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Continuous Homomorphisms on $\beta\mathbb{N}$ and Ramsey Theory

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Abstract. We consider the question of the existence of a nontrivial continuous homomorphism from $(\beta\mathbb{N}, +)$ into $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$. This problem is known to be equivalent to the existence of distinct p and q in $\mathbb{N}^*$ satisfying the equations $p + p = q = q + q = q + p = p + q$. We obtain certain restrictions on possible values of p and q in these equations and show that the existence of such p and q implies the existence of p, q, and r satisfying the equations above and the additional equations $r = r + r$, $p = p + r = r + p$, and $q = q + r = r + q$. We show that the existence of solutions to these equations implies the existence of triples of subsets of $\mathbb{N}$ satisfying an unusual Ramsey Theoretic property. In particular, they imply the existence of a subset A with the property that whenever it is finitely colored, there is a sequence in the *complement* of A, all of whose sums two or more terms at a time are monochrome. Finally we show that there do exist sets satisfying finite approximations to this latter property.

Contents

1. Introduction

In 1979, van Douwen asked (in [5], published much later) whether there are topological and algebraic copies of the right topological semigroup $(\beta\mathbb{N}, +)$ contained in $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$. This question was answered in [15], where it was in fact established that if φ is a continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$, then $\varphi[\beta\mathbb{N}]$ is finite. Whether one can have such a continuous homomorphism with $|\varphi[\beta\mathbb{N}]| > 1$ is a difficult open question which we address in this paper.

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Another old and difficult problem in the algebra of $\beta\mathbb{N}$ was solved in 1996 by E. Zelenuk [16] who showed that there are no nontrivial finite groups contained in $\mathbb{N}^*$. (See [10, Section 7.1] for a presentation of this proof.) Using Zelenuk's Theorem, it is not hard to show that there is a nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$ if and only if there exist distinct p and q in $\mathbb{N}^*$ such that $p+p=q=q+q=q+p=p+q$. (See [10, Corollary 10.20].) It is in this guise that we shall be investigating the continuous homomorphism problem.

The question of which finite semigroups can exist in $\mathbb{N}^*$ has implications for a large class of semigroups of the form βS. It is not hard to prove that any finite semigroup in $\mathbb{N}^*$ is contained in $\mathbb{H}=\bigcap_{n\in\mathbb{N}}\mathrm{cl}_{\beta\mathbb{N}}(2^n\mathbb{N})$. Now if S is any infinite discrete semigroup which is right cancellative and weakly left cancellative, S^* contains copies of $\mathbb{H}$ [10, Theorem 6.32]. Thus a finite semigroup which occurs in $\mathbb{N}^*$ also occurs in S^*, if S is any semigroup of this kind.

The conjecture that $\mathbb{N}^*$ contains no elements of finite order, other than idempotents, has implications about the nature of possible continuous homomorphisms from βS into $\mathbb{N}^*$, where S is any semigroup at all. If C is any compact subsemigroup of $\mathbb{N}^*$, its topological center $\Lambda(C)=\{x\in C:\lambda_x:C\to C$ is continuous$\}$ contains only elements of finite order[12, Corollary 6.8]. It follows that, if this conjecture is true, then any continuous homomorphism from βS into $\mathbb{N}^*$ must map all the elements of S to idempotents.

We write $\mathbb{N}$ for the positive integers and ω for the nonnegative integers. Given a set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X. The points of $\beta\mathbb{N}$ are the ultrafilters on $\mathbb{N}$ and the topology of $\beta\mathbb{N}$ is defined by choosing the sets of the form $\overline{A}=\{p\in\beta\mathbb{N}:A\in p\}$, where $A\subseteq\mathbb{N}$, as a basis for the open sets. Then each set $\overline{A}$ is clopen in $\beta\mathbb{N}$ and $\overline{A}=\mathrm{cl}_{\beta\mathbb{N}}A$. The operation $+$ on $\beta\mathbb{N}$ is the extension of ordinary addition on $\mathbb{N}$ making $(\beta\mathbb{N},+)$ into a right topological semigroup (meaning that for all $p\in\beta\mathbb{N}$, the operation $\rho_p:\beta\mathbb{N}\to\beta\mathbb{N}$ defined by $\rho_p(q)=q+p$ is continuous) with $\mathbb{N}$ as its topological center (which is the set of points x such that the function $\lambda_x:\beta\mathbb{N}\to\beta\mathbb{N}$ defined by $\lambda_x(q)=x+q$ is continuous). Given $p,q\in\beta\mathbb{N}$ and $A\subseteq\mathbb{N}$, $A\in p+q$ if and only if $\{x\in\mathbb{N}:-x+A\in q\}\in p$, where $-x+A=\{y\in\mathbb{N}:x+y\in A\}$. See [10] for an elementary introduction to the compact right topological semigroup $(\beta\mathbb{N},+)$.

In Section 2 we present some restrictions on possible values of p and q solving the equations $p+p=q=q+q=q+p=p+q$. We further establish that the existence of such a two element semigroup in $\mathbb{N}^*$ implies the existence of a three element semigroup $\{p,q,r\}$ where $p+p=q=q+q=q+p=p+q$, $r+r=r$, $p=p+r=r+p$, and $q=q+r=r+q$ (and consequently, the existence of a nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$ implies the existence of such a semigroup).

In Section 3 we show that the existence of the three element semigroup described above implies the existence of a triple of disjoint subsets of $\mathbb{N}$ satisfying a strong infinitary Ramsey Theoretic property. A simpler consequence of this property is the following assertion. *If there is a nontrivial continuous homomorphism from $\beta\mathbb{N}$ into $\mathbb{N}^*$, then there is a subset A of $\mathbb{N}$ with the property that, whenever A is finitely colored, there must exist a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}\backslash A$ such that*

$$\{\sum_{t\in F} x_t : F\in\mathcal{P}_f(\mathbb{N}) \text{ and } |F|\geq 2\}$$

is a monochrome subset of A. (When we refer to a "k-coloring" of a set X we mean a function $\phi : X \to \{1, 2, \ldots, k\}$. The assertion that a set B is "monochrome" is the assertion that ϕ is constant on B.)

We concentrate on this simple assertion for the remainder of the paper. While it is not known whether any set satisfying such a property exists, we hope to popularize the search for a set satisfying that property (or the search for a proof that no such set exists). We do establish that certain classes of sets do not have that property.

In Section 4, we show that sets satisfying arbitrary finite approximations to this simpler Ramsey Theoretic property do exist.

