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Test Elements and the Retract Theorem in Hyperbolic Groups

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Abstract. We prove that in many, perhaps all, torsion free hyperbolic groups, test elements are precisely those elements not contained in proper retracts. We also show that all Fuchsian groups have this property. Finally, we show that all surface groups except $\mathbb{Z} \times \mathbb{Z}$ have test elements.

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0. Introduction

Definition 1. If G is a group then $g \in G$ is a *test element* if for any endomorphism $\phi : G \to G$, $\phi(g) = g$ implies that ϕ is an automorphism.

The notion of test elements was first considered in the context of free groups, in which they are called *test words*. The first example was provided in 1918 by Nielsen [N], who showed that the basic commutator $[a, b] = aba^{-1}b^{-1}$ is a test word in the free group $F(a, b)$. Considerable progress has been made recently in understanding both test elements and test words—see [Tu], for example, and the references cited there. The following reformulation of the definition makes clear how test elements are used to recognize automorphisms: g is a test element if $\phi(g) = \alpha(g)$ for *some* automorphism α implies that ϕ is an automorphism. Thus the issue of deciding whether ϕ is an automorphism is replaced by that of deciding whether $\phi(g)$ and g are equivalent under the action of the automorphism group $Aut(G)$. The classic algorithm of J. H. C. Whitehead [W] decides very effectively when two elements of a free group F are equivalent under the action of $Aut(F)$—in a forthcoming

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paper, D. J. Collins and the second author will show how to do the same thing in a torsion-free hyperbolic group.

Producing non-test elements is quite easy. Suppose, for example, that the subgroup R of G is a proper *retract*, i.e., the image of a non-surjective map $\rho : G \to G$, called a *retraction*, with the property that $\rho(r) = r$ for all $r \in R$. Then no element of R can be a test element. In [Tu] the following theorem—the *Retract Theorem for free groups*—was proven, characterizing test words in free groups.

Theorem. *A word w in a free group F is a test word if and only if w is not in any proper retract.*

We are interested in deciding whether the Retract Theorem is true for general hyperbolic groups. (By hyperbolic we mean word hyperbolic in the sense of Gromov—see, e.g., [GH]. In particular, hyperbolic groups are finitely generated.) We succeed in proving it for torsion free hyperbolic groups that are *stably hyperbolic* in the following sense.

Definition 2. A hyperbolic group is *stably hyperbolic* if for every endomorphism $\varphi : G \to G$, there are arbitrarily large values of n so that $\varphi^n(G)$ is hyperbolic.

Any hyperbolic group with the property that *every* finitely generated subgroup is hyperbolic (hyperbolic surface groups, for example) clearly satisfies this property. A famous application of the Rips construction [R] shows that some hyperbolic groups have finitely generated non-hyperbolic subgroups. We modify this construction in Section 1 to produce an example in which such a subgroup is the image of an endomorphism. It's relatively easy to extend this to find endomorphisms of hyperbolic groups that have arbitrarily many non-hyperbolic forward images, but in all cases we've studied, the images are eventually hyperbolic. It seems quite possible that all hyperbolic groups are actually stably hyperbolic.

Our specific results are the following.

Theorem 1. *If G is a torsion free stably hyperbolic group and $g \in G$, then g is a test element if and only if g is not in any proper retract.*

Theorem 2. *If H is a finitely generated Fuchsian group and $h \in H$, then h is a test element if and only if h is not in any proper retract. This applies in particular if H is a finite free product of cyclic groups.*

Theorem 3. *If G is a surface group other than $\mathbb{Z} \times \mathbb{Z}$ then G has test elements.*

The situation for surface groups is interesting—all surface groups except $\mathbb{Z} \times \mathbb{Z}$ have test elements (explicit examples given in Section 3) and all except the fundamental group of the Klein bottle $\langle a, b \mid aba^{-1} = b^{-1} \rangle$ satisfy the Retract Theorem. (In [V2] it was shown that b lies in no proper retract but is nevertheless not a test word since it is fixed by $\varphi(a) = a^3, \varphi(b) = b$.)

We will use the following terminology.

Definition 3. If G is a group and $\varphi : G \to G$ is an endomorphism, then

$$\varphi_n = \varphi\,|_{\varphi^n(G)} \colon \varphi^n(G) \to \varphi^n(G), \quad \text{and}$$
$$\varphi_\infty = \varphi\,|_{\varphi^\infty(G)} \colon \varphi^\infty(G) \to \varphi^\infty(G),$$

where $\varphi^\infty = \bigcap_{i=1}^{\infty} \varphi^n(G)$.

The group G is *Hopfian* if it is not isomorphic to any of its proper quotients and is *co-Hopfian* if it is not isomorphic to any of its proper subgroups.

