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The Chromatic E_1-term $H^0M_1^2$ for $p > 3$

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Abstract. In this paper, study the module structure of

$$\mathrm{Ext}^0_{BP_*BP}(BP_*, BP_*/(p, v_1^\infty, v_2^\infty)),$$

which is regarded as the chromatic E_1-term converging to the second line of the Adams-Novikov E_2-term for the Moore spectrum. The main difficulty here is to construct elements $x(sp^r/j;k)$ from the Miller-Ravenel-Wilson elements $(x_{3,r}^s/v_2^j)^{p^k} \in H^0M_2^1$. We achieve this by developing some inductive methods of constructing $x(sp^r/j;k)$ on k.

Contents

1. Introduction

Let BP be the Brown-Peterson spectrum for a fixed prime p. As is well known, the pair of homotopy groups BP_* and the BP_*-homology BP_*BP forms a Hopf-algebroid

$$(BP_*, BP_*BP) = (\mathbb{Z}_{(p)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots]).$$

The Adams-Novikov spectral sequence (ANSS) is one of the fundamental tools to compute the p-component of the stable homotopy groups $\pi_*^S X_{(p)}$ for a spectrum X:

$$E_2^{*,*} = \mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*X) \implies \pi_*^S X_{(p)}.$$

Here, for any BP_*BP-comodule M, $\mathrm{Ext}^*_{BP_*BP}(BP_*, M)$ is regarded as the right derived functor of $\mathrm{Hom}_{BP_*BP}(BP_*, M)$. We abbreviate $\mathrm{Ext}^s_{BP_*BP}(BP_*, M)$ to H^sM as usual.

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As an important example of finite spectra, we have *the Smith-Toda spectrum* $V(n)$ for each prime p with $BP_*V(n) \cong BP_*/(p, v_1, \cdots, v_n)$, although its existence is verified only for $0 \le n \le 3$ and $2n+1 \le p$ so far. (Recently, Lee S. Nave has shown the non-existence of $V((p+3)/2)$ for $p > 5$.) Then the E_2-term of the ANSS for $V(n)$ is $H^*BP_*V(n)$. Miller, Ravenel and Wilson [2] have constructed an algebraic spectral sequence converging to $H^*BP_*V(n)$ as follows:

Denote the BP_*BP-comodules $BP_*/(p, v_1, \cdots, v_{n-1})$ by N_n^0. Then, define N_n^m $(m \ge 1)$ inductively on m by short exact sequences

$$0 \longrightarrow N_n^m \longrightarrow v_{m+n}^{-1} N_n^m \longrightarrow N_n^{m+1} \longrightarrow 0.$$

We also define M_n^m by $M_n^m = v_{m+n}^{-1} N_n^m$. Indeed, they can be described directly as

$$\begin{aligned} N_n^m &= BP_*/(p, \cdots, v_{n-1}, v_n^\infty, \cdots, v_{n+m-1}^\infty), \\ M_n^m &= v_{n+m}^{-1} BP_*/(p, \cdots, v_{n-1}, v_n^\infty, \cdots, v_{n+m-1}^\infty). \end{aligned}$$

Splicing the above short exact sequences, we get a long exact sequence:

$$0 \longrightarrow N_n^0 \longrightarrow M_n^0 \longrightarrow M_n^1 \longrightarrow M_n^2 \longrightarrow \cdots,$$

called *the chromatic resolution* of N_n^0. Applying $H^*(-)$ to the above long exact sequence, we obtain a spectral sequence converging to $H^*N_n^0$ with $E_1^{s,t} = H^tM_n^s$ and $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$, called *the chromatic spectral sequence*.

The simplest example in these E_1-terms is the 0-th cohomology of the n-th Morava stabilizer group $H^0M_n^0$, which is isomorphic to $\mathbb{Z}/p\,[v_n^{\pm 1}]$. Moreover, $H^tM_n^0$ $(1 \le t \le 2)$ has been computed by Ravenel [5]. In general the calculation of $H^tM_n^s$ becomes terribly difficult as $s+t$ increases, except for the following case:

Theorem (Morava's vanishing theorem). *If* $(p-1) \nmid n$ *and* $t > n^2$, *then*

$$H^tM_n^0 = 0.$$

Many chromatic E_1-terms are computed so far (cf. [12]), most of them are due to Miller-Ravenel-Wilson [2], Ravenel [5] and Shimomura's works. In particular, $H^0M_n^1$ is computed in [2, Theorem 5.10] and $H^0M_n^2$ $(n \ge 2, p > 2)$ is also done in [10, Theorem 1.2]. The purpose of this paper is to prove the following theorem about the $k(1)_*$-module structure of unknown $H^0M_1^2$ for $p > 3$.

Theorem. *For each Miller-Ravenel-Wilson element* $x_{3,r+k}^s/v_2^{jp^k} \in H^0M_2^1$ $(p \nmid j$ *and* $p \nmid s)$ *and* $1/v_2^{mp^r} \in H^0M_2^1$ $(p \nmid m)$, *there exists an element*

$$x(sp^r/j;k) \in v_3^{-1}BP_*/(p, v_2^\infty)$$

which is congruent to $(x_{3,r}^s/v_2^j)^{p^k}$ *mod* (v_1), *and*

$$1/X_{2,r}^m \in BP_*/(p, v_2^\infty)$$

congruent to $1/v_2^{mp^r}$ *mod* (v_1), *so that as a* $k(1)_*$*-module*

$$\begin{aligned} H^0M_1^2 \cong \\ & k(1)_* \left\{ x(sp^r/j;k)/v_1^{N(s,r,j;k)} \;\middle|\; k \ge 0,\ r \ge 0,\ p \nmid s \in \mathbb{Z} \text{ and } p \nmid j \le a_{3,r} \right\} \\ & \quad \oplus\ k(1)_* \left\{ 1/v_1^{a_{2,r}} X_{2,r}^m \;\middle|\; r \ge 0 \text{ and } p \nmid m \ge 1 \right\}, \end{aligned}$$

where the integers $a_{2,r}$ and $a_{3,r}$ are defined in 2.3.1 *and* 3.3.1, *and the integers $N(s,r,j;k)$ are given as follows:*

$$
\begin{array}{lrl}
(1) & [(p^3+p^2-p-1)p^{k-3}]+[p^{k-3}] & \text{for } r=0 \text{ and } p\nmid(s-1),\\
(2) & (p^{k+1}-1)/(p-1) & \text{for } r=0,\ p\mid(s-1) \text{ and } s\notin\mathbb{N}_1,\\
(3) & [(2p^2-1)p^{k-2}]+[p^{k-2}] & \text{for } r=0 \text{ and } s\in\mathbb{N}_1,\\
(4) & (p^{k+1}-1)/(p-1)-[p^{k-1}] & \text{for odd } r\geq 1,\ s\notin\mathbb{N}_0 \text{ and } j=a_{3,r}-1,\\
(5) & p^k+p^{k-1}-1 \quad (=a_{2,k}) & \text{for odd } r\geq 1,\ s\notin\mathbb{N}_0 \text{ and } j\leq a_{3,r}-2\\
 & & \quad \text{or even } r\geq 2 \text{ and } s\notin\mathbb{N}_0,\\
(6) & p^{k+1}+p^k-1 \quad (=a_{2,k+1}) & \text{for } r=1,\ s\in\mathbb{N}_0 \text{ and } j=p-1,\\
(7) & [(2p^3+p^2-p-1)p^{k-3}]+[p^{k-3}] & \text{for } r=1,\ s\in\mathbb{N}_0 \text{ and } j\leq p-2,\\
(8) & [(p^2+p-1)p^{k-2}]+[p^{k-2}] & \text{for even } r\geq 2 \text{ and } s\in\mathbb{N}_0,\\
(9) & [(p^4+p^2+p-1)p^{k-4}]+[p^{k-4}] & \text{for odd } r\geq 3 \text{ and } j=a_{3,r},
\end{array}
$$

and for odd $r\geq 3$, $s\in\mathbb{N}_0$ and $j\leq a_{3,r}-1$, we have

$$
\begin{array}{lrl}
(10) & [(p^3+p^2-1)p^{k-3}]+[p^{k-3}] & \text{for } j=a_{3,r}-1,\\
(11) & \text{same as the case } (1) & \text{for } a_{3,r}-p\leq j\leq a_{3,r}-2,\\
(12) & 2p^k & \text{for } j=a_{3,r}-p-1,\\
(13) & \text{same as the case } (7) & \text{for } j=a_{3,r}-p+2,\\
 & & \quad \text{or } p\nmid(j+1) \text{ and } j\leq a_{3,r}-p-2,\\
(14) & p^k+(p^{k+1}-1)/(p-1) & \text{for } p\mid(j+1)\\
 & & \quad \text{and } a_{3,r}-p^2\leq j\leq a_{3,r}-2p,\\
(15) & [(p^2+p-1)p^{k-1}]+[p^{k-1}] & \text{for } j=a_{3,r}-p^2-p,\\
 & & \quad \text{or } p\mid(j+1),\ p^3\nmid(j+1)(j+p+1)\\
 & & \quad\quad \text{and } j\leq a_{3,r}-p^2-3p,\\
(16) & [(p^4+p^3-p^2+1)p^{k-3}]+[2p^{k-1}] & \text{for } j\leq a_{3,r}-2p^2-p \text{ and } p^2\mid(j+1),\\
(17) & p^{k+1}+p^k & \text{for } j\leq a_{3,r}-p^2-2p\\
 & & \quad \text{and } p^2\mid(j+p+1),
\end{array}
$$