2. Special Two and Three Element Subsemigroups of $\beta\mathbb{N}$

Given idempotents r and s in $\mathbb{N}^*$, one says that $r \leq_R s$ if and only if $r = s+r$, $r \leq_L s$ if and only if $r = r + s$, and $r \leq s$ if and only if $r = s + r = r + s$. If $k \in \mathbb{N}$ and one has distinct idempotents $r_1 \leq r_2 \leq \ldots \leq r_k$ (which exist by [10, Theorem 9.23]), then $\{r_1, r_2, \ldots, r_k\}$ is a k element subsemigroup of $\mathbb{N}^*$. However, by [10, Corollary 10.20], the existence of a nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$ is equivalent to the existence of a finite subsemigroup of $\mathbb{N}^*$ whose elements are not all idempotents, and also equivalent to the existence of a two element subsemigroup of $\mathbb{N}^*$ whose elements are not all idempotents. Such a subsemigroup is necessarily of the form $\{p, q\}$, where $p + p = q = q + q = q + p = p + q$. Curiously, Zelenuk's Theorem implies that if $p + p = q$, then one of the equations $p + q = q$, $q + p = q$, or $q + q = q$ implies the others.

Lemma 2.1. *Let p and q be elements of $\mathbb{N}^*$. If $p + p = q$ and any one of $p + q$, $q + p$, or $q + q$ is equal to q, then $p + p = q = q + q = p + q = q + p$.*

Proof. If $p = q$, then the conclusion is trivial, so assume that $p \neq q$. Note that $q+p = p+p+p = p+q$. If $q = q+p$, then $q+q = q+p+p = q+p = q$. So assume that $q = q + q$. Then $\{q, q + p\}$ is a subgroup of $\mathbb{N}^*$, so by Zelenuk's Theorem [10, Theorem 7.17], $q = q + p$. □

Next we show that the existence of such a two element subsemigroup of $\mathbb{N}^*$ implies the existence of a particular three element subsemigroup of $\mathbb{N}^*$, exactly two of whose members are idempotents. The existence of this semigroup yields the sets with the strong Ramsey Theoretic property that we have discussed. To say that an idempotent r is $\leq_R$-maximal means that whenever $r \leq_R s$, one also has $s \leq_R r$.)

Theorem 2.2. *Let p and q be distinct elements of $\mathbb{N}^*$ such that $p + p = q = q + q = p + q = q + p$. Then there exist distinct p', q', and r' in $\mathbb{N}^*$ such that $p' + p' = q' = q' + q' = p' + q' = q' + p'$, $r' + r' = r'$, $p' + r' = r' + p' = p'$, and $q' + r' = r' + q' = q'$. Further, the idempotent r' can be chosen to be $\leq_R$-maximal in $\mathbb{Z}^* = \beta\mathbb{Z}\backslash\mathbb{Z}$.*

Proof. We observe that it suffices to produce distinct p', q', and r' in $\mathbb{Z}^*$, with r' $\leq_R$-maximal in $\mathbb{Z}^*$, satisfying the specified equations. To see this notice that by [10, Exercise 4.3.5], both $\mathbb{N}^*$ and $-\mathbb{N}^*$ are left ideals of $\beta\mathbb{Z}$. Thus, either p', q', and r' are all in $\mathbb{N}^*$ as desired, or they are all in $-\mathbb{N}^*$. In the latter event, let $p'' = -p'$, $q'' = -q'$, and $r'' = -r'$. Then p'', q'', and r'' are distinct members of $\mathbb{N}^*$ and, by

[10, Lemma 13.1], they satisfy the specified equations. It is easy to verify that r'' is also $\leq_R$-maximal in $\mathbb{Z}^*$.

Since $q+p=p+p$ we have that p is not right cancelable in $\beta\mathbb{Z}$ so by [10, Theorem 8.18], there is an idempotent $r\in\mathbb{Z}^*$ such that $r+p=p$. Notice that if s is any idempotent with $r\leq_R s$, then $s+p=s+r+p=r+p=p$.

We consider first the possibility that there is some $\leq_R$-maximal idempotent $s\geq_R r$ in $\mathbb{Z}^*$ such that $p+s=p$. Then $s+p=p$, $s+q=s+p+q=p+q=q$, and $q+s=q+p+s=q+p=q$. Further, since $s+p=p$, $s\neq p$ and $s\neq q$. Thus, letting $r'=s$, $p'=p$, and $q'=q$, we are done.

Now assume that for every $\leq_R$-maximal idempotent $s\geq_R r$ in $\mathbb{Z}^*$, $p+s\neq p$. Pick by [10, Theorem 2.12], a $\leq_R$-maximal idempotent r' such that $r\leq_R r'$. Then, as noted above, $r'+p=p$. Let $p'=p+r'$ and let $q'=q+r'$. Then immediately $p'+r'=p'$ and $q'+r'=q'$. Also $r'+p'=r'+p+r'=p+r'=p'$ and $r'+q'=r'+q+r'=r'+p+q+r'=p+q+r'=q+r'=q'$. Further $p'+p'=p+r'+p+r'=p+p+r'=q+r'=q'$ and $q'+p'=q+r'+p+r'=q+p+r'=q+r'=q'$. Consequently, by Lemma 2.1, $p'+q'=q'+q'=q'$.

To complete the proof we need to show that p', q', and r' are all distinct. For this, it suffices to show that $p'\neq q'$. (For then $r'+p'=p'$, $p'+p'=q'$, and $q'+p'=q'$ so $r'\neq q'$ and $r'\neq p'$.) So suppose instead that $p'=q'$, that is, $p+r'=q+r'$. Then by [10, Theorem 9.4 and Lemma 9.5] there is an idempotent $s\geq_R r'$, necessarily $\leq_R$-maximal, such that either $p=q+s$ or $q=p+s$. If we had $q+s=p$, then we would have $q+p=q+q+s=q+s=p$, a contradiction. Thus $q=p+s$ and so $p+p=p+s$.

Now, by [10, Corollary 6.20] there is some $x\in\beta\mathbb{Z}$ such that $p=x+s$ or $s=x+p$. Suppose first that $p=x+s$. Then $p+s=x+s+s=x+s=p$, a contradiction since $s\geq_R r'$ and we are assuming that for every $\leq_R$-maximal idempotent $s\geq_R r'$, $p+s\neq p$. Thus, $s=x+p$ so, since $r\leq_R s$, $p=s+p=x+p+p=x+q$ and so $p+q=x+q+q=x+q=p$, a contradiction. □

Theorem 2.3. *Let p and q be distinct elements of $\mathbb{N}^*$ such that $p+p=q=q+q=p+q=q+p$. Then p is not a member of any subgroup of $\beta\mathbb{N}$. In particular, p is not a member of the smallest ideal $K(\beta\mathbb{N})$ of $\beta\mathbb{N}$.*

Proof. Suppose that p is a member of a subgroup G of $\mathbb{N}^*$ with identity r. Then q is an idempotent in G and so $q=r$ and thus $q+p=p$, a contradiction. For the "in particular" conclusion, recall that the smallest ideal of any compact right topological semigroup is the union of groups [10, Theorems 2.7 and 2.8]. □

We have not been able to show that $q\notin K(\beta\mathbb{N})$. However, it is not even known whether there is any $p\in\beta\mathbb{N}\backslash K(\beta\mathbb{N})$ with $p+p\in K(\beta\mathbb{N})$. See [9] for information about the question of whether $K(\beta\mathbb{N})$ is prime or semiprime.

