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A Model for the Universal Space for Proper Actions of a Hyperbolic Group

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Abstract. Let G be a word hyperbolic group in the sense of Gromov and P its associated Rips complex. We prove that the fixed point set P^H is contractible for every finite subgroup H of G. This is the main ingredient for proving that P is a finite model for the universal space $\underline{E}G$ for proper actions. As a corollary we get that a hyperbolic group has only finitely many conjugacy classes of finite subgroups.

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1. Introduction

The aim of this note is to prove the following theorem.

Theorem 1. *Let G be a δ-hyperbolic group relative to a set of generators S. Let P be the second barycentric subdivision of its Rips complex $P_d(G,S)$, with $d \geq 16\delta+8$. Then P is a finite G-CW-model for the universal space $\underline{E}G$ for proper actions, i.e., P consists of finitely many G-equivariant cells, P has only finite stabilizers and P^H is contractible for any finite subgroup $H \subset G$.*

This has been claimed in [BCH94, Section 2], but there does not seem to be a proof in the literature. We will give a complete and detailed proof. We are grateful for fruitful discussions with Martin Bridson. We thank the referee for some useful suggestions about the presentation.

We start with some basic definitions about universal spaces for proper actions. For a more detailed introduction see [LM00]. Let G be a discrete group. A G-CW-complex X is a CW-complex with a G-action that is cellular and whenever

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a cell is mapped on itself by some $g \in G$, the restriction of the action of g to this cell is the identity map. We call a G-CW-complex X *proper* if the stabilizers $G_x := \{g \in G \mid gx = x\}$ are finite for all $x \in X$. A G-CW-complex is called *finite* if X/G is compact.

Definition 2. A *G-CW-model for the universal space $\underline{E}G$ for proper actions of G* is a proper G-CW-complex such that the fixed point sets $\underline{E}G^H$ are contractible for every finite subgroup H of G.

For every group G there is such a model $\underline{E}G$ and any two such models are G-homotopy equivalent. The space $\underline{E}G$ has the following universal property: For any proper G-CW-complex X, there is a unique G-homotopy class of G-maps $X \to \underline{E}G$. We mention that the space $\underline{E}G$ plays an important role in the formulation of the Baum-Connes Conjecture [BCH94] and for the generalization of the Atiyah-Segal completion theorem [LO98]. In particular, finiteness conditions for $\underline{E}G$ have been studied for discrete groups, e.g., in [KM98], [Lüc00], and for locally compact groups in [LM00].

2. Hyperbolicity and the Rips Theorem

Let G be a finitely generated group and S a finite set of generators. We will always assume that the identity element is not contained in S, $e \notin S$ and that S is symmetric, that is $S = S^{-1}$. Given $x, y \in G$, write $x^{-1}y = s_1 \cdots s_d$ with a minimal number d of $s_i \in S$. Then $d_S(x, y) := d$ gives the left-invariant word metric on G. We set $[x, y] := \{x, xs_1, xs_1s_2, \ldots, xs_1 \cdots s_d = y\}$. This is a "geodesic" joining x and y. Note that $[x, y]$ is not unique. See [GdlH90, 1.2] for more details.

Definition 3. Let $d \in \mathbb{N}$. The *Rips complex* $P_d(G, S)$ is the simplicial complex whose k-simplices are given by $(k+1)$-tuples $(g_0, \ldots, g_k)$ of pairwise distinct elements of G with $\max\limits_{0 \le i,j \le k} d_S(g_i, g_j) \le d$. Observe that the 0-skeleton of $P_d(G, S)$ coincides with G.

If no confusion is possible, we well omit the notation of the set of generators, and simply write $d(.,.)$ instead of $d_S(.,.)$ for the word metric and $P_d(G)$ instead of $P_d(G, S)$ for the Rips complex. For finite subsets K, L of vertices we will use the notation

$$d(K, L) := \max_{k \in K, l \in L} d(k, l)$$

for the maximal distance between K and L. The *diameter* of K is given by

$$d(K) := d(K, K) = \max_{k, k' \in K} d(k, k').$$

Typically, we will look at diameters of orbits Hx of finite subgroups H. Notice that the H-invariance of the word metric implies the identity $d(Hx) = d(\{x\}, Hx)$. Since the word metric is left invariant, we have a simplicial action of G on $P_d(G, S)$ given by $g \cdot (g_0, \ldots, g_k) = (gg_0, \ldots, gg_k)$.