1. Torsion free hyperbolic groups

This section is devoted to a proof of Theorem 1. Our proof of Theorem 1 is modeled on the proof of the Retract Theorem in [Tu], the main technical tool being Proposition 1, whose proof we defer to the end of the section.

Proposition 1. *Suppose that $G = G_1 * G_2 * \cdots * G_m$ is a free product of infinite cyclic and freely indecomposable co-Hopfian groups. If $\varphi : G \to G$ is a monomorphism, then $\varphi^{\infty}(G)$ is a free of factor G.*

Proposition 2. *If G is a torsion-free hyperbolic group and $\varphi : G \to G$ is an endomorphism with the property that $\varphi^n(G)$ is hyperbolic for arbitrarily large n, then $\varphi^{\infty}(G)$ is a free factor of $\varphi^N(G)$ for some N.*

Proof. It was proven by Sela [Se2] that if $\varphi : G \to G$ is an endomorphism of a torsion-free hyperbolic group G then $\varphi|_{\varphi^n(G)} : \varphi^n(G) \to \varphi^n(G)$ is a monomorphism for large enough n. Let n be large enough so that $\varphi|_{\varphi^n(G)}$ is a monomorphism and that $\varphi^n(G)$ is hyperbolic and consider the free product decomposition $\varphi^n(G) = H_1 * \cdots * H_m$ into freely indecomposable hyperbolic factors. Sela [Se1] has also proven that every freely indecomposable torsion free hyperbolic group is either co-Hopfian or infinite cyclic. Proposition 2 now follows from Proposition 1. □

Proof of Theorem 1. The fact that elements of proper retracts are not test elements is trivially true in all groups since a proper retraction is not an automorphism. To prove the converse, we begin with the following general observation: *if $\mu : G \to G$ is a monomorphism of a group G, then μ_{∞} is an automorphism of $\mu^{\infty}(G)$.* It is clear that μ_{∞} is injective. To see that μ_{∞} is surjective, let $g \in \mu^{\infty}(G)$; then for every n, there exists $g_n \in \mu^n(G)$ such that $g = \mu(g_n)$; since μ is injective, $g_m = g_n$ for all m, n. Hence $g_1 \in \mu^{\infty}(G)$, and μ is surjective.

Now suppose that G is a stably hyperbolic group, that g is not a test element and that φ is a endomorphism which is not an automorphism so that $\varphi(g) = g$—we show that g lies in a proper retract by showing that $\varphi^{\infty}(G)$ is a proper retract.

For large enough n, φ is a monomorphism by [Se2]; since $(\varphi^n)_{\infty} = \varphi_{\infty}$, φ_{∞} is an automorphism by the observation above. According to Proposition 2, $\varphi^{\infty}(G)$ is a free factor of $\varphi^N(G)$ for some N: choose such an N and let $\pi : \varphi^N(G) \to \varphi^{\infty}(G)$ be a free factor projection mapping. Then

$$\rho = \varphi_{\infty}^{-N} \circ \pi \circ \varphi^N : G \to \varphi^{\infty}(G)$$

is a retraction mapping. Thus φ^{∞} is a retract. □

It may be that the Retract Theorem is true for general hyperbolic groups: Theorem 2 is a partial result in this direction. It may also be the case that all hyperbolic groups are stably hyperbolic. However, the following example, suggested by G. A. Swarup, shows that it may be that $\varphi(G)$ is not hyperbolic (but in this case $\varphi^2(G)$ is trivial).

Example. Suppose that

$$F_2 \oplus F_2 = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4] \rangle$$

and let

$$\psi : F_2 \oplus F_2 \to F_2 \oplus F_2 \text{ by } \psi(x_i) = x_1 \text{ for } 1 \le i \le 4.$$

In [Dr] it was shown that $\ker(\psi)$ has rank 4 and is not finitely presented and is therefore not a hyperbolic group.

Now let $G = (F_2 \oplus F_2) * F_r$, $F_r = \langle y_1, \ldots, y_r \rangle$ and $\tilde{\psi} : G \to F_{r+1} = \langle x_1, y_1, \ldots, y_r \rangle$ by

$$\tilde{\psi}(x_i) = \psi(x_i),$$
$$\tilde{\psi}(y_i) = y_i.$$

and let $P = \ker\left(\tilde{\psi}\right)$. Then P has rank 4.

Now the Rips construction [R] produces a hyperbolic group H with rank $6 + r$ that maps onto G by a map ϵ with kernel generated by two added generators a and b. Then $H' = \epsilon^{-1}(P)$ is a non-hyperbolic subgroup of H of rank at most 6.

$$\begin{array}{ccccccccc}
 & & & & & & 1 & & \\
 & & & & & & \downarrow & & \\
 & & & & H' & \xrightarrow{\epsilon} & P & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \to & \langle a, b \rangle & \to & H & \xrightarrow{\epsilon} & G & \to & 1 \\
 & & & & & & \downarrow \tilde{\psi} & & \\
 & & & & & & F_{r+1} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 1 & &
\end{array}$$

Now let $r \geq 5$, $\pi : F_{r+1} \to H'$ be a surjection and $\varphi = \pi \circ \tilde{\psi} \circ \epsilon : H \to H'$. Then H is hyperbolic and ϕ is an endomorphism whose image H' is not hyperbolic.