In [1, Theorem 2.2] it was shown that if $p\in\mathbb{N}^*$ and p generated a finite subsemigroup of $\mathbb{N}^*$, then p could not be distinguished from an idempotent by means of a continuous homomorphism into a compact topological group. In the current context we have a stronger, yet very simple, result. Notice for example that the requirement that "every idempotent is a right identity" is satisfied by any minimal left ideal [10, Lemma 1.30].

Lemma 2.4. *Let p and q be distinct elements of $\mathbb{N}^*$ such that $p+p=q=q+q=p+q=q+p$ and let $(T,\cdot)$ be a semigroup in which every idempotent is a left* *identity or every idempotent is a right identity. If φ is a homomorphism from any semigroup containing $\{p,q\}$ to T, then $\varphi(p) = \varphi(q)$.*

Proof. Assume without loss of generality that every idempotent in T is a right identity of T. Then $\varphi(p) = \varphi(p) \cdot \varphi(q) = \varphi(p+q) = \varphi(q)$. □

3. Ramsey Theoretic Consequences

We show in this section that the existence of a continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$ implies the existence of disjoint subsets of $\mathbb{N}$ satisfying a strong infinitary Ramsey Theoretic property.

Definitions 3.1. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$, then

(1) $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$.
(2) $FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})$ and $|F| \geq 2\}$.
(3) If $k \in \mathbb{N}$, $FS_k(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})$ and $|F| = k\}$.

Similarly, if $\langle x_t \rangle_{t=1}^{m}$ is a finite sequence, we shall let

$$FS_{\geq 2}(\langle x_t \rangle_{t=1}^{m}) = \{\sum_{t\in F} x_t : \emptyset \neq F \subseteq \{1,2,\ldots,m\} \text{ and } |F| \geq 2\}.$$

Recall [10, Theorem 5.12] that a set $A \subseteq \mathbb{N}$ is a member of some idempotent in $\beta\mathbb{N}$ if and only if there is some sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$. Part of this assertion is imitated in the following theorem.

Theorem 3.2. *Let p, q, and r be elements of $\mathbb{N}^*$ such that $p+p = q = q+q = q+p = p+q$, $r+r = r$, $r+p = p+r = p$, and $q+r = r+q = q$. Let $A \in q$, $B \in r$, and $C \in p$. Then there exist sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that*

(1) *$FS(\langle y_n \rangle_{n=1}^{\infty}) \subseteq B$,*
(2) *for all $n \in \mathbb{N}$ and all $z \in FS(\langle y_n \rangle_{n=1}^{\infty}) \cup \{0\}$, $x_n + z \in C$ (and in particular $\{x_n : n \in \mathbb{N}\} \subseteq C$), and*
(3) *for all $w \in FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty})$ and all $z \in FS(\langle y_n \rangle_{n=1}^{\infty}) \cup \{0\}$, $w + z \in A$ (and in particular $FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$).*

Proof. Let

$$\begin{aligned}
\widehat{A} &= \{a \in A : -a + A \in p \cap q \cap r\}, \\
\widehat{B} &= \{b \in B : -b + A \in q\,,\, -b + B \in r, \text{ and } -b + C \in p\}, \text{ and} \\
\widehat{C} &= \{c \in C : -c + A \in p \cap q \text{ and } -c + C \in r\}.
\end{aligned}$$

Since $q = q+p = q+q = q+r$, we have that $\widehat{A} \in q$. Since $q = r+q$, $r = r+r$, and $p = r+p$, we have that $\widehat{B} \in r$. Since $q = p+p = p+q$ and $p = p+r$, we have that $\widehat{C} \in p$. Next we claim that

$$\begin{aligned}
&\text{if } a \in \widehat{A}, \text{ then } -a + \widehat{A} \in p \cap r\,; \\
&\text{if } b \in \widehat{B}, \text{ then } -b + \widehat{C} \in p \text{ and } -b + \widehat{B} \in r\,; \text{ and} \\
&\text{if } c \in \widehat{C}, \text{ then } -c + \widehat{A} \in p \text{ and } -c + \widehat{C} \in r\,.
\end{aligned}$$

We verify the second of these assertions, the other two being similar.

Let $b \in \widehat{B}$. Given $x \in \mathbb{N}$, $x \in -b + \widehat{C}$ if and only if $b + x \in C$, $-(b+x) + A \in p \cap q$, and $-(b+x) + C \in r$. That is

$$-b + \widehat{C} = -b + C \cap \{x : -x + (-b + A) \in p \cap q\} \cap \{x : -x + (-b + C) \in r\}\,.$$

Now $-b+C \in p$. Also, $-b+A \in q = p+q = p+p$, so that

$$\{x : -x + (-b+A) \in p \cap q\} \in p.$$

And $-b+C \in p = p+r$, so that $\{x : -x + (-b+C) \in r\} \in p$.

Similarly $-b+\widehat{B} = -b+B \cap \{x : -x+(-b+A) \in q\} \cap \{x : -x+(-b+B) \in r\} \cap \{x : -x+(-b+C) \in p\}$. We have that $-b+B \in r$. Also $\{x : -x+(-b+A) \in q\} \in r$ because $-b+A \in q = r+q$; $\{x : -x+(-b+B) \in r\} \in r$ because $-b+B \in r = r+r$; and $\{x : -x+(-b+C) \in p\} \in r$ because $-b+C \in p = r+p$.

We now construct the sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$. Choose $x_1 \in \widehat{C}$. Then $-x_1 + \widehat{C} \in r$ so choose $y_1 \in \widehat{B} \cap (-x_1 + \widehat{C})$. Inductively, let $m \in \mathbb{N}$ and assume that we have chosen $\langle x_t\rangle_{t=1}^{m}$ and $\langle y_t\rangle_{t=1}^{m}$ so that

(i) for each $n \in \{1,2,\ldots,m\}$, $x_n \in \widehat{C}$,
(ii) for each $F \subseteq \{1,2,\ldots,m\}$ with $|F| \geq 2$, $\sum_{n\in F} x_n \in \widehat{A}$,
(iii) for $\emptyset \neq G \subseteq \{1,2,\ldots,m\}$, $\sum_{t\in G} y_t \in \widehat{B}$,
(iv) for each $n \in \{1,2,\ldots,m\}$ and $\emptyset \neq G \subseteq \{1,2,\ldots,m\}$, $x_n + \sum_{t\in G} y_t \in \widehat{C}$,
(v) for $\emptyset \neq F, G \subseteq \{1,2,\ldots,m\}$ with $|F| \geq 2$, $\sum_{t\in F} x_t + \sum_{t\in G} y_t \in \widehat{A}$.

