ and the antipode of $\mathcal{B}(E)$ has order $\leq 2m$. Now consider the 6-dimensional comodule associated to the elements a_{ij}: this comodule corresponds, via the category equivalence of [1], to the simple 2-dimensional $\mathcal{O}(SL_q(2))$-comodule. It follows that the elements a_{ij}, $1 \leq i, j \leq 6$, are linearly independent. Let $G \in M_6(k)$ be such that $aG = G^t a$. The linear independence of the a_{ij}'s forces $G = 0$, and thus it is clear that the antipode of $\mathcal{B}(E)$ has even order. Now let $k \in \mathbb{N}^*$ be such that $S^{2k}(a) = F^k a F^{-k} = a$. Again by the linear independence of the a_{ij}'s, there exists $\lambda \in k^*$ such that $F^k = \lambda I$, and hence $\xi^k = 1$. Since ξ is a primitive m-th root of unity, m divides k and we conclude that the antipode of $\mathcal{B}(E)$ has order $2m$. □

Theorem 1 is an immediate consequence of Proposition 2.

Remark 3. The cosemisimplicity of $\mathcal{B}(E)$ cannot be proved using a compactness-like argument (we assume in this remark that $k = \mathbb{C}$). Indeed $\mathcal{B}(E)$ does not admit a CQG algebra structure [7]. This is easily seen examining the eigenvalues of the matrix F (see [7], Lemma 30 of Chapter 11).

Remark 4. The matrix E just considered does not have the smallest possible size for particular values of m. Here are the smallest sizes we have found. Again ξ is a primitive m-th root of unity.

- Assume that $m \geq 5$. Put $E = \mathrm{AD}(1, 1, \xi)$. Then $\mathcal{B}(E)$ is a cosemisimple Hopf algebra with antipode of order $2m$.
- Assume that $m = 4$. Put $E = \mathrm{AD}(1, 1, 1, \xi)$. Then $\mathcal{B}(E)$ is a cosemisimple Hopf algebra with antipode of order 8.
- Assume that $m = 3$. Put $E = \mathrm{AD}(1, 1, \xi, \xi)$. Then $\mathcal{B}(E)$ is a cosemisimple Hopf algebra with antipode of order 6, and is cotriangular since the corresponding q in Theorem 1.1 of [1] is $q = 1$.

Remark 5. Let $t \in k^*$ be such that $t^2 + 3t + 1 = 0$. Let $E = \mathrm{AD}(1, 1, t)$. The corresponding q in Theorem 1.1 of [1] is $q = 1$, and thus $\mathcal{B}(E)$ is a cotriangular Hopf algebra. Since t^2 is not a root of unity and is an eigenvalue of S^2, we have a partial negative answer to Question 7.4 in [5] (of course $\mathcal{B}(E)$ is not the twist of a function algebra).

3. The Schur indicator for cosemisimple Hopf algebras

Let G be a compact group and let V be a complex finite-dimensional representation of G. Following the notation of [11], we define the Schur indicator of V to be

$$\nu_2(V) = \int_G \chi_V(g^2) dg.$$

The classical Frobenius-Schur theorem (see [2]) states that for an irreducible representation V, then $\nu_2(V) = 0, 1$ or -1, with $\nu_2(V) \neq 0$ if and only if V is self-dual. The case $\nu_2(V) = 1$ corresponds to the existence of a G-invariant symmetric nondegenerate bilinear form on V, while the case $\nu_2(V) = -1$ corresponds to the existence of a G-invariant skew-symmetric nondegenerate bilinear form on V.

The Frobenius-Schur theorem for finite groups was generalized to finite-dimensional semisimple Hopf algebras by Linchenko-Montgomery [11]. We prove such a theorem for cosemisimple Hopf algebras with involutive antipode, and discuss the difficulty for predicting a more general theorem, even for cosemisimple Hopf algebras with antipode of order 4.