Proof of Proposition 1. We begin by observing that it suffices to show that $\varphi^\infty(G)$ is a free factor of $\varphi^n(G)$ for some n (in fact we will show that this is true for all sufficiently large n). For if $\varphi^n(G) = \varphi^\infty(G) * G'$ for some G', then the monomorphism

$$\varphi^n : G \to \varphi^n(G)$$

pulls this factorization back to G:

$$G = \varphi^{-n}(\varphi^\infty(G)) * \varphi^{-n}(G').$$

But it is straightforward to show that $\varphi^{-n}(\varphi^\infty(G)) = \varphi^\infty(G)$.

The proof is a geometric generalization of the Takahasi Theorem [Ta] and is modeled on the proof of the Takahasi result outlined in problem 33 on page 118 of [MKS]). We will need the following slight variant of the usual normal form measure of length in a free product (which depends on the product decomposition).

Definition 4. If $G = G_1 * G_2 * \cdots * G_m$ is a free product of freely indecomposable factors, then the *length* $|g|_G$ is

$$|g|_G = \sum_{j=i}^{t} |g_j|_{G_{i_j}}.$$

where g has free product normal form $g = g_1 g_2 \ldots g_t$, $1 \neq g_j \in G_{i_j}$ for some i_j, $|g_j|_{G_{i_j}} = 1$ if $G_{i_j} \not\cong \mathbb{Z}$ and $|g_j|_{G_{i_j}} = n$ if $G_{i_j} \cong \mathbb{Z}$ and g_j is the n^{th} power of a generator.

We order the factors of G so that $G_i \cong \mathbb{Z}$ for $1 \le i \le \ell$; thus

$$G = F_\ell * G_{\ell+1} * \cdots * G_m.$$

Let X be the natural 2-complex with fundamental group G which is the wedge of ℓ circles and 2-complexes $K_{\ell+1}, \dots, K_m$, $\pi_1(K_i) = G_i$, each of which is joined to the wedge point by a line segment.

The nested sequence of subgroups

$$G \ge \varphi(G) \ge \varphi^2(G) \ge \cdots \ge \varphi^n(G) \ge \cdots \ge \varphi^\infty(G)$$

determines a sequence of coverings of X

$$X_\infty \cdots \xrightarrow{p_{k+2}} X_{k+1} \xrightarrow{p_{k+1}} X_k \xrightarrow{p_k} \dots X_1 \xrightarrow{p_1} X$$

where $\pi_1(X_k) = \varphi^k(G)$. Let $P_k = p_1 \circ p_2 \circ \cdots \circ p_k : X_k \to X$.

In the covering space X_k, a connected component $\hat{K}_i$ of $P_k^{-1}(K_i)$ will be called an *essential K_i-country* if $\pi_1\left(\hat{K}_i\right) \ne 1$ and an *inessential K_i-country* otherwise. Let M_k be a contractable subspace of X_k which is the union of a maximal tree in X_k (fixed for the remainder of the argument) with all the the inessential countries in X_k. Then M_k—which we call a *representing subspace*—determines a free product representation for $\varphi^k(G)$ as in the Kurosh Subgroup Theorem.

We define a *unit path* in X_k to be a path in X_k that begins and ends at a lift of the basepoint of X and never passes through:

i) an intermediary basepoint,
ii) an inessential country,
iii) an edge of the maximal tree.

Clearly each path in X_k is uniquely a product of unit paths and paths without unit subpieces. If $g \in \varphi^k(G)$ and $\tilde{f}$ is the lift to X_k of a closed loop f representing g, then relative to the free product representation corresponding to M_k, $|g|_{\varphi^k(G)}$ is just the number of unit paths in $\tilde{f}$.

For $g \in \varphi^k(G)$, let $\|g\|_{\varphi^k(G)}$ be the length of the shortest representation of g in $\varphi^k(G)$ with respect to any free product representation for $\varphi^k(G)$ into freely indecomposable factors. Suppose that $\min\limits_{k\in\mathbb{N}}\left\{\|g\|_{\varphi^k(G)}\right\} = t$ is attained in $\varphi^N(G) = G_1' * G_2' * \cdots * G_m'$, with $G_i' \cong \mathbb{Z}$ for $1 \le i \le \ell$ and suppose that $g = g_1 g_2 \dots g_t$, where $g_j \in G_{i_j}'$ for some $1 \le i_j \le m$.