All hypotheses are satisfied for $m = 1$, (ii) and (v) vacuously.

Now, given $z \in FS(\langle x_t\rangle_{t=1}^{m})$ we have that either $z \in \widehat{C}$ or $z \in \widehat{A}$, and so $-z+\widehat{A} \in p$. Given $w \in FS(\langle y_t\rangle_{t=1}^{m})$, $w \in \widehat{B}$ so that $-w+\widehat{C} \in p$. Given $z \in FS(\langle x_t\rangle_{t=1}^{m})$ and $w \in FS(\langle y_t\rangle_{t=1}^{m})$, we have that either $z+w \in \widehat{C}$ or $z+w \in \widehat{A}$, and so $-(z+w)+\widehat{A} \in p$. Thus we may choose

$$\begin{aligned} x_{m+1} \;\in\;& \widehat{C} \cap \textstyle\bigcap_{z\in FS(\langle x_t\rangle_{t=1}^{m})}(-z+\widehat{A}) \\ & \cap \textstyle\bigcap_{w\in FS(\langle y_t\rangle_{t=1}^{m})}(-w+\widehat{C}) \\ & \cap \textstyle\bigcap_{z\in FS(\langle x_t\rangle_{t=1}^{m})} \bigcap_{w\in FS(\langle y_t\rangle_{t=1}^{m})}(-(z+w)+\widehat{A})\,. \end{aligned}$$

Given $n \in \{1,2,\ldots,m+1\}$, we have $x_n \in \widehat{C}$, and so $-x_n + \widehat{C} \in r$. Given $w \in FS(\langle y_t\rangle_{t=1}^{m})$, we have $w \in \widehat{B}$ and so $-w+\widehat{B} \in r$. Given $n \in \{1,2,\ldots,m+1\}$ and $w \in FS(\langle y_t\rangle_{t=1}^{m})$ we have that $x_n + w \in \widehat{C}$ and so $-(x_n+w)+\widehat{C} \in r$. Given $z \in FS_{\geq 2}(\langle x_t\rangle_{t=1}^{m+1})$ and $w \in FS(\langle y_t\rangle_{t=1}^{m})$ we have that $z+w \in \widehat{A}$ and so $-(z+w)+\widehat{A} \in r$. Thus we may choose

$$\begin{aligned} y_{m+1} \;\in\;& \widehat{B} \cap \textstyle\bigcap_{n=1}^{m+1}(-x_n+\widehat{C}) \cap \bigcap_{w\in FS(\langle y_t\rangle_{t=1}^{m})}(-w+\widehat{B}) \\ & \cap \textstyle\bigcap_{n=1}^{m+1} \bigcap_{w\in FS(\langle y_t\rangle_{t=1}^{m})}(-(x_n+w)+\widehat{C}) \\ & \cap \textstyle\bigcap_{z\in FS_{\geq 2}(\langle x_t\rangle_{t=1}^{m+1})} \bigcap_{w\in FS(\langle y_t\rangle_{t=1}^{m})}(-(z+w)+\widehat{A})\,. \end{aligned}$$

All induction hypotheses can be easily verified. □

The following is the strong Ramsey Theoretic property which we have discussed.

Corollary 3.3. *Assume that there is a continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$. Then there exist disjoint subsets A, B, and C of $\mathbb{N}$ such that, whenever $\mathcal{F}$ is a finite partition of A, $\mathcal{G}$ is a finite partition of B, and $\mathcal{H}$ is a finite partition of C, there exist $F \in \mathcal{F}$, $G \in \mathcal{G}$, and $H \in \mathcal{H}$, and sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that*

(1) $FS(\langle y_n\rangle_{n=1}^{\infty}) \subseteq G$,

(2) *for all $n \in \mathbb{N}$ and all $z \in FS(\langle y_n\rangle_{n=1}^\infty) \cup \{0\}$, $x_n + z \in H$ (and in particular $\{x_n : n \in \mathbb{N}\} \subseteq H$), and*

(3) *for all $w \in FS_{\geq 2}(\langle x_n\rangle_{n=1}^\infty)$ and all $z \in FS(\langle y_n\rangle_{n=1}^\infty) \cup \{0\}$, $w + z \in F$ (and in particular $FS_{\geq 2}(\langle x_n\rangle_{n=1}^\infty) \subseteq F$).*

Proof. Assume that there is a continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$. Then by [10, Corollary 10.20] and Theorem 2.2, we may pick distinct p, q, and r in $\mathbb{N}^*$ such that $p+p = q = q+q = q+p = p+q$, $r+r = r$, $r+p = p+r = p$, and $q+r = r+q = q$. Pick pairwise disjoint $A \in q$, $B \in r$, and $C \in p$. Let $\mathcal{F}$ be a finite partition of A, let $\mathcal{G}$ be a finite partition of B, and let $\mathcal{H}$ be a finite partition of C. Pick $F \in \mathcal{F}$, $G \in \mathcal{G}$, and $H \in \mathcal{H}$ such that $F \in q$, $G \in r$, and $H \in p$. Apply Theorem 3.2. □

We now consider a much simpler consequence of the statement in Corollary 3.3.

Definitions 3.4. For $A \subseteq \mathbb{N}$, let $\Psi(A)$ be the statement: "For each $k \in \mathbb{N}$, whenever A is k-colored, there exists an increasing sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}\backslash A$ such that $FS_{\geq 2}(\langle x_n\rangle_{n=1}^\infty)$ is monochrome."

Notice that any set A satisfying Ψ is automatically an IP set. That is, there is a sequence $\langle y_n\rangle_{n=1}^\infty$ such that $FS(\langle y_n\rangle_{n=1}^\infty) \subseteq A$. To see this note that, given any sequence $\langle x_n\rangle_{n=1}^\infty$, if $y_n = x_{2n} + x_{2n+1}$, then $FS(\langle y_n\rangle_{n=1}^\infty) \subseteq FS_{\geq 2}(\langle x_n\rangle_{n=1}^\infty)$.

Corollary 3.5. *If there exist distinct p and q in $\mathbb{N}^*$ such that $q = p+p = p+q = q+p = q+q$, then there is a set $A \subseteq \mathbb{N}$ such that $\Psi(A)$.*

Proof. This is an immediate consequence of Corollary 3.3 and [10, Corollary 10.20]. □

One could define a statement $\Gamma(A)$ to be the statement: "For each $k \in \mathbb{N}$, whenever A and $\mathbb{N}\backslash A$ are k-colored, there exists a monochrome increasing sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}\backslash A$ such that $FS_{\geq 2}(\langle x_n\rangle_{n=1}^\infty)$ is monochrome." The existence of a set A satisfying Γ also follows immediately from Corollary 3.3. However, this is not really a stronger conclusion, because $\Gamma(A)$ follows trivially from $\Psi(A)$ by applying the pigeon hole principle.