Definition 4. Let X be a metric space and $\delta \ge 0$. Then X is *δ-hyperbolic* if for any four points $x, y, z, t \in X$ the following inequality holds:

$$d(x, y) + d(z, t) \le \max\{d(x, z) + d(y, t), d(x, t) + d(y, z)\} + 2\delta.$$

A group G is δ-hyperbolic if it is δ-hyperbolic as metric space equipped with the word metric.

We quote the following theorem from [GdlH90, 12.- Théorème, p. 73]:

Theorem 5 (Rips Theorem)**.** *Let G be a δ-hyperbolic group for the set of generators S. Let $d \geq 4\delta + 2$ and P be the second barycentric subdivision of $P_d(G, S)$. Then P is a contractible, locally finite simplicial complex of finite dimension and with a simplicial action of G which is faithful and properly discontinuous. Moreover:*

(i) *The stabilizer of each simplex is finite.*
(ii) *If p is a vertex of P and $g \in G$ such that $p \neq gp$, then the stars of p and gp are disjoint.*
(iii) *The orbit space $G\backslash P$ is a finite simplicial complex and the projection $\pi : P \to G\backslash P$ is simplicial.*
(iv) *If G is torsion-free then the action is free.*

The Rips Theorem already proves some parts of Theorem 1. Also, we can conclude from parts (i) and (iv), that P is a finite G-CW-model for $\underline{E}G = EG$ in the torsion-free case. For the general case, the next lemma is an important tool, because it provides the existence of universally small orbits. In the following, for a real number r the notation $[r]$ will always mean the integral part of r.

Lemma 6. *Let G be a δ-hyperbolic group and y_0 a vertex of the Rips complex $P_d(G)$. Let H be a finite subgroup and $d(Hy_0) = R$.*

(a) *Then there is a vertex x of $P_d(G)$ such that*

$$d(Hx, Hy_0) \leq \left[\frac{R}{2}\right] + 2\delta + 1,$$
$$d(Hx) \leq 8\delta + 4.$$

(b) *If, in addition $R \geq 8\delta + 2$ and $d(y_0, x_0) = d(Hy_0, x_0)$ for some vertex x_0 of $P_d(G)$, then*

$$d(Hx, x_0) \leq d(x_0, y_0).$$

Proof. Let $y' \in Hy_0$ be a vertex on the orbit such that $d(y', y_0) = R$ and let x be a vertex on a geodesic $[y', y_0]$ such that $d(x, y_0) = \left[\frac{R}{2}\right]$. Then, using the definition of hyperbolicity for the points x, hy_0, y_0, y' we get:

$$\begin{aligned} d(x, hy_0) &\leq \max\{d(y', hy_0) + d(x, y_0), d(y_0, hy_0) + d(x, y')\} - d(y_0, y') + 2\delta \\ &\leq R + \left[\frac{R}{2}\right] + 1 - R + 2\delta \\ &\leq \left[\frac{R}{2}\right] + 2\delta + 1. \end{aligned}$$

Now the left invariance of the metric gives the first statement of (a). For the second part let $r := d(x, hx)$ for an $h \in H$ and v be a vertex on a geodesic $[x, hx]$ such that $d(v, x) = \left[\frac{r}{2}\right]$. Since $d(Hy_0) = R$, there is a vertex $z \in Hy_0$ such that $d(v, z) \geq \left[\frac{R}{2}\right]$. Applying the hyperbolicity to the points x, hx, v, z we get:

$$\begin{aligned} r = d(x, hx) &\leq \max\{d(x, z) + d(v, hx), d(hx, z) + d(v, x)\} - d(v, z) + 2\delta \\ &\leq \left[\frac{R}{2}\right] + 2\delta + 1 + \frac{r}{2} + 1 - \left[\frac{R}{2}\right] + 2\delta \\ &= \frac{r}{2} + 4\delta + 2. \end{aligned}$$

This implies $r \leq 8\delta + 4$ what we wanted to show.