Applying the Kurosh Theorem to $\varphi^{N+1}(G)$ as a subgroup of $\varphi^N(G)$ and using the fact that $\varphi^N(G) \cong \varphi^{N+1}(G)$ we get that

$$\varphi^{N+1}(G) = F_\ell' * \gamma_{\ell+1}\alpha_{\ell+1}\left(G_{\ell+1}'\right)\gamma_{\ell+1}^{-1} * \cdots * \gamma_m\alpha_m\left(G_m'\right)\gamma_m^{-1}$$

where $F_\ell' \cong F_\ell$, $\gamma_i \in \varphi^N(G)$ for $\ell < i \le m$ and $\alpha_j : G_j' \to G_{i_j}'$ is a monomorphism for $\ell < j, i_j < m$.

The factors of $\varphi^N(G)$ can be rearranged and φ iterated as often as needed so that

$$\begin{aligned}\varphi^N(G) &= F_\ell' * G_{\ell+1}' * G_{\ell+2}' * \cdots * G_p' * G_{p+1}' * \dots G_m' \\ \varphi^{N+r}(G) &= F_\ell'' * \gamma_{\ell+1}\alpha_{\ell+1}\left(G_{\ell+1}'\right)\gamma_{\ell+1}^{-1} * \cdots * \gamma_p\alpha_p\left(G_p'\right)\gamma_p^{-1} \\ &\quad * \delta_{p+1}\beta_{p+1}\left(G_{p+1}'\right)\delta_{p+1}^{-1} * \cdots * \delta_m\beta_m\left(G_m'\right)\delta_m^{-1}\end{aligned}$$

where $F''_\ell \cong F'_\ell$, $\alpha_i : G'_i \to G'_i$ and $\beta_k : G'_k \to G'_{i_k}$ are monomorphisms, and γ_i, δ_j are elements of $\varphi^N(G)$ for $\ell < i, i_k \leq p$, $p < k \leq m$.

Since G_i is co-Hopfian for $\ell < i \leq p$, we may write

$$\begin{aligned}\varphi^{N+r}(G) = F''_\ell &* \gamma_{\ell+1} G'_{\ell+1} \gamma_{\ell+1}^{-1} * \cdots * \gamma_p G'_p \gamma_p^{-1} \\ &* \delta_{p+1}\beta_{p+1}\left(G'_{p+1}\right)\delta_{p+1}^{-1} * \cdots * \delta_m \beta_m \left(G'_m\right)\delta_m^{-1}.\end{aligned}$$

We note that X_{N+r} is a covering space of X_N with the property that any essential country in X_{N+r} is a cover of a subcomplex K'_i for $\ell < i \leq p$.

Since $\varphi^\infty(G)$ is a subgroup of a finitely generated group, it is countable and we may list the elements. Let g be the first element in this list and denote by $I_g \subset \{1, 2, \ldots, k\}$ the set of subscripts appearing in the normal form representation for g. The first main step in the proof of the Proposition is the following claim.

Claim 1. *Suppose that* $\|g\|_{\varphi^N(G)} = \min_k \left\{ ||g||_{\varphi^k(G)} \right\} = t$ *and furthermore that* $g = g_1 g_2 \ldots g_t$*, where each* $g_j \in G'_{i_j}$*, for* $1 \leq i_j \leq m$*. Then* $\star_{k \in I_g} G'_k$ *is a free factor of* $\varphi^\infty(G)$*.*

Proof of Claim 1. Suppose that $g_1 \in G'_{i_1}$. There are three cases to consider, according to the index i_1.

Case 1. Suppose that $1 \leq i_1 \leq \ell$.

Since $G'_{i_1} \cong \langle x_i \rangle$, $g_1 = x_i^n$ for some n. Consider a loop f representing x_i in X_N and its lift $\hat{f}$ in X_k for $k > N$. If $\hat{f}$ is not a loop, then it is a contractible path in X_k which we can extend to a representing space, M_k. Note that $\|g\|_{\varphi^k(G)} \leq |g|_{M_k} < t$, contradicting the assumption that t was minimal. Hence $\hat{f}$ is a loop, and therefore G'_i is a free factor of $\varphi^k(G)$ for all $k \geq N$.

Case 2. Suppose that $\ell < i_1 \leq p$.

Let K_{i_1} be the subcomplex in X_N such that $\pi_1(K_{i_1}) = G'_{i_1}$. Consider the cover $\hat{K}_{i_1}$ of K_{i_1} in X_{N+r} which is adjacent to the basepoint. *A priori*, there are three possibilities for $\hat{K}_{i_1}$:

Subcase a. $\pi_1\left(\hat{K}_{i_1}\right) = 1$.

Subcase b. $\pi_1\left(\hat{K}_{i_1}\right)$ is a nontrivial subgroup of G'_{i_1}.

Subcase c. $\pi_1\left(\hat{K}_{i_1}\right) = G'_{i_1}$.