We have already noted that a point $p \in \mathbb{N}^*$ is an idempotent if and only if there is a sequence $\langle x_n\rangle_{n=1}^\infty$ such that $FS(\langle x_n\rangle_{n=1}^\infty) \in p$. We doubt that the existence of a set A satisfying Ψ can be shown to imply that there exist $p, q \in \beta\mathbb{N}$ such that $A \in q$, $\mathbb{N}\backslash A \in p$, and $p+p = q = q+q = p+q = q+p$ in any way short of proving that no such set A exists. We do have the following partial converse to Corollary 3.5.

Lemma 3.6. *Let $A \subseteq \mathbb{N}$. If $\Psi(A)$, then there exists $r \in \overline{A}$ with the property that for every $B \in r$, there exist $p \in \mathbb{N}^* \backslash \overline{A}$ and $q = q+q \in \overline{B}$ such that, whenever $k \geq 2$ and $u_1, u_2, \ldots, u_k \in \{p, q\}$, one has $u_1 + u_2 + \ldots + u_k \in \overline{B}$.*

Proof. Let $\mathcal{B} = \{B \subseteq A :$ there is an increasing sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}\backslash A$ such that $FS_{\geq 2}(\langle x_n\rangle_{n=1}^\infty) \subseteq B\}$. Since $\Psi(A)$, we have that whenever $\mathcal{F}$ is a finite partition of A, one must have $\mathcal{F} \cap \mathcal{B} \neq \emptyset$. Thus, by [10, Theorem 5.7], there exists $r \in \overline{A}$ with $\mathcal{B} \subseteq r$. Let $B \in r$ and pick an increasing sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}\backslash A$ such that $FS_{\geq 2}(\langle x_n\rangle_{n=1}^\infty) \subseteq B\}$.

By [10, Theorem 4.20], $\bigcap_{m=1}^{\infty} \overline{FS_{\geq 2}(\langle x_n \rangle_{n=m}^{\infty})}$ is a subsemigroup of $\beta\mathbb{N}$ so pick by [10, Theorem 2.5] an idempotent $q \in \bigcap_{m=1}^{\infty} \overline{FS_{\geq 2}(\langle x_n \rangle_{n=m}^{\infty})}$. Pick $p \in \mathbb{N}^*$ with $\{x_n : n \in \mathbb{N}\} \in p$.

To complete the proof it suffices to show

(1) $FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty}) \in p + p$ and

(2) if $u \in \beta\mathbb{N}$ and $FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty}) \in u$, then $FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty}) \in u + p$ and $FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty}) \in u + q$.

To establish (1), we show that

$$\{x_n : n \in \mathbb{N}\} \subseteq \{y \in \mathbb{N} : -y + FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty}) \in p\}.$$

Let $m \in \mathbb{N}$. Then $\{x_n : n \in \mathbb{N} \text{ and } n > m\} \subseteq -y + FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty})$.

To establish (2), assume that $FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty}) \in u$. We show that

$$FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq \{y \in \mathbb{N} : -y + FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty}) \in p\} \text{ and}$$
$$FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq \{y \in \mathbb{N} : -y + FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty}) \in q\}\,.$$

So let $y \in FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty})$ and pick $F \in \mathcal{P}_f(\mathbb{N})$ with $|F| > 1$ such that $y = \sum_{n \in F} x_n$. Let $m = \max F$. Then $\{x_n : n \in \mathbb{N} \text{ and } n > m\} \subseteq -y + FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty})$ and $FS_{\geq 2}(\langle x_n \rangle_{n=m+1}^{\infty}) \subseteq -y + FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty})$. □

Some of us would conjecture strongly that no set $A \subseteq \mathbb{N}$ satisfying $\Psi(A)$ exists. (At least one author disagrees.) In fact, we shall introduce a property Θ weaker than Ψ. We do not even know of a set $A \subseteq \mathbb{N}$ for which Θ holds. We shall give several examples of sets A for which Θ fails. These are, of course, examples for which Ψ fails as well.

Definitions 3.7. For each $A \subseteq \mathbb{N}$, $\Theta(A)$ is the statement: "For every finite coloring of A, there exists an infinite sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}\backslash A$ such that $FS_2(\langle x_n \rangle_{n=1}^{\infty}) \cup FS_3(\langle x_n \rangle_{n=1}^{\infty})$ is monochrome."

Theorem 3.8. *Let $\langle a_n \rangle_{n=1}^{\infty}$ be an increasing sequence in $\mathbb{N}$ such that $\lim_{n\to\infty} (a_{n+1} - a_n) = \infty$ and let $A = \mathbb{N}\backslash\{a_n : n \in \mathbb{N}\}$. Then $\neg\Theta(A)$. (In fact there is a counterexample using two colors.)*

Proof. Let $B = \{a_m + a_n : m < n \text{ and } a_{n+1} - a_n > a_m\}$ and let $C = \{a_r + a_s + a_t : r < s < t\,, a_{s+1} - a_s > a_r, \text{ and } a_{t+1} - a_t > a_r + a_s\}$. We claim that $B \cap C = \emptyset$. To see this, suppose that $a_m + a_n = a_r + a_s + a_t$, where $a_{n+1} - a_n > a_m$, $a_{s+1} - a_s > a_r$ and $a_{t+1} - a_t > a_r + a_s$. Then $a_n = a_t$, because $n > t$ implies that $a_n - a_t \geq a_{t+1} - a_t > a_r + a_s$, and $t > n$ implies that $a_t - a_n \geq a_{n+1} - a_n > a_m$. So $a_m = a_r + a_s$ and thus $m > s$. Therefore $a_m - a_s \geq a_{s+1} - a_s > a_r$, a contradiction.

In a similar fashion, one can show that $(B \cup C) \subseteq A$. Suppose that one has an infinite sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}\backslash A$ such that either $FS_2(\langle x_n \rangle_{n=1}^{\infty}) \cup FS_3(\langle x_n \rangle_{n=1}^{\infty}) \subseteq B$ or $FS_2(\langle x_n \rangle_{n=1}^{\infty}) \cup FS_3(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A\backslash B$. Since $FS_2(\langle x_n \rangle_{n=1}^{\infty}) \cap B \neq \emptyset$ and $FS_3(\langle x_n \rangle_{n=1}^{\infty}) \cap C \neq \emptyset$, we have a contradiction. □

In the following theorem, we give another simple example of a family of sets A for which $\Theta(A)$ fails.