For part (b) of the lemma we apply hyperbolicity to hx, x_0, y_0, y' and get

$$\begin{aligned} d(hx, x_0) &\leq \max\{d(hx, y_0) + d(x_0, y'), d(hx, y') + d(x_0, y_0)\} - d(y_0, y') + 2\delta \\ &\leq \left[\frac{R}{2}\right] + 2\delta + 1 + d(x_0, y_0) - R + 2\delta \\ &\leq d(x_0, y_0) \end{aligned}$$

because we assumed $R \geq 8\delta + 2$. □

The next proposition shows the contractibility of the fixed point sets. It is, roughly speaking, an H-invariant version of the proof of the non-equivariant contractibility of the Rips complex, cf. [GdlH90, 4.2].

Proposition 7. *Let G be δ-hyperbolic with set of generators S and $P_d(G)$ its Rips complex with $d \geq 16\delta + 8$. Let H be a finite subgroup of G. Then $P_d(G)^H$ is contractible.*

Proof. Let F be the following subcomplex of $P_d(G)$. A simplex σ of $P_d(G)$ belongs to F if σ contains an H-fixed point. This is the case if and only if H permutes the vertices of σ. Now add all the faces of these simplices to make F a subcomplex. Clearly, F is an H-invariant subcomplex and $F^H = P_d(G)^H$. To show the contractibility of F^H we proceed as follows: Let K' be a finite subcomplex of F, and set $K := HK'$. We will show that K is H-equivariantly contractible in F, i.e., the inclusion $K \hookrightarrow F$ is H-equivariantly homotopic to a constant map. This implies that $\pi_i(F^H) = 0$ for $i \geq 0$. Since F^H is a simplicial complex (after two barycentric subdivisions), this is all we need to prove.

By Lemma 6 we can find a vertex x_0 with $d(Hx_0) \leq 8\delta + 4$. Hence $x_0 \in F$, i.e., F is not empty. Without loss of generality we can assume that $x_0 \in K$. Let K_0 denote the finite set of vertices of K. We distinguish two cases:

Case 1. $\max_{y \in K_0} d(x_0, y) \leq \frac{d}{2}$. Then K_0 spans an H-invariant simplex σ that contains a fixed point. Any H-equivariant contraction of σ to this fixed point will also contract K in σ.

Case 2. $\max_{y \in K_0} d(x_0, y) > \frac{d}{2}$. Let $y_0 \in K_0$ be a point furthest away from x_0, i.e.,

$$d(y_0, x_0) = \max_{y \in K_0} d(x_0, y).$$

Let y_0' be the point on a geodesic $[x_0, y_0]$ with

$$d(y_0', x_0) = d(y_0, x_0) - \left[\frac{d}{4}\right].$$

Next, we want to define the function $f_0 : (K_0, x_0) \to (F, x_0)$, by

$$\begin{aligned} f(hy_0) &:= hy_0', \qquad h \in H, \\ f(y) &:= y, \qquad y \in K_0 \backslash Hy_0. \end{aligned}$$

Notice that H acts freely on K_0, hence the first part of the definition of f makes sense. We have to verify two more things to know that this function is well-defined. First that $x_0 \notin Hy_0$. This follows from the fact that $d(Hx_0) \leq 8\delta + 4 \leq \frac{d}{2}$, but $d(x_0, y_0) > \frac{d}{2}$. Secondly, that $y_0' \in F$. We will show this by proving $d(Hy_0') \leq d$. Then, by definition of F, we have $y_0' \in F$. Again, we have to look at two cases:

Case (a). $d(Hy_0) \leq \frac{d}{2}$: Then we have by the triangle inequality for each $h \in H$

$$\begin{aligned} d(hy_0', y_0') &\leq d(hy_0', hy_0) + d(hy_0, y_0) + d(y_0, y_0') \\ &\leq \left[\frac{d}{4}\right] + \frac{d}{2} + \left[\frac{d}{4}\right] \leq d, \end{aligned}$$

hence $d(Hy_0') \leq d$.