In all cases, let f be the loop in K_{i_1} representing g_1 and let $\hat{f}$ be the lift of f in $\hat{K}_{i_1}$.

In Subcase a, $\pi_1\left(\hat{K}_{i_1}\right) = 1$, and hence $\hat{K}_{i_1}$ is contractible; we include $\hat{K}_{i_1}$ in an representing space, M_{N+r} for X_{N+r}. Then $\|g\|_{\varphi^k(G)} \leq |g|_{M_{N+r}} < t$, contradicting the assumption that t was minimal.

In Subcase b, we assume that $\pi_1\left(\hat{K}_{i_1}\right) \cong H$, a nontrivial proper subgroup of G_{i_1}. Up to rearrangement of factors, we may assume that

$$H \cong G'_{p+1} * G'_{p+2} * \cdots * G'_{p+t}$$

and so that $\delta_{p+1} = \delta_{p+2} = \cdots = \delta_{p+t} = 1$. Therefore, we may rewrite $\varphi^{N+r}(G)$ as follows:

$$\begin{aligned}\varphi^{N+r}(G) = F''_\ell &* \gamma_{\ell+1} G'_{\ell+1} \gamma_{\ell+1}^{-1} * \cdots * \gamma_p G'_p \gamma_p^{-1} \\ &* \beta_{p+1}\left(G'_{p+1}\right) * \cdots * \beta_{p+t}\left(G'_{p+t}\right) \\ &* \delta_{p+t+1}\beta_{p+t+1}\left(G'_{p+t+1}\right)\delta_{p+t+1}^{-1} * \cdots * \delta_m \beta_m \left(G'_m\right)\delta_m^{-1}\end{aligned}$$

where $\gamma_{i_1} \neq 1$ (otherwise φ is not injective), and if $\delta_j = 1$, then $\beta_j(G_j)$ is not a subgroup of G_{i_1}.

We now turn our attention to

$$\begin{aligned}\varphi^{N+2r}(G) = F'''_\ell &* \varphi(\gamma_{\ell+1}) G'_{\ell+1} \varphi\left(\gamma_{\ell+1}^{-1}\right) * \varphi(\gamma_p) G'_{\ell+1} \varphi\left(\gamma_p^{-1}\right) \\ &* \gamma_{i_1}\beta_{p+1}\left(G'_{p+1}\right)\gamma_{i_1}^{-1} * \cdots * \gamma_{i_1}\beta_q\left(G'_q\right)\gamma_{i_1}^{-1} \\ &* \varphi(\delta_{q+1})\beta_{q+1}\left(G'_{q+1}\right)\varphi(\delta_{q+1}^{-1}) * \cdots * \varphi(\delta_m)\beta_m\left(G'_m\right)\varphi(\delta_m^{-1}),\end{aligned}$$

where $q = p + t$, and consider the cover of K_{i_1} in X_{N+2r} which is adjacent to the basepoint, say $\check{K}_{i_1}$. If $\check{K}_{i_1}$ were essential, then either $\varphi(\gamma_{i_1}) = 1$, or $\varphi(\delta_j) = 1$ for $q + 1 \leq j \leq m$. The former contradicts injectivity of φ. If the latter occurs, then in fact $\delta_j = 1$, but we previously assumed that in this case $\beta_j(G_j)$ was not a subgroup of G_{i_1}. Hence $\check{K}_{i_1}$ is inessential, and we find that $\|g\|_{\varphi^k(G)} \leq |g|_{M_{N+2r}} < t$, which is a contradiction.

Then Subcase c is the only possibility which does not contradict the minimality of t. Therefore $\pi_1\left(\hat{K}_{i_1}\right) = G'_{i_1}$ and G_{i_1} is a free factor of $\varphi^{N+r}(G)$. Similarly, G'_{i_1} is a free factor of $\varphi^{N+2r}(G)$ for all $p \in \mathbb{N}$. Hence, G_{i_1} is a free factor of $\varphi^k(G)$ for all $k \geq N$, and it is an easy exercise to prove then that G_{i_1} is a free factor of $\varphi^\infty(G)$.

Case 3. Suppose that $p < i_1 \leq m$.

We note that if $\min\limits_{k\in\mathbb{N}}\left\{|g|_{\varphi^k(G)}\right\} = t$ and $g = g_1 \ldots g_t$ is in $\varphi^N(G)$, then $g_i \notin G'_j$ for $p < j \leq m$; if K_j is the subspace of X_N with $\pi_1(K_j) = G'_j$, and $\hat{K}_j$ is the cover of K_j in X_{N+r} which is adjacent to the basepoint, then $\hat{K}_j$ is inessential.

This shows that in all cases G'_{i_1} is a free factor of $\varphi^\infty(G)$.