Theorem 3.9. *Let $B \subseteq \mathbb{N}$ be an infinite set with the property that, for some $k \in \mathbb{N}$, either $k\mathbb{N} \cap B$ is finite or $k\mathbb{N}\backslash B$ is finite. If $A = \mathbb{N}\backslash B$, then $\Theta(A)$ fails.*

Proof. For each $i \in \{0,1,2,\cdots,k-1\}$, let $A_i = \{a \in A : a \equiv i (\mathrm{mod}\ k)\}$. Suppose that there is an infinite sequence $\langle x_n\rangle_{n=1}^{\infty}$ in B and $i \in \{0,1,2,\cdots,k-1\}$ for which $FS_2(\langle x_n\rangle_{n=1}^{\infty}) \cup FS_3(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A_i$. We may suppose that there exists $j \in \{0,1,2,\cdots,k-1\}$ such that $x_n \equiv j(\mathrm{mod}\ k)$ for every $n \in \mathbb{N}$, because this could be achieved by replacing $\langle x_n\rangle_{n=1}^{\infty}$ by a subsequence. Note that $j \neq 0$. (If $j = 0$, then $\{x_n : n \in \mathbb{N}\} \subseteq k\mathbb{N}$ and $FS_2(\langle x_n\rangle_{n=1}^{\infty}) \subseteq k\mathbb{N}$.) Then $FS_2(\langle x_n\rangle_{n=1}^{\infty}) \subseteq \{a \in A : a \equiv 2j(\mathrm{mod}\ k)\}$ and $FS_3(\langle x_n\rangle_{n=1}^{\infty}) \subseteq \{a \in A : a \equiv 3j(\mathrm{mod}\ k)\}$. This is a contradiction, because $2j \not\equiv 3j(\mathrm{mod}\ k)$. □

Another natural candidate for a set satisfying Ψ is based on *spectra of numbers*. These are sets of the form $\{\lfloor n\alpha + \gamma\rfloor : n \in \mathbb{N}\}$, where α is a positive real number, usually irrational, and $0 \leq \gamma \leq 1$. These sets have been much studied. See for example [2, 3, 4, 6, 13, 14].

For large irrational α, the set $\mathbb{N}\backslash\{\lfloor n\alpha\rfloor : n \in \mathbb{N}\}$ seems as though it might satisfy the statement Ψ. If $0 < \gamma < 1$, it is immediate that $\mathbb{N}\backslash\{\lfloor n\alpha + \gamma\rfloor : n \in \mathbb{N}\}$ does not satisfy Ψ. In fact, $\{\lfloor n\alpha + \gamma\rfloor : n \in \mathbb{N}\}$ is an IP* set ([3, Theorem 6.1] or see [10, Theorem 16.42]). That is, for any sequence $\langle x_n\rangle_{n=1}^{\infty}$, $FS(\langle x_n\rangle_{n=1}^{\infty}) \cap \{\lfloor n\alpha + \gamma\rfloor : n \in \mathbb{N}\} \neq \emptyset$. On the other hand [3, Theorem 6.2], for irrational $\alpha > 1$, the sets $\{\lfloor n\alpha\rfloor : n \in \mathbb{N}\}$ and $\{\lfloor n\alpha + 1\rfloor : n \in \mathbb{N}\}$ are disjoint and each contain sets of the form $FS(\langle x_n\rangle_{n=1}^{\infty})$ so neither is an IP* set. (In fact, both of these sets are *central.* For a description of some of the properties of central sets see [10, Chapter 14].)

Theorem 3.10. *Let α be a positive real number and either let $A = \mathbb{N}\backslash\{\lfloor n\alpha\rfloor : n \in \mathbb{N}\}$ or let $A = \mathbb{N}\backslash\{\lfloor n\alpha + 1\rfloor : n \in \mathbb{N}\}$. Then $\neg\Theta(A)$.*

Proof. We do the proof for the case $A = \mathbb{N}\backslash\{\lfloor n\alpha\rfloor : n \in \mathbb{N}\}$. (The other case can be done in a similar way by using the mapping $m \mapsto m - \max\{n\alpha : n \in \omega$ and $n\alpha \leq m\}$.) Notice that if $\alpha \leq 1$, then $A = \emptyset$ so we may assume that $\alpha > 1$.

For each $m \in \mathbb{N}$, let $f(m) = \min\{n\alpha : n \in \mathbb{N}$ and $n\alpha \geq m\} - m$. Notice that $m \in A$ if and only if $f(m) \geq 1$.

Let $\epsilon = \min\left\{\frac{1}{8}, \frac{\alpha-1}{3}\right\}$. Since $[0,\alpha]$ is covered by a finite number of intervals of length ϵ, $\Theta(A)$ implies that there is an interval I of length ϵ and an infinite sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}\backslash A$ such that $FS_2(\langle x_n\rangle_{n=1}^{\infty}) \cup FS_3(\langle x_n\rangle_{n=1}^{\infty}) \subseteq f^{-1}[I] \cap A$. Choose $\beta \in I$.

We may suppose that $\langle f(x_n)\rangle_{n=1}^{\infty}$ converges to a limit γ and that $|f(x_n) - \gamma| < \epsilon$ for every $n \in \mathbb{N}$, because we can achieve this by replacing $\langle x_n\rangle_{n=1}^{\infty}$ by a subsequence. Note that $\gamma \leq 1$ since each $x_n \notin A$.

For any $m, n, r \in \mathbb{N}$, with $m < n < r$, each of $x_m + \gamma$, $x_n + x_r + \beta$ and $x_m + x_n + x_r + \beta$ is within ϵ of a number in $\mathbb{N}\alpha$. It follows that $|\gamma - t\alpha| < 3\epsilon$ for some $t \in \mathbb{Z}$. If $t < 0$, then $\gamma - t\alpha \geq \alpha > \epsilon$. If $t > 0$, we have $|t\alpha - \gamma| = t\alpha - \gamma \geq \alpha - 1 \geq 3\epsilon$. Thus $t = 0$ and hence $|\gamma| < 3\epsilon$.

Thus we have $x_m \leq k\alpha < x_m + \gamma + \epsilon < x_m + 4\epsilon$ and $x_n \leq l\alpha < x_n + \gamma + \epsilon < x_n + 4\epsilon$ for some $k, l \in \mathbb{N}$. So $x_m + x_n \leq (k + l)\alpha < x_m + x_n + 8\epsilon \leq x_m + x_n + 1$ and thus $x_m + x_n \notin A$, a contradiction. □

4. A Finitary Ramsey Theoretic Approximation

We have seen that the existence of distinct p and q in $\mathbb{N}^*$ satisfying the equations $p + p = q = q + q = p + q = q + p$ (equivalently the existence of a nontrivial continuous homomorphism from $\beta\mathbb{N}$ into $\mathbb{N}^*$) implies the existence of a set $A \subseteq \mathbb{N}$

such that whenever A is finitely colored, there must exist an increasing sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}\backslash A$ such that $FS_{\geq 2}(\langle x_n \rangle_{n=1}^{\infty})$ is monochrome. We have not been able to show that no such set exists, and certainly have not been able to produce one.

In this section we produce a set C satisfying the weaker conclusion that whenever it is finitely colored, there exist arbitrarily large *finite* sequences in $\mathbb{N}\backslash C$ with all sums of the form $\sum_{t\in F} x_t$ with $|F| > 1$ monochrome. We remark that the existence of such a set follows from Prömel's Induced Graham-Rothschild theorem [11]. However, the proof of Prömel's result is rather long and difficult, so we include here a relatively short proof of our result.