Here $[x]$ is the greatest integer which does not exceed x, and

$$
\begin{aligned}
\mathbb{N}_0 &= \{\, ap-1 \mid p\nmid a \,\},\\
\mathbb{N}_1 &= \left\{\, (ap^2-p-1)p^r+1 \mid p\nmid a,\ r\geq 1 : \mathit{odd} \,\right\}.
\end{aligned}
$$ □

In Section 2 we shall review BP-theory and the Bockstein spectral sequence and recall the structure of the chromatic E_1-term $H^0M_2^1$ computed in [2]. Then we change the $\mathbb{F}_p$-module basis of $H^0M_2^1$ and state the method of getting the structure of $H^0M_1^2$ (originally due to Miller-Ravenel-Wilson). It is enough to read this section for an idea of the theoretical part. In Section 3 we give the fundamental elements $u_{n,k}$, $w_{n,k}$ and $X_{3,r}$, construct the new element $1/X_{2,r}$, and give the differentials on these elements (some of them are introduced in [1] and [8]). In Section 4 we set up the elements $X_0(v_2^j, X_{3,r}^s)$ and $X(v_2^j, X_{3,r}^s)$, each of which is congruent to $X_{3,r}^s/v_2^j \bmod (v_1)$, and compute the differentials. We also introduce some inductive methods of constructing $x(k)$ for large k. This is the hardest part of this paper. Using these results, we construct the series of elements $x(sp^r/j; k)$ and complete the proof of the main theorem in Section 5.

We can deduce some applications to $H^2BP_*V(0)$ from this result. These will appear in the forthcoming paper [3].

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2. BP_*-homology and Bockstein spectral sequence

In this section we review several basic facts in BP-theory and explain how to determine the structure of $H^0M_1^2$.

2.1. **Summary of BP_*-homology and related maps.** For a fixed prime p, there is a spectrum BP called *the Brown-Peterson spectrum*, which is characterized by

$$\begin{aligned} BP_* &= \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n, \cdots], \\ BP_*BP &= BP_*[t_1, t_2, \cdots, t_n, \cdots], \end{aligned}$$

where $|v_n| = |t_n| = 2(p^n - 1)$. The pair $(A, \Gamma) = (BP_*, BP_*BP)$ form a Hopf algebroid with the following structure maps:

$$\eta_L \text{ (resp.} \eta_R) : A \to \Gamma \quad \text{(left (resp. right) unit)}, \qquad c : \Gamma \to \Gamma \quad \text{(conjugation)},$$
$$\epsilon : \Gamma \to A \quad \text{(augmentation)}, \qquad \Delta : \Gamma \to \Gamma \otimes_A \Gamma \quad \text{(coproduct)}.$$

Given a BP_*BP-comodule M and its BP_*BP-comodule structure $\phi : M \to M \otimes_A \Gamma$, we use the following notation as usual:

$$H^*M = \mathrm{Ext}_\Gamma^*(A, M) \;\cong\; H^*(\Omega^*M, d),$$

where the cobar complex (Ω^*M, d) is the double graded $\mathbb{Z}_{(p)}$-module with

$$\Omega^n M = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma \quad (n \text{ factors of } \Gamma),$$
$$\begin{aligned} d_n(m \otimes \gamma_1 \otimes \cdots \gamma_n) = \phi(m) &\otimes \gamma_1 \otimes \cdots \otimes \gamma_n \\ &+ \sum (-1)^k \, m \otimes \gamma_1 \otimes \cdots \otimes \Delta(\gamma_k) \otimes \cdots \otimes \gamma_n \\ &+ \; (-1)^{n+1} \, m \otimes \gamma_1 \otimes \cdots \otimes \gamma_n \otimes 1. \end{aligned}$$

In this paper we compute only the 0-th differential $d_0 = \eta_R - \eta_L : M_1^2 \to M_1^2 \otimes_A \Gamma$. By definition, d_0 satisfies

$$d_0(xy) = d_0(x)\, y + \eta_R(x)\, d_0(y)$$

for any $x, y \in A$. This formula is frequently used for computations in this paper. When $d_0(x) \equiv z \bmod (p, J)$ for an element $z \in \Gamma$ and an ideal $J \subset A$, we also have

$$d_0(x^p) \equiv z^p \mod (p, J^p).$$

See also [2, Observation 5.8].

To compute d_0, we summarize some known results about η_R. Ravenel [4] has shown the following congruence of formal group laws:

$$\sum_{i,j\geq 0}{}^F t_i\eta_R(v_j^{p^i}) \equiv \sum_{i,j\geq 0}{}^F v_i t_j^{p^i} \qquad \mathrm{mod}\ p.$$

In general $\eta_R(v_n)\equiv v_n+v_{n-1}t_1^{p^{n-1}}-v_{n-1}^p t_1 \bmod I_{n-1}$ for the invariant prime ideal $I_{n-1}=(p,v_1,\cdots,v_{n-2})$. Then, for instance, direct calculation shows that

(2.1.1) $\eta_R(v_2^i)\equiv(v_2+v_1t_1^p)^i - i\,v_1^pv_2^{i-1}t_1 - i(i-1)\,v_1^{p+1}v_2^{i-2}t_1^{p+1} \quad \mathrm{mod}\ p, v_1^{p+2}.$

More precisely, $\eta_R(v_3)$ and $\eta_R(v_4)$ satisfy the following congruences (cf. [6](4.3.21)):

(2.1.2) $\eta_R(v_3)\equiv v_3+v_2t_1^{p^2}-v_2^pt_1+v_1t_2^p-v_1^2v_2^{p-1}t_1^p \quad \mathrm{mod}\ p, v_1^3,$

(2.1.3) $\eta_R(v_4)\equiv v_4+v_3t_1^{p^3}-v_3^pt_1+v_2t_2^{p^2}-v_2^pt_1^{p^3+1} \quad \mathrm{mod}\ p, v_1, v_2^{p+1}.$

2.2. **Bockstein spectral sequence.** In this paper will deduce the $k(1)_*$-module structure of $H^0M_1^2$ from $H^0M_2^1$, which has already been computed in [2]. By definitions of comodules M_2^1 and M_1^2, we obtain the short exact sequence

$$0 \longrightarrow M_2^1 \xrightarrow{1/v_1} M_1^2 \xrightarrow{v_1} M_1^2 \longrightarrow 0.$$

Applying $H^*(-)$ to this sequence, we get the long exact sequence

$$0 \longrightarrow H^0M_2^1 \xrightarrow{1/v_1} H^0M_1^2 \xrightarrow{v_1} H^0M_1^2 \xrightarrow{\delta} H^1M_2^1 \xrightarrow{1/v_1} \cdots.$$

Regarding this long sequence as an exact couple, we get a Bockstein type spectral sequence in the usual way, leading from $H^*M_2^1$ to $H^*M_1^2$. But as in [2] we compute $H^0M_1^2$ directly by making use of the following lemma.

Lemma 2.2.1 (cf. [2], Remark 3.11). *Assume that there exists a $k(1)_*$-submodule B^t of $H^tM_1^2$ for each $t<N$, such that the following sequence is exact:*

$$0\to H^0M_2^1 \xrightarrow{1/v_1} B^0 \xrightarrow{v_1} B^0 \xrightarrow{\delta} H^1M_2^1 \xrightarrow{1/v_1} \cdots \xrightarrow{1/v_1} B^{N-1} \xrightarrow{v_1} B^{N-1} \xrightarrow{\delta} H^NM_2^1,$$

where $\delta: B^t \to H^{t+1}M_2^1$ is the restriction of the coboundary map $\delta: H^tM_1^2\to H^{t+1}M_2^1$. Then the inclusion map $i_t: B^t\to H^tM_1^2$ is an isomorphism between $k(1)_*$-modules for each $t<N$.