Let A be a cosemisimple Hopf algebra with Haar measure h, and let V be a finite-dimensional A-comodule with corresponding coalgebra map $\Phi_V : V^* \otimes V \longrightarrow A$. The character of V [8] is defined to be $\chi_V := \Phi_V(\mathrm{id}_V)$. Dualizing [11], we define the Schur indicator of V to be

$$\nu_2(V) := h(\chi_{V(1)}\chi_{V(2)}),$$

with Sweedler's notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

Theorem 6. *Let A be a cosemisimple Hopf algebra, and let V be a finite-dimensional irreducible A-comodule.*

1) *If the A-comodule V is not self-dual, then $\nu_2(V) = 0$.*

Assume now that V is self-dual.

2) *Let $\beta : V \otimes V \longrightarrow k$ be a nondegenerate A-colinear bilinear form. Let E be the matrix of β in some basis of V. Then*

$$\nu_2(V) = \frac{\dim(V)}{\mathrm{tr}(E{}^tE^{-1})}.$$

3) *Assume that $S^2 = \mathrm{id}$. Then $\nu_2(V) = \pm 1$. The case $\nu_2(V) = 1$ corresponds to the existence of an A-colinear symmetric nondegenerate bilinear form on V, while the case $\nu_2(V) = -1$ corresponds to the existence of an A-colinear skew-symmetric nondegenerate bilinear form on V.*

Proof. Let $e_1, \ldots, e_n$ be a basis of V, and let a_{ij}, $1 \leq i, j \leq n$, be the corresponding matrix coefficients of the comodule V. Then $\chi_V = \sum_i a_{ii}$ and $\nu_2(V) = \sum_{i,j} h(a_{ji}a_{ij})$. By the orthogonality relations [7], Proposition 15 of Section 11 (the orthogonality relations first appeared in [8]), we have $\nu_2(V) = 0$ if V is not self-dual.

Assume now that V is self-dual. Let E be the matrix of $\beta : V \otimes V \longrightarrow k$ in the fixed basis of V. Since β is A-colinear, we have $S(a) = E^{-1}{}^taE$ and $S^2(a) = E^{-1}{}^tEa^tE^{-1}E$. Then again by the orthogonality relations [7], we have

$$h(a_{kl}S(a_{ij})) = \delta_{kj}\frac{(E^{-1}{}^tE)_{il}}{\mathrm{tr}(E{}^tE^{-1})},\ 1 \leq i, j, k, l \leq n.$$

Using $S(a) = E^{-1}{}^taE$, a direct computation gives

$$h(a_{kl}a_{ij}) = \frac{E^{-1}_{ki}E_{lj}}{\mathrm{tr}(E{}^tE^{-1})},\ 1 \leq i, j, k, l \leq n.$$

This leads to

$$\nu_2(V) = \sum_{i,j}\frac{E^{-1}_{ji}E_{ij}}{\mathrm{tr}(E{}^tE^{-1})} = \frac{\dim(V)}{\mathrm{tr}(E{}^tE^{-1})},$$

as claimed. Assume finally that $S^2 = \mathrm{id}$. The linear independence of the a_{ij}'s forces $E^{-1}{}^tE = \lambda I$ for $\lambda \in k^*$, and necessarily $\lambda = \pm 1$. This gives $\nu_2(V) = \pm 1$. The last assertion is immediate. □

Using the Larson-Radford's theorems [9, 10], it is clear that Theorem 6 implies Theorem 3.1 of [11].

The Hopf algebras $\mathcal{O}(SL_q(2))$ show that the Schur indicator may take various possible values. Even in the case of cosemisimple Hopf algebras with antipode of finite order > 2, it seems difficult to predict all the possible values of the Schur indicator. The following example might convince the reader.

Example 7. Let $n \geq 3$, with n odd. We claim that there exists a cosemisimple Hopf algebra with antipode of order 4 having an irreducible comodule V with $\nu_2(V) = n$.