We shall use the Hales-Jewett Theorem. Given a finite alphabet A, a *variable word* is a word over the alphabet $A \cup \{v\}$ in which v occurs, where v is a "variable" which is not a member of A. Given a variable word w and a letter $a \in A$, the word $w(a)$ is the result of replacing each occurrence of v by a.

Theorem 4.1 (Hales-Jewett)**.** *Let $k, m \in \mathbb{N}$. There exists some $d \in \mathbb{N}$ such that, whenever A is an alphabet with m letters and the length d words over A are k-colored, there exists a variable word w of length d such that $\{w(a) : a \in A\}$ is monochrome.*

Proof. [8]. Or see [7, Section 2.2] or [10, Section 14.2]. □

Corollary 4.2. *Let $k, m \in \mathbb{N}\backslash\{1\}$. There is some $d \in \mathbb{N}$ such that, whenever A is an alphabet with m letters and the length d words over A are k-colored, there exists a variable word $w = l_0 l_1 \ldots l_{d-1}$ with each $l_i \in A \cup \{v\}$ so that:*

(1) *there exist $i \neq j$ such that $l_i, l_j \in A$ and $l_i \neq l_j$ and*
(2) *$\{w(a) : a \in A\}$ is monochrome.*

Proof. Pick d as guaranteed by the Hales-Jewett Theorem for an alphabet of size m and $k+m$ colors. Assume without loss of generality that $A = \{1, 2, \ldots, m\}$. Let $\phi : A^d \to \{1, 2, \ldots, k\}$ and define $\gamma : A^d \to \{1, 2, \ldots, k+m\}$ by $\gamma(w) = \phi(w)$ unless w is constant and $\gamma(ii\ldots i) = k+i$. Pick a variable word w such that γ is constant on $\{w(i) : i \in A\}$. □

We let $k \in \mathbb{N}$ be fixed throughout the remainder of this section. We inductively define for each $m \in \mathbb{N}$ numbers d_m and e_m and sets A_m, B_m, and C_m as follows.

$d_1 = e_1 = 1$, $A_1 = B_1 = \{1\}$, $C_1 = \emptyset$ (or anything else – C_1 is not used).
$d_2 = e_2 = 2$, $A_2 = \{1, 2, 3\}$, and $B_2 = C_2 = \{3\}$. (Or, thinking in binary, as is appropriate, $A_2 = \{01, 10, 11\}$ and $B_2 = C_2 = \{11\}$.)

Inductively, assume $m \geq 3$ and we have defined numbers d_{m-1} and e_{m-1} and sets A_{m-1}, B_{m-1}, and C_{m-1}. Pick by Corollary 4.2, a number d_m so that whenever the length d_m words over the alphabet A_{m-1} are k-colored there is a variable word $w(v) = l_0 l_1 \ldots l_{d_m-1}$ (where each $l_i \in A_{m-1} \cup \{v\}$) such that

(1) there exist $i, j \in \{0, 1, \ldots, d_m - 1\}$ with $l_i, l_j \in A_{m-1}$ and $l_i \neq l_j$ and
(2) $\{w(a) : a \in A_{m-1}\}$ is monochrome.

Let $e_m = d_m \cdot e_{m-1}$ and let

$$A_m = \{1, 2, \ldots, 2^{e_m} - 1\} \ (= \{\textstyle\sum_{t\in F} 2^t : \emptyset \neq F \subseteq \{0, 1, \ldots, e_m - 1\}\}) .$$

Let

$$\begin{aligned}B_m &= \{\textstyle\sum_{i=0}^{d_m-1} a_i \cdot 2^{i\cdot e_{m-1}} : \text{ each } a_i \in A_{m-1}\}\\ &= \{\textstyle\sum_{t\in F} 2^t : F \subseteq \{0,1,\ldots,e_m-1\} \text{ and for each } i \in \{0,1,\ldots,d_m-1\},\\ &\qquad F \cap \{i\cdot e_{m-1}, i\cdot e_{m-1}+1,\ldots,(i+1)\cdot e_{m-1}-1\} \neq \emptyset\}.\end{aligned}$$

Let

$$C_m = B_m \cup \big\{w \cdot \textstyle\sum_{i\in F} 2^{i\cdot e_{m-1}} : w \in C_{m-1} \text{ and } \emptyset \neq F \subsetneq \{0,1,\ldots,d_m-1\}\big\}.$$

For sufficiently large m, the set C_m will be our desired set, so let us describe how we can recognize members of C_m. We do this by induction, so we assume that when a number is written in binary, you can recognize the members of C_{m-1}. Let a number x be given. Since $C_m \subseteq A_m$, if $x \notin A_m$ we know $x \notin C_m$. Thus assume that $x \in A_m$ so $x = \sum_{t\in F} 2^t$ for some $F \subseteq \{0,1,\ldots,e_m-1\}$. That is the length of the binary expansion of x is at most e_m which we view as d_m blocks of length e_{m-1}. If there is a 1 in each such block, that is if for each $i \in \{0,1,\ldots,d_m-1\}$,

$$F \cap \{i\cdot e_{m-1}, i\cdot e_{m-1}+1,\ldots,(i+1)\cdot e_{m-1}-1\} \neq \emptyset\,,$$

then $x \in B_m$ so $x \in C_m$. So assume at least one such block has all 0's. On the other hand, $x \neq 0$ so at least one such block has a 1. If two non-zero blocks look different, then we know $x \notin C_m$. So assume all non-zero blocks are the same and let G be the set of non-zero blocks. Then there is some $w \in A_{m-1}$ such that $x = w \cdot \sum_{i\in G} 2^{i\cdot e_{m-1}}$. If $w \in C_{m-1}$, then $x \in C_m$ and if $w \notin C_{m-1}$, then $x \notin C_m$.

Lemma 4.3. *Let $m \in \mathbb{N}$ with $m \geq 2$ and let $\psi : C_m \to \{1,2,\ldots,k\}$. Then there exist $x_1,x_2,\ldots,x_m$ in $A_m\backslash C_m$ and $\varphi : \{2,3,\ldots,m\} \to \{1,2,\ldots,k\}$ such that whenever $G \subseteq \{1,2,\ldots,m\}$ and $|G| > 1$, one has $\sum_{t\in G} x_t \in C_m$ and $\psi(\sum_{t\in G} x_t) = \varphi(\max G)$.*

Proof. We proceed by induction, so first assume $m = 2$. Let $x_1 = 1$, $x_2 = 2$, and let $\varphi(2) = \psi(3)$.