Sketch of the proof. Because $H^tM_1^2$ is a v_1-torsion module, we can filter B^t and $H^tM_1^2$ as

$$P_t(m)=\{x\in B^t \mid v_1^mx=0\} \quad \text{and} \quad Q_t(m)=\{x\in H^tM_1^2 \mid v_1^mx=0\}.$$

Assume that the inclusion i_k is an isomorphism for $k\leq t-1$ (the $t=0$ case is obvious), and consider the following commutative ladder diagram:

$$\begin{array}{ccccccccc}
B^{t-1} & \xrightarrow{\delta} & H^tM_2^1 & \xrightarrow{1/v_1} & P_t(m) & \xrightarrow{v_1} & P_t(m-1) & \xrightarrow{\delta} & H^{t+1}M_2^1 \\
\cong\downarrow i_{t-1} & & \| & & \downarrow i_t & & \downarrow i_t & & \| \\
H^{t-1}M_1^2 & \xrightarrow{\delta} & H^tM_2^1 & \xrightarrow{1/v_1} & Q_t(m) & \xrightarrow{v_1} & Q_t(m-1) & \xrightarrow{\delta} & H^{t+1}M_2^1.
\end{array}$$

Using the Five Lemma, we can show that $P_t(m)\cong Q_t(m)$ $(m\geq 1)$ by induction on m. $\square$

We shall construct B^0 satisfying the above lemma to determine the $k(1)_*$-module structure of $H^0M_1^2$. In order to construct a $k(1)_*$-module basis of B^0, it is natural to push each element of $H^0M_2^1$ to $H^0M_1^2$ and to divide it by v_1 as many times as possible. So we need to review the module structure of $H^0M_2^1$.

2.3. **$H^0M_2^1$ and changing its basis.** We first recall some notations defined in [2] to write down a $k(2)_*$-module basis of $H^0M_2^1$. Hereafter we assume that $p > 2$.

Definition 2.3.1 ([2], (5.11) and (5.13))**.** Define integers $a_{3,k}$ by

$$a_{3,0} = 1, \quad a_{3,1} = p, \quad a_{3,2t} = p\,a_{3,2t-1}, \quad a_{3,2t+1} = p\,a_{3,2t} + p - 1 \quad \text{for} \quad t \geq 1$$

and elements $x_{3,k} \in v_3^{-1}BP_*$ by

$$x_{3,0} = v_3, \qquad x_{3,1} = v_3^p - v_2^p v_3^{-1} v_4, \qquad x_{3,2t} = x_{3,2t-1}^p,$$
$$x_{3,2t+1} = x_{3,2t}^p - v_2^{a_{3,2t+1}-p} v_3^{(p-1)p^{2t}+1}$$

for $t \geq 1$.

Using these notations, Miller-Ravenel-Wilson ([2, Theorem 5.10]) have shown the following:

Theorem 2.3.2. *As a $k(2)_*$-module,*

$$H^0M_2^1 \cong k(2)_* \left\{ x_{3,k}^m / v_2^{a_{3,k}} \;\middle|\; k \geq 0,\, p \nmid m \in \mathbb{Z} \right\} \oplus \mathbb{F}_p \left\{ 1/v_2^i \;\middle|\; i \geq 1 \right\}.$$

In this paper we will consider an analogue to Miller-Ravenel-Wilson construction of the elements $x_{3,k}^m \in v_3^{-1}BP_*$ ([2, Section 5]): The elements $x_{3,k}$ have been defined inductively on k with $x_{3,0} = v_3$, and each of them has the relation $x_{3,k} \equiv v_3^{p^k}$ mod (p, v_2^N) for a small enough integer N. Motivated by this, we shall construct elements $x(k) \in v_3^{-1}BP_*/(p, v_2^\infty)$ inductively on k with $x(0) = x_{3,r}^s/v_2^j$ so that $x(k) \equiv (x_{3,r}^s/v_2^j)^{p^k}$ mod (p, v_1^N) for a small enough N. Keeping it in mind, we now change the $\mathbb{F}_p$-basis of $H^0M_2^1$ to "a p^k-power basis".

Lemma 2.3.3. *As a $k(2)_*$-module,*

$$\begin{aligned} H^0M_2^1 \cong \mathbb{F}_p &\left\{ (x_{3,r}^s/v_2^j)^{p^k} \;\middle|\; k \geq 0,\ r \geq 0,\ p \nmid s \in \mathbb{Z},\ \text{and}\ p \nmid j \leq a_{3,r} \right\} \\ &\oplus \mathbb{F}_p \left\{ 1/v_2^i \;\middle|\; i \geq 1 \right\}. \end{aligned}$$

Proof. It is sufficient to prove that any $\mathbb{F}_p$-module base $x_{3,u}^m/v_2^l$ $(1 \leq l \leq a_{3,u})$ displayed in Theorem 2.3.2 can be written as a linear sum

$$x_{3,u}^m/v_2^l = \sum_\lambda a_\lambda x_\lambda, \tag{2.3.4}$$

where $a_\lambda \in \mathbb{F}_p$ and each x_λ has a form $(x_{3,r}^s/v_2^j)^{p^k}$ with $p \nmid s \in \mathbb{Z}$ and $p \nmid j \leq a_{3,r}$.

We shall show it by induction on mp^u. Notice that

$$x_{3,u}^m/v_2^l = v_3^{mp^u}/v_2^l + (\text{elements with } v_3\text{-exponents less than } mp^u).$$

When $mp^u = 1$, the only such a base is v_3/v_2, so it is clear. Suppose that each base $x_{3,b}^a/v_2^c$ $(1 \leq ap^b < mp^u)$ can be expressed as (2.3.4) and $l = dp^e$ with $p \nmid d$. Because $l \leq a_{3,u} < p^{u+1}$, we may assume that $e \leq u$. Define $y = x_{3,u}^m/v_2^l - \left(x_{3,u-e}^m/v_2^d\right)^{p^e} \in H^0M_2^1$. Since the maximal v_3-exponent in y is less than mp^u, y has a form (2.3.4) by inductive assumption, hence so does $x_{3,u}^m/v_2^l$. $\square$

2.4. **The construction of the module B^0.** Let $x(sp^r/j;k)$ and $y(mp^r)$ be elements of $v_3^{-1}BP_*/(p,v_2^\infty)$ satisfying

$$x(sp^r/j;k)/v_1=(x_{3,r}^s/v_2^j)^{p^k}/v_1 \qquad \text{and} \qquad y(mp^r)/v_1=1/v_1v_2^{mp^r}$$

in $H^0M_1^2$, and $N(s,r,j;k)$ (resp. $N(m;r)$) be the maximal integer such that each $x(sp^r/j;k)/v_1^i$ (resp. $y(mp^r)/v_1^i$) with $1\le i\le N(s,r,j;k)$ (resp. $1\le i\le N(m;r)$) is a cycle of $H^0M_1^2$.

Proposition 2.4.1. *As a $k(1)_*$-module,*

$$\begin{aligned} B^0 = {} & k(1)_*\left\{x(sp^r/j;k)/v_1^{N(s,r,j;k)} \mid k\ge 0,\ r\ge 0,\ p\nmid s\in\mathbb{Z} \ \textit{and}\ p\nmid j\le a_{3,r}\right\} \\ & \oplus k(1)_*\left\{y(mp^r)/v_1^{N(m;r)} \mid r\ge 0 \ \textit{and}\ p\nmid m\ge 1\right\} \end{aligned}$$

is isomorphic to $H^0M_1^2$ if it satisfies the following condition for the coboundary map $\delta: B^0\to H^1M_2^1$ in Lemma 2.2.1:

$$\left\{\delta\left(x(sp^r/j;k)/v_1^{N(s,r,j;k)}\right)\right\}\cup\left\{\delta\left(y(mp^r)/v_1^{N(m;r)}\right)\right\} \textit{ is } \mathbb{F}_p\textit{-linearly independent.}$$

Proof. All exactness of the sequence $0\to H^0M_2^1 \xrightarrow{1/v_1} B^0 \xrightarrow{v_1} B^0 \xrightarrow{\delta} H^1M_2^1$ is obvious, but $\operatorname{Ker}\delta\subset\operatorname{Im}v_1$. So we need to show only this inclusion. Separate the $\mathbb{F}_p$-basis of B^0 into two parts

$$\begin{aligned} A &= \left\{x(sp^r/j;k)/v_1^{N(s,r,j;k)}\right\}\cup\left\{y(mp^r)/v_1^{N(m;r)}\right\} \\ B &= \left\{x(sp^r/j;k)/v_1^l \mid 1\le l< N(s,r,j;k)\right\}\cup\left\{y(mp^r)/v_1^l \mid 1\le l<N(m;r)\right\}. \end{aligned}$$

Then it is obvious that $\delta(x_\lambda)\ne 0\in H^1M_2^1$ for $x_\lambda\in A$, and that $\delta(y_\mu)=0\in H^1M_2^1$ for $y_\mu\in B$. Thus for any element $z=\sum_\lambda a_\lambda x_\lambda+\sum_\mu b_\mu y_\mu$ of B^0 ($a_\lambda,b_\mu\in\mathbb{Z}/p$), we have $\delta(z)=\sum_\lambda a_\lambda\delta(x_\lambda)$. The condition implies that all a_λ are zero when $\delta(z)=0$, and so $v_1\sum_\mu b_\mu y_\mu/v_1=z$. This completes the proof. □

We will construct the element $x(sp^r/j;k)$ from $(x_{3,r}^s/v_2^j)^{p^k}$ in Section 5, and the element $1/X_{2,r}^m$ from $1/v_2^{mp^r}$ as a candidate for $y(mp^r)$ in Section 3.