Case (b). $d(Hy_0) > \frac{d}{2}$: Since $y_0 \in F$ we know that $d(Hy_0) \leq d$. Thus we can apply Lemma 6(a) to Hy_0 to obtain a vertex x with orbit Hx satisfying

$$d(Hx) \leq 8\delta + 4$$

and

$$d(Hx, y_0) \leq \frac{d}{2} + 2\delta + 1.$$

Since we have $d(Hy_0) > \frac{d}{2} \geq 8\delta + 4$ and $d(x_0, y_0) = d(x_0, Hy_0)$ by the choice of y_0, Lemma 6(b) also gives

$$d(Hx, x_0) \leq d(x_0, y_0).$$

Hence, applying hyperbolicity to the points hx, y_0', y_0, x_0 we get:

$$\begin{aligned} d(hx, y_0') &\leq \max \left\{ \begin{array}{l} d(hx, y_0) + d(y_0', x_0) - d(y_0, x_0), \\ d(hx, x_0) + d(y_0, y_0') - d(y_0, x_0) \end{array} \right\} + 2\delta \\ &\leq \max \left\{ \frac{d}{2} + 2\delta + 1 - \left[\frac{d}{4}\right], \left[\frac{d}{4}\right] \right\} + 2\delta \\ &\leq \frac{d}{2} - \left[\frac{d}{4}\right] + 4\delta + 1 \\ &\leq \frac{d}{2}. \end{aligned}$$

The last estimation holds because we assumed $d \geq 16\delta + 8$, hence $\left[\frac{d}{4}\right] \geq 4\delta + 1$. Now we can again use the triangle inequality:

$$d(hy_0', y_0') \leq d(hy_0', hx) + d(hx, y_0') \leq \frac{d}{2} + \frac{d}{2} = d.$$

and we get $d(Hy_0') \leq d$. This completes Case (b).

Now that we know that f_0 is well-defined we claim:

Claim: f_0 can be extended to a simplicial map $f : (K, x_0) \to (F, x_0)$.

We have to show that for $x, y \in K_0$, $d(x, y) \leq d$ implies $d(f_0(x), f_0(y)) \leq d$. Since we only moved the orbit of y_0, there is only one non-trivial case to check, the implication:

$$d(y, hy_0) \leq d \Rightarrow d(y, hy_0') \leq d. \tag{1}$$

Because of the left-invariance of the metric (replace y by $h^{-1}y$) this is equivalent to:

$$d(y, y_0) \leq d \Rightarrow d(y, y_0') \leq d.$$

Applying hyperbolicity to y, y_0', y_0, x_0 we get

$$\begin{aligned} d(y,y_0') &\leq \max\left\{ \begin{array}{l} d(y,y_0)+d(x_0,y_0')-d(y_0,x_0), \\ d(y_0',y_0)+d(y,x_0)-d(y_0,x_0) \end{array} \right\} + 2\delta \\ &\leq \max\left\{ d-\left[\frac{d}{4}\right], \left[\frac{d}{4}\right] \right\} + 2\delta \\ &= d-\left[\frac{d}{4}\right] + 2\delta \leq d, \end{aligned}$$

since $d \geq 16\delta + 8$ implies $\left[\frac{d}{4}\right] \geq 2\delta$. Thus, the claim is verified.

Now f is by definition an H-equivariant map. It remains to show that f is H-homotopic to the inclusion map. But this follows by noticing that for any simplex σ of K the set $f(\sigma) \cup \sigma$ is contained in a simplex of F. This is clear except for the case where one point is in the orbit of y_0', but then it follows from implication (1).

Continuing this process with the finite complex $f(K)$ and iterating finitely many times leads to a finite subcomplex that will satisfy Case 1. This ends the proof. □

Proof of Theorem 1. Since P is a second barycentric subdivision of a simplicial complex with a simplicial G-action, P is by [Bre72, p. 117] a G-CW-complex. Theorem 5(i) states that all stabilizers are finite, (iii) proves that P is finite as G-CW-complex. The contractibility of P^H for finite groups H is shown in Proposition 7. □

The next corollary generalizes [GdlH90, Prop. 4.13, p. 73]. An alternative proof can be found in [BH99, Theorem 3.2].

Corollary 8. *A δ-hyperbolic group G has a finite number of conjugacy classes of finite subgroups.*

Proof. This is true for every discrete group with a model for $\underline{E}G$ of finite type, as is shown in [Lüc00, Theorem 4.2]. □

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