Now assume $m > 2$ and the lemma is valid for $m-1$. Let

$$\psi : C_m \to \{1,2,\ldots,k\}\,.$$

Define a k-coloring γ of the length d_m words over A_{m-1} as follows. Given $u = l_0l_1\ldots l_{d_m-1}$ with each $l_i \in A_{m-1}$, let $\gamma(u) = \psi(\sum_{i=0}^{d_m-1} l_i \cdot 2^{i\cdot e_{m-1}})$.

By the choice of d_m, pick a variable word $w(v) = l_0l_1\ldots l_{d_m-1}$ and $r \in \{1,2,\ldots,k\}$ such that there exist $i,j \in \{0,1,\ldots,d_m-1\}$ with $l_i, l_j \in A_{m-1}$ and $l_i \neq l_j$ and for each $a \in A_{m-1}$, $\gamma\big(w(a)\big) = r$. Define $\varphi(m) = r$.

Let $F = \{i \in \{0,1,\ldots,d_m-1\} : l_i = v\}$. Let x_m be the member of A_m corresponding to $w(0)$. That is, $x_m = \sum_{i\in\{0,1,\ldots,d_m-1\}\backslash F} l_i \cdot 2^{i\cdot e_{m-1}}$. We claim that $x_m \notin C_m$. Indeed pick $i,j \in \{0,1,\ldots,d_m-1\}$ with $l_i,l_j \in A_{m-1}$ and $l_i \neq l_j$. Since x_m has some blocks of 0's (corresponding to elements of F) we have $x_m \notin B_m$. Since $l_i \neq l_j$, we have $x_m \neq b \cdot \sum_{i\in G} 2^{i\cdot e_{m-1}}$ for any $b \in C_{m-1}$ and any $G \subseteq \{0,1,\ldots,d_m-1\}$.

Now define a k-coloring μ of C_{m-1} by letting $\mu(y) = \psi(y \cdot \sum_{i\in F} 2^{i\cdot e_{m-1}})$. By the induction hypothesis, pick $y_1,y_2,\ldots,y_{m-1} \in A_{m-1}\backslash C_{m-1}$ and

$$\delta : \{2,3,\ldots,m-1\} \to \{1,2,\ldots,k\}$$

such that whenever $G \subseteq \{1,2,\ldots,m-1\}$ and $|G|>1$, $\sum_{t\in G} y_t \in C_{m-1}$ and $\mu(\sum_{t\in G} y_t) = \delta(\max G)$. For $t \in \{2,3,\ldots,m-1\}$, let $\varphi(t) = \delta(t)$. For $t \in \{1,2,\ldots,m-1\}$, let $x_t = y_t \cdot \sum_{i\in F} 2^{i\cdot e_{m-1}}$.

We claim that for each $t \in \{2,3,\ldots,m-1\}$, $x_t \notin C_m$. Indeed, since $F \neq \{0,1,\ldots,d_m-1\}$ we have $x_t \notin B_m$ and since $y_t \notin C_{m-1}$, we have $x_t \neq b \cdot \sum_{i\in G} 2^{i\cdot e_{m-1}}$ for any $b \in C_{m-1}$ and any $G \subseteq \{0,1,\ldots,d_m-1\}$.

Now let $G \subseteq \{1,2,\ldots,m\}$ with $|G|>1$ and let $p=\max G$. Assume first that $p<m$. Then we have that $\sum_{t\in G} y_t \in C_{m-1}$ and $\mu(\sum_{t\in G} y_t) = \delta(p) = \varphi(p)$. Thus $(\sum_{t\in G} y_t)\cdot(\sum_{i\in F} 2^{i\cdot e_{m-1}}) \in C_m$ and $\psi\big((\sum_{t\in G} y_t)\cdot(\sum_{i\in F} 2^{i\cdot e_{m-1}})\big) = \mu(\sum_{t\in G} y_t) = \varphi(p)$. Since

$$\begin{aligned}(\sum_{t\in G} y_t)\cdot(\sum_{i\in F} 2^{i\cdot e_{m-1}}) &= \sum_{t\in G}(y_t \cdot \sum_{i\in F} 2^{i\cdot e_{m-1}})\\ &= \sum_{t\in G} x_t\ ,\end{aligned}$$

we have $\psi(\sum_{t\in G} x_t) = \varphi(p)$ as required.

Finally, assume $\max G = m$. Let $H = G\backslash\{m\}$. Let $z = \sum_{t\in H} y_t$ and note that $z \in A_{m-1}$. (This is immediate if $|H|=1$ and if $|H|>1$, then $z \in C_{m-1} \subseteq A_{m-1}$.) Now

$$\begin{aligned}\sum_{t\in G} x_t &= x_m + \sum_{t\in H} x_t\\ &= x_m + \sum_{t\in H}(y_t \cdot \sum_{i\in F} 2^{i\cdot e_{m-1}})\\ &= x_m + (\sum_{t\in H} y_t)\cdot(\sum_{i\in F} 2^{i\cdot e_{m-1}})\\ &= x_m + z\cdot\sum_{i\in F} 2^{i\cdot e_{m-1}}\ .\end{aligned}$$

Let $w(z) = u_0u_1\ldots u_{d_m-1}$, where, recall, $w(v) = l_0l_1\ldots l_{d_m-1}$. Then

$$u_i = \begin{cases} l_i & \text{if } i \notin F\\ z & \text{if } i \in F\ .\end{cases}$$

So

$$\begin{aligned}\gamma\big(w(z)\big) &= \psi(\sum_{i=0}^{d_m-1} u_i \cdot 2^{i\cdot e_{m-1}})\\ &= \psi(\sum_{i\in\{0,1,\ldots,d_m-1\}\backslash F} l_i\cdot 2^{i\cdot e_{m-1}} + \sum_{i\in F} z\cdot 2^{i\cdot e_{m-1}})\\ &= \psi(x_m + z\cdot\sum_{i\in F} 2^{i\cdot e_{m-1}})\\ &= \psi(\sum_{t\in G} x_t)\ .\end{aligned}$$

Thus $\psi(\sum_{t\in G} x_t) = \gamma\big(w(z)\big) = \varphi(m)$. $\square$

Theorem 4.4. *For each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that whenever C_m is k-colored, there exist $z_1, z_2, \ldots, z_n$ in $\mathbb{N}\backslash C_m$ such that*

$$\{\sum_{t\in F} z_t : F \subseteq \{1,2,\ldots,n\} \text{ and } |F| > 1\}$$

is contained in C_m and is monochrome.

Proof. Let $m = (n-2)\cdot k+2$. Given a k-coloring ψ of C_m pick $x_1, x_2, \ldots, x_m$ and φ as guaranteed by Lemma 4.3. By the pigeon hole principle, pick $G \subseteq \{2, 3, \ldots, m\}$ with $|G| = n-1$ and $\varphi(i) = \varphi(j)$ for all $i, j \in G$. Let $z_1 = x_1$ and let $z_2, z_3, \ldots, z_n$ enumerate $\{x_i : i \in G\}$. □

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