2.5. **$\mathbb{F}_p$-linear independence and Coker δ.** When we compute the coboundaries $\delta: B^0\to H^1M_2^1$ of $x(sp^r/j;k)/v_1^{N(s,r,j;k)}$ and $y(mp^r)/v_1^{N(m;r)}$, we expect each of these images to have an appropriate form so that we can judge whether the set of δ-images is $\mathbb{F}$-linearly independent or not in $H^1M_2^1$ for use in Proposition 2.4.1. Though the structure of $H^1M_2^1$ ($p>3$) has already been computed by Shimomura ([9] and [11]), we don't need the whole structure of $H^1M_2^1$ for our purpose. We follow the same method as in the proof of [2, Theorem 6.1 (p. 500)].

Consider the long exact sequence

$$\cdots\longrightarrow H^0M_2^1\xrightarrow{\delta} H^1M_3^0\xrightarrow{1/v_2} H^1M_2^1\xrightarrow{v_2} H^1M_2^1\xrightarrow{\delta}\cdots.$$

Since the coboundaries on elements of $H^0M_2^1$ have already been computed in [2, Proposition 5.17], we can obtain generators of $\operatorname{Ker}\left(v_2\mid H^1M_2^1\right)$. An easy calculation shows that $\operatorname{Coker}(\delta: H^0M_2^1\to H^1M_3^0)$ is isomorphic to the following direct sum as a $\mathbb{F}_p$-vector space:

$$\begin{aligned}(2.5.1)\qquad &\mathbb{F}_p\left\{v_3^{sp^r}h_0 \ \mid\ \text{either } s\notin\mathbb{N}_0 \text{ and even } r\geq 0, \text{ or } p\nmid s \text{ and odd } r\geq 1\right\}\\ &\oplus\mathbb{F}_p\left\{v_3^{sp^r}h_1 \ \mid\ \text{either } s\notin\mathbb{N}_0 \text{ and odd } r\geq 1, \text{ or } p\nmid s \text{ and even } r\geq 0\right\}\\ &\oplus\mathbb{F}_p\{\ h_1\ \}\ \oplus\ \mathbb{F}_p\left\{v_3^{tp-1}h_2\ \mid\ t\in\mathbb{Z}\right\}\ \oplus\ K(3)_*\{\ \zeta_3\ \},\end{aligned}$$

where $\mathbb{N}_0=\{\ s\in\mathbb{Z}\ \mid\ p\mid s+1\ \text{ and }\ p^2\nmid s+1\ \}$, and h_i is the cohomology class of $t_1^{p^i}$.

Let S be the set of elements

$$\{A_k=v_3^{b_k}h_{a_k}/v_2^{c_k}\mid 0\leq k\leq n,\ 0\leq a_k\leq 2,\ c_1\geq\cdots\geq c_n,\ v_3^{b_k}h_{a_k}\in\operatorname{Coker}\delta\,\},$$

and consider the following condition on S:

$$(2.5.2)\qquad (a_i,b_i,c_i)\neq(a_j,b_j,c_j)\ \text{ for any two elements with } i\neq j.$$

For a linear sum $A=\sum_{k=1}^n\lambda_kA_k$, assume that $A=0$ in $H^1M_2^1$ and $c_1=c_2=\cdots=c_m$ for some $m\leq n$. Because $H^1M_2^1$ is a v_2-torsion module, we can consider the multiplication by $v_2^{c_1-1}$, so that $v_2^{c_1-1}A=\sum_{k=1}^m\lambda_k(v_3^{b_k}h_{a_k}/v_2)=0$ in $\operatorname{Ker}v_2$ ($\cong\operatorname{Coker}\delta$). By the condition (2.5.2), the set $\{v_3^{b_k}h_{a_k}\}$ is linearly independent in $\operatorname{Coker}\delta$ and thus all λ_k ($1\leq\lambda_k\leq m$) should be zero. Iterating this, we conclude that all coefficients λ_k are zero and S is linearly independent in $H^1M_2^1$.

3. Definitions of some elements

In this section we introduce the elements $u_{n,k}$, $w_{n,k}$, $X_{3,r}$ and $1/X_{2,r}$. We will use these to define many elements in Section 4.

3.1. Moreira's element $u_{n,k}$ and Shimomura's element $w_{n,k}$.

Definition 3.1.1. As is done in [1, §6(4)] or [8, 2.8], we define the element $u_{n,k}\in v_n^{-1}BP_*$ by the following recursive formula:

$$u_{n,0}=v_n^{-1}\qquad\text{and}\qquad\sum_{i+j=k}v_{n+i}u_{n,j}^{p^i}=0\quad\text{for } k\geq 1.$$

Remark 3.1.2. In [1] $u_{n,k}$ is defined only for $k\leq n$ and denoted as u_{n+k}. By definition

$$\begin{aligned}u_{n,1}&=-\,v_n^{-p-1}v_{n+1},\\ u_{n,2}&=v_n^{-p^2-p-1}v_{n+1}^{p+1}-v_n^{-p^2-1}v_{n+2},\\ u_{n,3}&\equiv-\,v_n^{-p^3-p^2-p-1}v_{n+1}^{p^2+p+1}+v_n^{-p^3-p^2-1}(v_{n+1}^{p^2}v_{n+2}+v_{n+1}v_{n+2}^p)\\ &\qquad-v_n^{-p^3-1}v_{n+3}\mod (p),\end{aligned}$$

and so on.

Computing the right unit η_R on $u_{n,n}$, Moreira has shown that the $K(n)_*$-module base ζ_n of $H^1M_n^0$ is homologous to ζ_n^p:

Proposition 3.1.3 ([1], Theorem 6.2.1.1). $\eta_R(u_{n,n})-\eta_L(u_{n,n})\equiv\zeta_n-\zeta_n^p\mod I_n$.

We now recall the elements $w_{n,k}$ introduced by Shimomura [8]. Define elements T_j ($j \ge 0$) by $T_0 = 1$ and $\sum_{i=0}^{j-1} t_i \eta_R(v_{j-i}^{p^i}) \equiv \sum_{i=1}^{j} v_i T_{j-i}^{p^{i-1}}$ mod (p) for $j \ge 1$, and an element $e_n(x) \in v_n^{-1}BP_*BP/I_n$ for $x \in v_n^{-1}BP_*$ by the congruence $\eta_R(x) \equiv e_n(x)$ mod I_n.

Definition 3.1.4 (cf. [8], 2.10)**.** Define elements $w_{n,k} \in v_n^{-1}BP_*BP$ ($n \ge 2$) inductively on k by

$$w_{n,0} = 0 \quad \text{and} \quad w_{n,k} = \sum_{j=1}^{k} e_n(u_{n,k-j}^{p^{j-1}}) T_j^{p^{n-2}} \quad \text{for } k \ge 1.$$

Remark 3.1.5. By definition,

$$\begin{aligned} w_{n,1} &= v_n^{-1} t_1^{p^{n-1}}, \\ w_{n,2} &\equiv -\, v_n^{-p-1} v_{n+1} t_1^{p^{n-1}} + v_n^{-p}(t_2^{p^{n-1}} - t_1^{p^n + p^{n-1}}) + v_n^{-1} t_1^{p^{n-1}+1} \mod p, v_1^{p^{n-2}}. \end{aligned}$$

In general, as an element of $M_2^1 \otimes_A \Gamma$

$$w_{2,k} = (-1)^{k-1} \frac{v_3^{(p^{k-1}-1)/(p-1)} t_1^p}{v_2^{(p^k-1)/(p-1)}} + (\text{elements killed by } v_2^{(p^k-1)/(p-1)-1}).$$

We note that $w_{2,2}$ is similar to $\zeta_2 = -\, v_2^{-p-1} v_3 t_1^p + v_2^{-p}(t_2^p - t_1^{p^2+p}) + v_2^{-1} t_2$. In fact $w_{n,n}$ ($n \ge 2$) is related to ζ_n by the following congruence (cf. [8, 4.8]):

$$\zeta_n - w_{n,n} \equiv \sum_{1 \le i < j \le n} u_{n,n-j}^{p^{j-i-1}} \left(\sum_{i \le k \le j} t_k\, c(t_{j-k}^{p^k}) \right)^{p^{n-i-1}} \mod I_n.$$

In particular we have $\zeta_2 - w_{2,2} \equiv v_2^{-1}(t_2 - t_1^{p+1})$ mod I_2 for $n = 2$.