Put $k = \frac{n-1}{2}$. Consider the matrices

$$E = \mathrm{AD}(\underbrace{-1, -1, \ldots, -1}_{k \text{ times}}, \underbrace{1, 1, \ldots, 1}_{2n-k \text{ times}}) \quad \text{and}$$

$$E^{-1t}E = \mathrm{D}(\underbrace{-1, -1, \ldots, -1}_{k \text{ times}}, \underbrace{1, 1, \ldots, 1}_{2n-2k \text{ times}}, \underbrace{-1, -1, \ldots, -1}_{k \text{ times}}).$$

Then $\mathrm{tr}(E^tE^{-1}) = 2n - 4k = 2$, and hence $\mathcal{B}(E)$ is cosemisimple by [1]. Consider now the fundamental $2n$-dimensional $\mathcal{B}(E)$-comodule (irreducible by [1]), denoted by V. By Theorem 6 we have $\nu_2(V) = \frac{2n}{2} = n$, as claimed.

Remark 8. ($k = \mathbb{C}$) If A is cosemisimple Hopf algebra with antipode of finite order, it is easily seen, using Theorem 6, that $|\nu_2(V)| \geq 1$ for any self-dual irreducible comodule V.

References

[1] J. Bichon, *The representation category of the quantum group of a nondegenerate bilinear form*, Comm. Algebra, to appear.

[2] T. Bröcker and T. t. Dieck, *Representations of Compact Lie Groups*, Graduate Texts in Math., **98**, Springer, New York, 1985, MR 86i:22023, Zbl 0581.22009.

[3] M. Dubois-Violette and G. Launer, *The quantum group of a nondegenerate bilinear form*, Phys. Lett. B, **245(2)** (1990), 175–177, MR 91j:16049.

[4] P. Etingof and S. Gelaki, *On finite-dimensional semisimple and cosemisimple Hopf algebras in positive characteristic*, Int. Math. Res. Not., **16** (1998), 851–864, MR 99i:16068, Zbl 0918.16027.

[5] P. Etingof and S. Gelaki, *On cotriangular Hopf algebras*, Amer. J. Math., **123(4)** (2001), 699–713, MR 2002e:16057, Zbl 0990.16030.

[6] I. Kaplansky, *Bialgebras*, Lecture Notes in Mathematics, Department of Mathematics, University of Chicago, Chicago, 1975, MR 55 #8087.

[7] A. Klimyk and K. Schmüdgen, *Quantum Groups and their Representations*, Texts and Monographs in Physics, Springer, Berlin, 1997, MR 99f:17017, Zbl 0891.17010.

[8] R.G. Larson, *Characters of Hopf algebras*, J. Algebra, **17** (1971), 352–368, MR 44 #287, Zbl 0217.33801.

[9] R.G. Larson and D.E. Radford, *Semisimple cosemisimple Hopf algebras*, Amer. J. Math., **110** (1988), 187–195, MR 89a:16011, Zbl 0637.16006.

[10] R.G. Larson, D.E. Radford, *Finite dimensional cosemisimple Hopf algebras in characteristic* 0 *are semisimple*, J. Algebra, **117** (1988), 267–289, MR 89k:16016, Zbl 0649.16005.

[11] V. Linchenko and S. Montgomery, *A Frobenius-Schur theorem for Hopf algebras*, Algebr. Represent. Theory, **3(4)** (2000), 347–355, MR 2001k:16073, Zbl 0971.16018.

[12] D.E. Radford, *The order of the antipode of a finite-dimensional Hopf algebra is finite*, Amer. J. Math., **98** (1976), 333–355, MR 53 #10852, Zbl 0332.16007.

[13] E.J. Taft, *The order of the antipode of finite-dimensional Hopf algebras*, Proc. Nat. Acad of Sci. USA, **68(11)** (1971), 2631–2633, MR 44 #4075, Zbl 0222.16012.

Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l'Adour, IPRA, Avenue de l'université, 64000 Pau, France.
Julien.Bichon@univ-pau.fr

This paper is available via http://nyjm.albany.edu:8000/j/2002/8-15.html.