We repeat the argument for all g_i involved in g. Each g_i is represented by a loop f_i that lifts to a path $\hat{f}_i^k$ in X_k which is either a loop adjacent to the basepoint, or a non-contractible path that lies in a inessential country adjacent to the basepoint. These essential countries and loops are actually homeomorphic to those countries they cover. Thus $*_{I_j} G_{i_j}$ for $i_j \in I_j$ is actually a free factor of X_k for all $k \geq n$, and hence a free factor of $\varphi^\infty(G)$, as well. This completes the proof of Claim 1. □

We now reorder the factors of $\varphi^N(G)$ in the following way:

$$\varphi^N(G) = G'_1 * \cdots * G'_L * G'_{L+1} * \cdots * G'_Q * G'_{Q+1} * \ldots G'_m$$

where

1) for $1 \leq i \leq L$, G'_i is a free factor of $\varphi^k(G)$ for all $k \geq N$ and for each further iterate of φ, the corresponding Kurosh conjugator $\gamma_i = 1$,
2) for $L < i \leq Q$, $G'_i \cong \mathbb{Z}$,
3) for $Q < i \leq m$, G'_i is a co-Hopfian group.

Now let g' be the first element in the ordering of $\varphi^{\infty}(G)$ so that g' is not contained in $\Gamma = *_{i=1}^{L} G'_i$. If one cannot do this, then $\varphi^{\infty}(G) = *_{i=1}^{L} G'_i$ is a free factor of $\varphi^k(G)$ for all $k \geq N$, and the Proposition is complete. We define $|g'|_{\varphi^k(G)|\Gamma}$ to be the minimal length of g' with respect to all free product representations of $\varphi^k(G)$ which contain Γ as a free factor. Let

$$\|g'\|_{\varphi^k(G)|\Gamma} = \min_{k\in\mathbb{N}} \left\{ |g'|_{\varphi^k(G)|\Gamma} \right\} = s$$

and suppose that this minimum is attained in

$$\varphi^{N'}(G) = G''_1 * G''_2 * \cdots * G''_L * G''_{L+1} * \cdots * G''_Q * G''_{Q+1} * \cdots * G''_m$$

where $G''_i = G'_i$ if $1 \leq i \leq L$, $G''_i \cong \mathbb{Z}$ if $L < i \leq Q$, and G''_i is co-Hopfian if $Q < i \leq m$.

Let $g' = g'_1 \cdots g'_s$ be the normal form of g' with respect to this representation, where $g'_j \in G''_{i_j}$. All of the arguments pertaining to the length of g in Part I of the argument directly apply to g'; for each G''_{i_j} containing g'_j in the element g', we have a corresponding subspace K_{i_j} in $X_{N'}$ that lifts to a homeomorphic copy of itself in X_k, for $k \geq N'$. If this fails to be the case for some $k_0 > N'$, then we can find some $k' \geq k_0$ for which $|g'|_{\varphi^{k'}(G)|\Gamma} < \|g'\|_{\varphi^k(G)|\Gamma}$, which contradicts the assumption of the minimality.

In this way, we obtain $*_{i=1}^{L'} G''_i = \Gamma'$ which is a free factor of $\varphi^k(G)$ for all $k \geq N'$, and is hence a free factor of $\varphi^{\infty}(G)$. This completes the proof of the Proposition since the process must terminate; this is since the free factors of $\varphi^{\infty}(G)$ are also free factors of $\varphi^M(G)$ for large enough M, and the number of these free factors is bounded by m. $\square$

2. The Retract Theorem in Fuchsian groups

In this section, we show that the Retract Theorem holds for a finitely generated Fuchsian group G generalizing results of Voce [V1] and [V2]. See [B] for background information on Fuchsian groups. (This theorem also holds, by the same techniques, for groups of isometries of the hyperbolic plane $\mathbb{H}$ that include orientation reversing isometries.) We begin with the special case of finite free products of cyclic groups.

Lemma 1. *If G is a finite free product of cyclic groups, and φ is an endomorphism of G then $\varphi^{\infty}(G)$ is a retract. The Retract Theorem therefore holds for G.*

Proof. We begin by showing that φ_N is a monomorphism for sufficiently large N. If $G = F_r * \mathbb{Z}_{k_1} * \cdots * \mathbb{Z}_{k_s}$ then $rank(G) = r + s$. By the Kurosh Theorem, the images $\varphi^n(G)$ all have the same form as well and $rank(\varphi^n(G)) = r_n + s_n$ is a non-increasing function of n. In fact, the values of r_n are non-increasing by the following argument. Abstractly, φ is a surjection from $F_r * \mathbb{Z}_{k_1} * \cdots * \mathbb{Z}_{k_s}$ to $F_{r_1} * \mathbb{Z}_{k'_1} * \cdots * \mathbb{Z}_{k'_{s_1}}$. Following by projection onto F_{r_1}, we get a map which must be trivial on each factor $\mathbb{Z}_{k_i}$, inducing a surjection of F_r onto F_{r_1}; so $r_1 \leq r$; similarly, $r_{n+1} \leq r_n$. The values of r_n are therefore eventually constant and so the values of s_n are also eventually constant. Now replace (G, φ) with $\left(\varphi^M(G), \varphi_M\right)$ for sufficiently large M—here the values of r_n and s_n are constant and it suffices to prove the Lemma in this case.