Using the notation $w_{n,k}$, Shimomura [8] has proved the following proposition, which is a generalization of Moreira's result:

Proposition 3.1.6 ([8], Proposition 2.2)**.** *For $n \ge 2$,*

$$\eta_R(u_{n,k}) \equiv \sum_{i+j=k} u_{n,i} t_j^{p^i} - w_{n,k}^p - v_{n-1} w_{n,k+1} \eta_R(v_n^{-1}) \quad \mathit{mod}\ I_{n-1} + (v_{n-1}^p).$$

□

3.2. **Shimomura's element $X_{3,r}$.** The elements $x_{n,r} \in v_n^{-1}BP_*$ ($n \ge 1$ and $r \ge 0$) have been defined in [2, (5.11)] to express a basis of $H^0M_{n-1}^1$ ($x_{3,r}$ was listed in 2.3.1). As is shown in [2, Proposition 5.17], the differentials $d_0 : v_n^{-1}BP_* \to v_n^{-1}BP_* \otimes_A \Gamma$ mod $(p, v_1, v_2^{1+a_{3,r}})$ are given by

$$d_0(x_{3,0}) \equiv v_2 t_1^{p^2} \qquad \text{and} \qquad d_0(x_{3,r}) \equiv v_2^{a_{3,r}} x_{3,r-1}^{p-1} t_1^{p^{\delta(r)}} \quad \text{for} \quad r \ge 1,$$

where $\delta(r) = 0$ for odd r and 1 for even r. However, we need to calculate $d_0(x_{3,r})$ mod (p, v_1, v_2^t) with $t > 1 + a_{3,r}$ for our computation. So we use the following elements $X_{3,r}$ defined by Shimomura [8] instead of $x_{3,r}$.

Definition 3.2.1. cf. [8, (3.3)]] Define elements $X_{3,r} \in v_3^{-1}BP_*$ by

$$
\begin{aligned}
&X_{3,0} = v_3, \qquad X_{3,1} = X_{3,0}^p + v_2^p v_3^p u_{3,1}, \qquad X_{3,2} = X_{3,1}^p - v_2^{p^2+p} X_{3,1}^{p-1} u_{3,2},\\
&X_{3,3} = X_{3,2}^p + v_2^{1+a_{3,3}} X_{3,2}^{p-1} u_{2,1} + v_2^{b(3)} X_{3,1}^{p^2-p-1} u_{3,3},\\
&X_{3,4} = X_{3,3}^p - v_2^{b(4)} X_{3,2}^{p^2-p-1}(u_{2,2} - u_{3,3}),\\
&X_{3,r} = X_{3,r-1}^p + v_2^{1+a_{3,r}} X_{3,r-1}^{p-1} u_{2,1} - v_2^{b(r)} X_{3,r-2}^{p^2-p-1}(u_{2,2} - u_{3,3}) \quad \text{for odd } r \geq 5,\\
&X_{3,r} = X_{3,r-1}^p - v_2^{b(r)} X_{3,r-2}^{p^2-p-1}(2\, u_{2,2} - u_{3,3}) \quad \text{for even } r \geq 6.
\end{aligned}
$$

with $b(0) = 1$, $b(1) = p+1$ and $b(r) = a_{3,r} + a_{3,r-1} + 1 = (p^2+p+1)p^{r-2}$ for $r \geq 2$.

Remark 3.2.2. By definition, it is obvious that $X_{3,r} \equiv x_{3,r} \mod (v_2^{1+a_{3,r}})$. In addition, they satisfy the congruences

$$
X_{3,1} \equiv v_3^p \mod v_2^p, \qquad X_{3,2} \equiv X_{3,1}^p \mod v_2^{p^2+p},
$$

$$
X_{3,2i+1} \equiv X_{3,2i}^p \mod v_2^{a_{3,2i+1}-p}, \qquad X_{3,2i+2} \equiv X_{3,2i+1}^p \mod v_2^{b(2i+2)-p^2-p-1}
$$

for $i \geq 1$.

Then $d_0(X_{3,r})$ is computed as follows. Let s be an integer with $p \nmid s$.

Proposition 3.2.3. *Mod $(p, v_1, v_2^{b(r)+1})$, $d_0(X_{3,r}^s)$ may be expressed as follows. For small values of r,*

$$
d_0(X_{3,r}^s) = \begin{cases} s v_2 v_3^s w_{3,1} \quad (\equiv s v_2 v_3^{s-1} t_1^{p^2}), & \text{for } r = 0\\ s v_2^p v_3^{sp-1} t_1 - s\, v_2^{p+1} v_3^{sp-1} w_{3,2}, & \text{for } r = 1\\ s v_2^{a_{3,2}} X_{3,1}^{sp-1} t_1^p - s v_2^{b(2)} v_3^{sp^2-p-1}(u_{2,0} t_2 - w_{3,3}), & \text{for } r = 2. \end{cases}
$$

For $r = 3$,

$$
d_0(X_{3,r}^s) = s v_2^{a_{3,3}} X_{3,2}^{sp-1} t_1 - s\, v_2^{b(3)} X_{3,1}^{sp^2-p-1}\{w_{2,2} + u_{2,0} t_2 - w_{3,3} + u_{3,0}(t_1^{p^2} t_2 - t_3)\}.
$$

For $r \geq 4$ and even,

$$
d_0(X_{3,r}^s) = s v_2^{a_{3,r}} X_{3,r-1}^{sp-1} t_1^p - s v_2^{b(r)} X_{3,r-2}^{sp^2-p-1}\,(\,\zeta_2 + u_{2,0} t_2 - \zeta_3\,)\,.
$$

For $r \geq 5$ and odd,

$$
d_0(X_{3,r}^s) = s v_2^{a_{3,r}} X_{3,r-1}^{sp-1} t_1 - s v_2^{b(r)} X_{3,r-2}^{sp^2-p-1}(\,\zeta_2 + w_{2,2} - \zeta_3\,).
$$

Proof. Originally, this is proved in [8, Proposition 3.1]. Notice that it is sufficient to prove the $s = 1$ case because $d_0(X_{3,r}^s) \equiv s X_{3,r}^{s-1} d_0(X_{3,r})$.

For $r = 0$, $d_0(X_{3,0}) \equiv v_2 t_1^{p^2} \mod p, v_1, v_2^2$ by (2.1.2). For $r = 1$, Proposition 3.1.6 shows that

$$
d_0(v_2^p v_3^p u_{3,1}) \equiv v_2^p v_3^{p-1} t_1 - v_2^p t_1^{p^2} - v_2^{p+1} v_3^{p-1} w_{3,2}
$$

mod (p, v_1, v_2^{p+2}). Summing this with $d_0(X_{3,0}^p) \equiv v_2^p t_1^{p^3}$ gives $d_0(X_{3,1})$. For $r = 2$, we have

$$
d_0(-v_2^{p^2+p} v_3^{p^2-p} u_{3,2}) \quad \equiv \quad -v_2^{p^2+p} v_3^{p^2-p}(u_{3,1} t_1^p + u_{3,0} t_2 - w_{3,2}^p - v_2 v_3^{-1} w_{3,3})
$$

mod $(p, v_1, v_2^{p^2+p+2})$. On the other hand, using the congruence $v_3^{p^2-p} \equiv X_{3,1}^{p-1} + v_2^p v_3^{p^2-p} u_{3,1}$ we have

$$d_0(X_{3,1}^p) \equiv v_2^{p^2} X_{3,1}^{p-1} t_1^p + v_2^{p^2+p} v_3^{p^2-p}(u_{3,1}t_1^p - w_{3,2}^p).$$

Again, summing the terms gives $d_0(X_{3,2})$.