It's not hard to see that $k'_1 \cdots k'_s \leq k_1 \cdots k_s$. (Consider the abelianization $\mathbb{Z}^r \bigoplus \mathbb{Z}_{k_1} \cdots \mathbb{Z}_{k_s} \to \mathbb{Z}^r \bigoplus \mathbb{Z}_{k'_1} \cdots \mathbb{Z}_{k'_s}$.) This is true as well for all φ_n, so eventually this product stops decreasing, at which point φ_N becomes a monomorphism.

The Lemma now follows by the same arguments as in Proposition 1 and Theorem 1. □

Proof of Theorem 2. Suppose that G is a finitely generated Fuchsian group with fundamental polygon P and orbifold $\mathbb{H}/G$. If there are no cusps, then G is torsion free and the Retract Theorem holds by Theorem 1. If G is not co-compact (i.e., if $\mathbb{H}/G$ is not compact) or if the genus n is 0, then G is a product of cyclic groups and the Retract Theorem holds by Lemma 1. We may therefore assume that P is compact and that G has presentation

$$G = \langle a_1, b_1, \ldots, a_n, b_n, c_1, \ldots, c_t \mid [a_1, b_1] \ldots [a_n, b_n]\, c_1 \ldots c_t, {c_i}^{k_i}\ \forall i \rangle$$

with $t > 0$ and $n > 0$. By considering the abelianization of G, it is easy to see that $rank(G) = 2n + t - 1$. The orbifold $\mathbb{H}/G$ has genus n topologically and has t cone points with cone angles $\frac{2\pi}{k_1}, \ldots, \frac{2\pi}{k_t}$. Denote the total cone angle of $\mathbb{H}/G$ by

$$Cone_G = \sum_{j=1}^{t} \frac{2\pi}{k_j}.$$

Then by [B, p269], P has area

$$A = 2\pi \left[(2n-2) + \sum_{j=1}^{t} \left(1 - \frac{1}{k_j} \right) \right] = 2\pi(rank(G) - 1) - Cone_G.$$

As before, it suffices to show that for any endomorphism φ of G, that $\varphi^\infty(G)$ is a retract. Consider first the case in which the index $|G : \varphi(G)|$ of $\varphi(G)$ in G is infinite, namely $|G : \varphi(G)| = \infty$. In [HKS], it is proven that a subgroup of infinite index in a Fuchsian group is a free product of cyclic groups—replacing G and φ with $\varphi(G)$ and φ_1 completes the argument in this case.

Now assume that $\varphi(G)$ has finite index in G and consider the family of subgroups

$$G \supseteq \varphi(G) \supseteq \varphi^2(G) \supseteq \cdots \supseteq \varphi^\ell(G) \supseteq \ldots$$

each of which is Fuchsian. If for any ℓ, $\varphi^\ell(G)$ is either not co-compact or has genus 0 or has no cone points, then the Retract Theorem holds for $\varphi^\ell(G)$ and $\varphi^\infty(G)$ is a retract of both $\varphi^\ell(G)$ and of G. So we can assume that for all ℓ, $\varphi^\ell(G)$ is co-compact, has genus $n_\ell > 0$, has $t_\ell > 0$ cone points, has rank $r_\ell = 2n_\ell + t_\ell - 1 > 1$, has total cone angle $Cone_\ell$ and area $A_\ell = 2\pi(r_\ell - 1) - Cone_\ell$.

The sequence r_ℓ is non-increasing, so it eventually stabilizes. The sequence of indices $|\varphi^\ell(G) : \varphi^{\ell+1}(G)|$ is also non-increasing and eventually stabilizes at a value $q > 1$. (If at any point, $\varphi^\ell(G) = \varphi^{\ell+1}(G)$ then $\varphi^\ell(G) = \varphi^\infty(G)$ and we are done.) So by replacing G and φ by $\varphi^\ell(G)$ and $\varphi\mid_{\varphi^\ell(G)}$ for suitably large ℓ, we may assume that all the ranks and all the indices are equal.