For $r=3$, we have

$$\begin{aligned} d_0(v_2^{p^3+p} X_{3,2}^{p-1} u_{2,1}) &\equiv v_2^{1+a_{3,3}} X_{3,2}^{p-1}(u_{2,0}t_1 - w_{2,1}^p) \\ &\quad - v_2^{b(3)} X_{3,1}^{p^2-p-1}(e_2(u_{2,1})t_1^p + u_{2,0}t_2 + u_{3,0}t_1^{p^2}t_2 - w_{3,3}), \\ d_0(v_2^{b(3)} X_{3,1}^{p^2-p-1} u_{3,3}) &\equiv v_2^{b(3)} X_{3,1}^{p^2-p-1}(\sum_{i=0}^{2} u_{3,i} t_{3-i}^{p^i} - w_{3,3}^p). \end{aligned}$$

mod $(p, v_1, v_2^{b(3)+1})$. On the other hand, by the congruences $X_{3,1}^{p^2-p} \equiv X_{3,2}^{p-1} - v_2^{p^2+p} X_{3,1}^{p^2-p-1} u_{3,2}$ and $v_3^{p^3-p^2-p} \equiv X_{3,1}^{p^2-p-1} + v_2^p X_{3,1}^{p^2-p-1} u_{3,1}$, we have

$$d_0(X_{3,2}^p) \equiv v_2^{p^3} X_{3,2}^{p-1} t_1^{p^2} - v_2^{b(3)} X_{3,1}^{p^2-p-1}(u_{2,0}^p t_2^p - w_{3,3}^p + \sum_{i=1}^{2} u_{3,i} t_{3-i}^{p^i}).$$

Using a congruence for $w_{2,2}$ of Remark 3.1.5, we obtain $d_0(X_{3,3})$.

For $r=4$, we have

$$\begin{aligned} &d_0(-v_2^{b(4)} X_{3,2}^{p^2-p-1} u_{2,2}) \equiv \\ &\quad - v_2^{b(4)} X_{3,2}^{p^2-p-1}(-\zeta_2^p - u_{2,0}^p t_2^p + 2\zeta_2 - w_{3,3} - \sum_{1\le i<j\le 3} u_{3,3-j}^{p^{j-i-1}} (t_i \cdot c(t_{j-i}^{p^i}))^{p^{2-i}}) \end{aligned}$$

and $d_0(v_2^{b(4)} X_{3,2}^{p^2-p-1} u_{3,3}) \equiv v_2^{b(4)} X_{3,2}^{p^2-p-1}(\zeta_3 - \zeta_3^p)$ mod $(p, v_1, v_2^{b(4)+1})$. On the other hand, by the congruences $X_{3,2}^{p^2-p} \equiv X_{3,3}^{p-1} + v_2^{p^3+p} X_{3,2}^{p^2-p-1} u_{2,1}$ and $X_{3,1}^{p^3-p^2-p} \equiv X_{3,2}^{p^2-p-1} - v_2^{p^2+p} X_{3,1}^{p^3-p^2-p-1} u_{3,2}$, $d_0(X_{3,3}^p)$ is congruent to

$$v_2^{a_{3,4}} X_{3,3}^{p-1} t_1^p - v_2^{b(4)} X_{3,2}^{p^2-p-1}\{-u_{2,1}t_1^p + u_{3,2}t_1^{p^2} + w_{2,2}^p + u_{2,0}^p t_2^p - w_{3,3}^p + u_{3,0}^p(t_1^{p^3} t_2^p - t_3^p)\}$$

mod $(p, v_1, v_2^{b(4)+1})$. Moreover, noticing the congruences

$$\begin{aligned} -w_{3,3}^p &\equiv \zeta_3 - \zeta_3^p - \sum_{i=0}^{2} u_{3,i} t_{3-i}^{p^i}, \\ -(u_{3,1}t_2^p + u_{3,0}t_3) + u_{3,0}^p t_1^{p^3} t_2^p &\equiv - \sum_{1\le i<j\le 3} u_{3,3-j}^{p^{j-i-1}} [t_{i+1} \cdot c(t_{j-i-1}^{p^{i+1}})]^{p^{2-i}}, \\ -u_{3,0}^p t_3^p &\equiv - \sum_{1\le i<j\le 3} u_{3,3-j}^{p^{j-i-1}} [t_{i+2} \cdot c(t_{j-i-2}^{p^{i+2}})]^{p^{2-i}}, \end{aligned}$$

we have

$$\begin{aligned} d_0(X_{3,3}^p) &\equiv v_2^{a_{3,4}} X_{3,3}^{p-1} t_1^p - v_2^{b(4)} X_{3,2}^{p^2-p-1}\left(-u_{2,1}t_1^p + w_{2,2}^p + u_{2,0}^p t_2^p\right) \\ &\quad - v_2^{b(4)} X_{3,2}^{p^2-p-1}(\zeta_3 - \zeta_3^p - \sum_{1\le i<j\le 3} u_{3,3-j}^{p^{j-i-1}} [\sum_{i+1\le k\le j} t_k \cdot c(t_{j-k}^{p^k})]^{p^{2-i}}). \end{aligned}$$

We obtain $d_0(X_{3,4})$ in the desired expression by summing the terms and using the congruences

$$\begin{aligned} -u_{2,1}t_1^p + w_{2,2}^p - \zeta_2^p &\equiv -\zeta_2 + u_{2,0}t_2, \\ \sum_{1\le i<j\le 3} u_{3,3-j}^{p^{j-i-1}} \big(\sum_{i\le k\le j} t_k \cdot c(t_{j-k}^{p^k})^{p^{2-i}} \big) &\equiv \zeta_3 - w_{3,3}. \end{aligned}$$

The $r \geq 5$ cases are proved by induction on r. By Proposition 3.1.3, we have congruences

$$\begin{aligned} d_0(v_2^{b(r)} X_{3,r-2}^{p^2-p-1} u_{2,2}) &\equiv v_2^{b(r)} X_{3,r-2}^{p^2-p-1}(\zeta_2 - \zeta_2^p), \\ d_0(v_2^{b(r)} X_{3,r-2}^{p^2-p-1} u_{3,3}) &\equiv v_2^{b(r)} X_{3,r-2}^{p^2-p-1}(\zeta_3 - \zeta_3^p). \end{aligned}$$

We also have the congruences $X_{3,r-2}^{(p-1)p} \equiv X_{3,r-1}^{p-1}$ and

$$d_0(v_2^{1+a_{3,r}} X_{3,r-1}^{p-1} u_{2,1}) \equiv v_2^{1+a_{3,r}} X_{3,r-1}^{p-1}(u_{2,0}t_1 - w_{2,1}^p) - v_2^{b(r)} X_{3,r-2}^{p^2-p-1} t_1 \cdot e_2(u_{2,1})$$

for odd r, and $X_{3,r-2}^{(p-1)p} \equiv X_{3,r-1}^{p-1} + v_2^{1+a_{3,r-1}} X_{3,r-2}^{p^2-p-1} u_{2,1}$ for even r. Easy calculation shows the desired results for $r \geq 5$. □

Moreover, we can obtain the following result in the same way as the proof of Proposition 3.2.3.

Proposition 3.2.4. *Mod $(p, v_1^2, v_2^{b(r)-1})$, $d_0(X_{3,r}^s)$ is expressed as follows. For small values of r, we get*

$s v_2^{a_{3,2}} X_{3,1}^{sp-1} t_1^p, \qquad r = 2;$

$s v_2^{a_{3,3}} X_{3,2}^{sp-1}(t_1 - v_1 w_{2,2}) - s v_2^{b(3)-1} X_{3,1}^{sp^2-p-1} \left\{ v_2\zeta_2 - v_1 t_1^p (w_{2,2} + v_2^{-1} t_2) \right\}, \quad r = 3;$

$s v_2^{a_{3,4}} X_{3,3}^{sp-1} t_1^p - s v_2^{b(4)-1} X_{3,2}^{sp^2-p-1} \left\{ v_2\zeta_2 - v_1(w_{2,3} + t_1^p \zeta_2) \right\}, \qquad r = 4.$

For $r \geq 5$ and odd, we get

$$s v_2^{a_{3,r}} X_{3,r-1}^{sp-1}(t_1 - v_1 w_{2,2}) - s v_2^{b(r)-1} X_{3,r-2}^{sp^2-p-1} \left\{ 2\, v_2\zeta_2 - v_1(w_{2,3} + t_1^p w_{2,2}) \right\}.$$

For $r \geq 6$ and even, we get

$$s\, v_2^{a_{3,r}} X_{3,r-1}^{sp-1} t_1^p - s\, v_2^{b(r)-1} X_{3,r-2}^{sp^2-p-1} \left\{ v_2\zeta_2 - 2\, v_1 w_{2,3} \right\}.$$

3.3. **The element $1/X_{2,r}^m$.** Here we modify $x_{2,r}$ into $X_{2,r}$ in the analogous way of modifying $x_{3,r}$ into $X_{3,r}$, and introduce new elements $1/X_{2,r}^m$.

Definition 3.3.1. Define integers $a_{2,r}$ by $a_{2,0} = 1$ and $a_{2,r} = p^r + p^{r-1} - 1$ for $r \geq 1$, and elements $X_{2,r} \in v_2^{-1}BP_*$ by

$$\begin{aligned} &X_{2,0} = v_2, \quad X_{2,1} = X_{2,0}^p + v_1^p v_2^p u_{2,1}, \quad X_{2,2} = X_{2,1}^p + v_1^{1+a_{2,2}} X_{2,1}^{p-1}(u_{1,1} - u_{2,2}), \\ &X_{2,r} = X_{2,r-1}^p + v_1^{1+a_{2,r}} X_{2,r-1}^{p-1}(2\, u_{1,1} - u_{2,2}) \quad \text{for } r \geq 3. \end{aligned}$$

Then we have:

Proposition 3.3.2. *Mod $(p, v_1^{2+a_{2,r}})$,*

$$d_0(X_{2,r}^m) \equiv \begin{cases} m v_1 v_2^{m-1} t_1^p + \binom{m}{2} v_1^2 v_2^{m-2} t_1^{2p} & \text{for } r = 0 \\ m v_1^p v_2^{mp-1}(t_1 - v_1 w_{2,2}) & \text{for } r = 1 \\ v_1^{a_{2,r}} v_2^{(mp-1)p^{r-1}}(2\, t_1 - v_1\zeta_2) & \text{for } r \geq 2. \end{cases}$$

Proof. It is sufficient to prove the $s = 1$ case because $d_0(X_{2,r}^m) \equiv mv_2^{(m-1)p^r} d_0(X_{2,r})$.

The $r = 0$ case is given by (2.1.1). For $r = 1$, we have

$$\begin{aligned} d_0(v_1^p v_2^p u_{2,1}) &\equiv v_1^p \{ d_0(u_{2,1}) v_2^p + \eta_R(u_{2,1}) d_0(v_2^p) \} \\ &\equiv v_1^p v_2^p d_0(u_{2,1}) \\ &\equiv v_1^p (v_2^{p-1} t_1 - t_1^{p^2}) - v_1^{p+1} v_2^{p-1} w_{2,2} \end{aligned}$$

mod p, v_1^{p+2}. Summing this with $d_0(X_{2,0}^p)$ gives $d_0(X_{2,1}) \equiv v_1^p v_2^{p-1}(t_1 - v_1 w_{2,2})$. (See also [2, Proposition 5.4].) For $r = 2$, we have

$$d_0(v_1^{1+a_{2,r}} X_{2,1}^{p-1}(u_{1,1} - u_{2,2})) \equiv -v_1^{p^2} X_{2,1}^{p-1} \{ (2v_1^{p-1} t_1 - t_1^p) + v_1^p (w_{2,2}^p - \zeta_2) \}$$

mod $p, v_1^{2+a_{2,2}}$. Again, summing this with $d_0(X_{2,1}^p)$ gives

$$d_0(X_{2,2}) \equiv v_1^{a_{2,2}} v_2^{p^2-p}(2t_1 - v_1\zeta_2).$$

For $r \geq 3$, we have

$$d_0(v_1^{1+a_{2,r}} X_{2,r-1}^{p-1}(2u_{1,1} - u_{2,2})) \equiv v_1^{a_{2,r}-p+1} X_{2,r-1}^{p-1} \{ 2(v_1^{p-1} t_1 - t_1^p) + v_1^p (\zeta_2^p - \zeta_2) \}$$

mod $p, v_1^{2+a_{2,r}}$. This gives $d_0(X_{2,r})$ inductively on r. □

Shimomura [7] has replaced $1/v_1^{mp}$ with $1/x_{1,1}^m$ in computing $H^0M_0^2$ for $p = 2$. Analogously, we construct elements $1/X_{2,r}^m$ as a substitute for $1/v_2^{mp^r}$. Notice that our $1/X_{2,r}^m$ should be different from $X_{2,r}^{-m}$ because there is no inverse element of $X_{2,r}$ $(r \geq 1)$ in $v_2^{-1}BP_*$.

Definition 3.3.3. Define elements $1/X_{2,r}^m \in v_2^{-1}BP_*$ $(r \geq 0$ and $p \nmid m \geq 1)$ by

$$1/X_{2,0}^m = 1/v_2^m \quad \text{and} \quad 1/X_{2,r}^m = 2\,(1/X_{2,r-1}^{mp}) - X_{2,r}^m\,(1/X_{2,r-1}^{2mp}) \qquad \text{for } r \geq 1.$$

For example, easy calculation shows the congruence

$$1/X_{2,1}^m \equiv 1/v_2^{mp} + m\, v_1^p v_3 / v_2^{(m+1)p+1} \mod (v_1^{2p}).$$

Although our $1/X_{2,r}^m$ is not equal to $(X_{2,r}^m)^{-1} = \sum_{k=0}^{\infty}(1 - X_{2,r}^m)^k$, the above definition is justified by the congruences $(X_{2,1}^m - v_2^{mp})^2 \equiv 0 \bmod (p, v_1^{2p})$ and $(X_{2,r}^m - X_{2,r-1}^{mp})^2 \equiv 0 \bmod (p, v_1^{2(a_{2,r}-p)})$ for $r \geq 2$. Hereafter we denote $Y(1/X_{2,r}^m)$ by $Y/X_{2,r}^m$ and $(1/X_{2,r}^m)/v_1^l$ by $1/v_1^l X_{2,r}^m$ for simplicity.

Proposition 3.3.4. *Mod* $(p, v_1^{2+a_{2,r}})$,

$$d_0(1/X_{2,r}^m) \equiv \begin{cases} -\, m\; v_1 t_1^p / v_2^{m+1} + \binom{m+1}{2} v_1^2 t_1^{2p} / v_2^{m+2} & \textit{for } r = 0 \\ -\, m\; v_1^p (t_1 - v_1 w_{2,2}) / v_2^{mp+1} & \textit{for } r = 1 \\ -\, m\; v_1^{a_{2,r}} (2\,t_1 - v_1 \zeta_2) / v_2^{(mp+1)p^{r-1}} & \textit{for } r \geq 2. \end{cases}$$

Proof. We prove that the all above differentials have the form $-\,d_0(X_{2,r}^m)/v_2^{2mp^r}$.

The case $r=0$ is easy. For $r \geq 1$, it is shown by induction on r. In fact, we have

$$\begin{aligned} d_0(1/X_{2,r}^m) &= 2\, d_0(1/X_{2,r-1}^{mp}) - d_0(X_{2,r}^m)/X_{2,r-1}^{2mp} - \eta_R(X_{2,r}^m) d_0(1/X_{2,r-1}^{2mp}) \\ &\equiv 2\, d_0(1/X_{2,r-1}^m)^p - d_0(X_{2,r}^m)/v_2^{2mp^r} - v_2^{mp^r} d_0(1/X_{2,r-1}^{2m})^p \\ &\equiv -2\, d_0(X_{2,r-1}^m)^p/v_2^{2mp^r} - d_0(X_{2,r}^m)/v_2^{2mp^r} + d_0(X_{2,r-1}^{2m})^p/v_2^{3mp^r} \\ &\equiv -d_0(X_{2,r}^m)/v_2^{2mp^r} + \{\, d_0(X_{2,r-1}^{2m}) - 2\, v_2^{mp^{r-1}} d_0(X_{2,r-1}^m)\,\}^p/v_2^{3mp^r} \end{aligned}$$

mod $p, v_1^{2+a_{2,r}}$. By the congruence $d_0(X_{2,r}^m) \equiv m v_2^{(m-1)p^r} d_0(X_{2,r})$, the second term is trivial. □

Then we directly obtain the v_1-divisibilities of $1/v_1 X_{2,r}^m$ $(= 1/v_1 v_2^{mp^r})$ in $H^0 M_1^2$.

Corollary 3.3.5. *$1/v_1 X_{2,r}^m$ can be divided by $v_1^{a_{2,r}-1}$ in $H^0 M_1^2$, and the image of $1/v_1^{a_{2,r}} X_{2,r}^m$ under the coboundary map $\delta : H^0 M_1^2 \to H^1 M_2^1$ is*

$$\delta(1/v_1^{a_{2,r}} X_{2,r}^m) = \begin{cases} -\, m\ h_1/v_2^{m+1} & \text{for } r=0 \\ -\, m\ h_0/v_2^{mp+1} & \text{for } r=1 \\ -\, 2m\ h_0/v_2^{(mp+1)p^{r-1}} & \text{for } r \geq 2. \end{cases}$$

□

This corollary asserts that $N(m;r) = a_{2,r}$ when we choose $1/X_{2,r}^m$ to be $y(mp^r)$ in Proposition 2.4.1.