Now $A_{\ell+1} = qA_\ell$ since a fundamental polygon for $\varphi^{\ell+1}(G)$ is q non-overlapping fundamental polygons for $\varphi^\ell(G)$. Thus

$$\lim_{\ell\to\infty} (A_\ell) = \infty \qquad \Longrightarrow \qquad \lim_{\ell\to\infty} (Cone_\ell) = -\infty.$$

This is a clear contradiction, completing the argument. □

3. Test elements in surface groups

In this section, we will prove the existence of test elements in the fundamental groups of orientable surfaces other than the torus. For the remainder of this section, we will consider the surface S_n of genus n with fundamental group

$$\Gamma_n = \langle x_1, x_2, \ldots, x_{2n} \mid [x_1, x_2]\,[x_3, x_4] \ldots [x_{2n-1}, x_{2n}] \rangle \,.$$

Theorem 3. *The group Γ_n, for $n \geq 2$, contains test elements. In particular the words*

$$w_k = x_1^k x_2^k \ldots x_{2n}^k, \qquad k > 1$$

are test elements.

Proof. The Retract Theorem holds for Γ_n since it is Fuchsian, so it suffices to show that for $k > 1$, w_k lies in no proper retract of Γ_n.

Suppose that $\rho : \Gamma_n \to \Gamma_n$ is a proper retraction with $H = \rho(\Gamma_n)$ and that $w_k \in H$. In general a subgroup K of Γ_n is a surface group and if $[\Gamma_n, K] = k < \infty$, then the Euler characteristics and ranks are related by

$$\chi(K) = k\chi(\Gamma_n), \qquad rank(K) = k(2n-2) + 2 > 2n = rank(\Gamma_n).$$

Since the $rank(H) \leq rank(\Gamma_n)$, H has infinite index. Let S_H be the covering space of S_n corresponding to H. Since S_H is a non-compact surface, H is free.

Consider the standard CW complex structure on S_n, with one 0-cell, 2n 1-cells and one 2-cell, and the map $f : S^1 \to S_n^{(1)}$ that represents $w_k \in \pi_1\left(S_n^{(1)}\right) \cong F_{2n}$ (where S^1 is the circle and $S_n^{(1)}$ is the 1-skeleton of S_n). Since $[f] = w_k \in H$, f lifts to a map $\widetilde{f}$ as indicated.

$$\begin{tikzcd}
 & {S_H}^{(1)} \arrow[r] \arrow[d] & S_H \arrow[d] \\
S^1 \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & {S_n}^{(1)} \arrow[r] & S_n
\end{tikzcd}$$

Claim 2. *There is a map $f' : S^1 \to S_n^{(1)}$ homotopic in S_n to f so that the image $\widetilde{f'}\left(S^1\right)$ of the lift $\widetilde{f'}$ is contained in a topological retract V of S_H. The subgroup $K \subset F_{2n}$ represented by V is a retract of F_{2n}.*

$$\begin{tikzcd}
 & S_H \arrow[d] \\
S^1 \arrow[ur, "\widetilde{f'}"] \arrow[r, "f'"] & S_n
\end{tikzcd}$$

Proof of Claim 2. The non-compact surface S_H retracts onto a homotopy equivalent compact subsurface (with boundary) T which contains $\widetilde{f}(S^1)$. Then T strong deformation retracts onto a subset V of its 1-skeleton $T^{(1)}$; for example, perform the sequence of simple homotopies that push in on a free edge of any remaining 2-cell. This strong deformation retract will homotop the map $\widetilde{f}$ to a map $\widetilde{f'}$ whose image is again in the 1-skeleton; f' is defined by projecting to S_n. Since a graph retracts onto any of its connected subgraphs, $\widetilde{f'}\left(S^1\right)$ is a retract of V, which is in turn a retract of S_H; thus $\widetilde{f'}\left(S^1\right)$ is a retract of S_H.

The diagram of spaces on the left determines the diagram of groups on the right below, in which ε is the presentation map, α is the inverse of the isomorphism induced by the deformation retraction, and i_K and i_H are inclusion maps.

$$\begin{tikzcd} V \arrow[r] \arrow[d] & S_H \arrow[d] \\ {S_n}^{(1)} \arrow[r] & S_n \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "\alpha"] \arrow[d, "i_K"] & H \arrow[d, "i_H"] \\ F_{2n} \arrow[r, "\varepsilon"] & \Gamma_n \end{tikzcd}$$

The retraction of F_{2n} to K is $\alpha^{-1} \circ \rho \circ \varepsilon$. This completes the proof of the Claim. $\square$

Now let $v_k \in K \subset F_{2n}$ be the element represented by f'. Since K is a proper retract of F_{2n}, the retract index $\delta(v_k)$ of v_k is 1 (see [Tu]). On the other hand, $\delta(v_k) = \delta(w_k)$ since the definition of δ depends only on the image in $\mathbb{Z}^{2n}$ and $\delta(v_k)$ and $\delta(w_k)$ have the same image in Γ_n ($F_{2n} \to \Gamma_n \to \mathbb{Z}^{2n}$). But $\delta(w_k)$ is a multiple of k [Tu, page 262]. This contradicts the existence of the retraction ρ. $\square$

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