Using 2.5, we easily see that $\{\, \delta(1/v_1^{a_{2,r}} X_{2,r}^m)\,\}$ is linearly independent.

4. Preliminary calculations

In Section 5 we shall construct elements $x(sp^r/j;k) \in v_3^{-1}BP_*/(p, v_2^\infty)$ and compute the differentials on them. For the sake of this, we define some elements and compute differentials in Subsection 4.1. Based on these results, we describe some inductive methods of constructing the elements $x(sp^r/j;k)$ in Subsection 4.2. This is the hardest part in this paper.

Hereafter we assume that $p > 3$, and that all elements are either in $v_3^{-1}BP_*/(p)$ or in $v_3^{-1}BP_*/(p, v_2^\infty)$. We shall compute the differential d_0 for positive integers e_1 and e_2 in two ways:

$$\begin{aligned} &d_0 : v_3^{-1}BP_*/(p) \longrightarrow v_3^{-1}BP_*/(p) \otimes_A \Gamma && \bmod v_1^{e_1}, v_2^{e_2}, \\ &d_0 : v_3^{-1}BP_*/(p, v_2^\infty) \longrightarrow v_3^{-1}BP_*/(p, v_2^\infty) \otimes_A \Gamma && \bmod v_1^{e_1}. \end{aligned}$$

4.1. Some lemmas.

Lemma 4.1.1. *Assume that an element X satisfies*

$$d_0(X) \equiv v_1^{e_1} X_{3,r}^s t_1^p / v_2^{e_2+1} \qquad \bmod v_1^{e_1+1},$$

where $r \geq 1$, $e_1 \geq 1$ and $1 \leq e_2 \leq a_{3,r} - 2$. Then $Y = X^p - v_1^{pe_1} v_3 X_{3,r}^{sp}/v_2^{(e_2+1)p+1}$ satisfies

$$d_0(Y) \equiv v_1^{pe_1} X_{3,r}^{sp}(t_1 - v_1 w_{2,2})/v_2^{pe_2+1} \qquad \bmod v_1^{pe_1+2}.$$

Proof. We observe that

$$\begin{aligned} d_0(v_3X_{3,r}^{sp}/v_2^{(e_2+1)p+1}) &= d_0(1/v_2^{(e_2+1)p+1})\, v_3X_{3,r}^{sp} + \eta_R(1/v_2^{(e_2+1)p+1})\, d_0(v_3X_{3,r}^{sp}) \\ &\equiv -\, v_1v_3X_{3,r}^{sp}t_1^p/v_2^{(e_2+1)p+2} \\ &\qquad + (1/v_2^{(e_2+1)p+1} - v_1t_1^p/v_2^{(e_2+1)p+2})\, d_0(v_3X_{3,r}^{sp}) \end{aligned}$$

mod p, v_1^2. Since $(e_2+1)p+2 < pa_{3,r}$, it is sufficient to compute $d_0(v_3X_{3,r}^{sp})$ mod $v_1^2, v_2^{pa_{3,r}}$. Using (2.1.2) and Proposition 3.2.3, we have

$$\begin{aligned} d_0(v_3X_{3,r}^{sp}) &= d_0(v_3)\, X_{3,r}^{sp} + \eta_R(v_3)\, d_0(X_{3,r}^{sp}) \equiv d_0(v_3)\, X_{3,r}^{sp} \\ &\equiv (v_2t_1^{p^2} - v_2^pt_1 + v_1t_2^p)X_{3,r}^{sp}. \end{aligned}$$

Observing the congruence $w_{2,2} \equiv -\, v_3t_1^p/v_2^{p+1} + (t_2^p - t_1^{p^2+p})/v_2^p + t_1^{p+1}/v_2$ (Remark 3.1.5), we obtain

$$d_0(v_3X_{3,r}^{sp}/v_2^{(e_2+1)p+1}) \equiv X_{3,r}^{sp}t_1^{p^2}/v_2^{(e_2+1)p} - X_{3,r}^{sp}t_1/v_2^{pe_2+1} + v_1X_{3,r}^{sp}w_{2,2}/v_2^{pe_2+1}.$$

Now the result follows easy calculation. □

Given a positive integer $j = cp - b \le a_{3,r}$ $(0 \le b < p)$, it is convenient to work with the element $v_2^bX_{3,r-1}^{sp}/X_{2,1}^c$ instead of $X_{3,r}^s/v_2^j$. Define the integer a by $a \equiv c$ (p) with $0 \le a < p$.

Lemma 4.1.2. *Assume that $c \le p$ for $r=2$, that $c \le a_{3,r-1} - 2$ for odd $r \ge 3$, or that $c \le a_{3,r-1} - p - 1$ for even $r \ge 4$. Then $d_0(v_2^bX_{3,r-1}^{sp}/X_{2,1}^c)$ is expressed as*

$$\begin{aligned} &\frac{X_{3,r-1}^{sp}}{v_2^{j+b}}\{\sum_{i=1}^{b}E(b,i)\} \\ &\qquad - \frac{v_1^pX_{3,r-1}^{sp}}{v_2^{j+1}}\{(a+b)\, t_1 - v_1[a\, w_{2,2} + ab\, \frac{v_3t_1^p}{v_2^{p+1}} - b(a+b-1)\, \frac{t_1^{p+1}}{v_2}]\} \end{aligned}$$

mod v_1^{p+2}, where $E(m,n) = \binom{m}{n}v_2^{m-n}(v_1t_1^p)^n$. In particular,

$$d_0(X_{3,r-1}^{sp}/X_{2,1}^c) \equiv -\, a\, v_1^pX_{3,r-1}^{sp}(t_1 - v_1w_{2,2})/v_2^{j+1} \quad \text{mod } (v_1^{p+2}).$$

Proof. Notice that

$$d_0(v_2^bX_{3,r-1}^{sp}/X_{2,1}^c) = d_0(v_2^b)\, X_{3,r-1}^{sp}/X_{2,1}^c + \eta_R(v_2^b)\, d_0(X_{3,r-1}^{sp}/X_{2,1}^c),$$

and that $\eta_R(v_2^b)$ is given in (2.1.1). By the assumption on c, we have

$$d_0(X_{3,r-1}^s/v_2^c) \equiv -\, a\, v_1X_{3,r-1}^st_1^p/v_2^{c+1} \quad \text{mod } (v_1^2).$$

Then, Lemma 4.1.1 gives $d_0(X_{3,r-1}^{sp}/X_{2,1}^c)$ mod (v_1^{p+2}). □

Similarly to the $r \ge 2$ case, we may consider the element $X_{3,1}^s/X_{2,1}$ instead of $X_{3,1}^s/v_2^p$ for $r=1$.

Lemma 4.1.3.

$$d_0\left(X_{3,1}^s/X_{2,1}\right) \equiv (s-1)\, v_1^pX_{3,1}^st_1/v_2^{p+1} + \binom{s}{2}v_1^pv_3^{sp-1}t_1^2/v_2 \quad \text{mod } (v_1^{p+1}).$$

Proof. Notice that $d_0(X^s_{3,1}/X_{2,1}) = d_0\,(1/X_{2,1})\,X^s_{3,1} + \eta_R\,(1/X_{2,1})\,d_0(X^s_{3,1})$, and that $d_0\,(1/X_{2,1})\,X^s_{3,1} \equiv -\,v_1^p X^s_{3,1} t_1/v_2^{p+1} \bmod (v_1^{p+1})$ by Proposition 3.3.4. On the other hand, we have

$$\begin{aligned}
\eta_R\,(1/X_{2,1}) &\equiv 1/v_2^p + v_1^p(v_3 - v_2^p t_1)/v_2^{2p+1} && \bmod v_1^{p+1},\\
d_0(X^s_{3,1}) &\equiv s\,v_1^p X^{s-1}_{3,1}\left\{t_2^{p^2} - t_1^{p^2}\eta_R(v_3^{-1}v_4)\right\} && \bmod v_1^{p+1}, v_2^p,\\
&\equiv s\,v_2^p X^{s-1}_{3,1}\left\{t_1^{p^3} - d_0(v_3^{-1}v_4)\right\} + \tbinom{s}{2} v_2^{2p} v_3^{sp-2} t_1^2 && \bmod v_1, v_2^{2p+1}.
\end{aligned}$$

Using these congruences, (2.1.2) and (2.1.3), we obtain

$$\eta_R\,(1/X_{2,1})\,d_0\left(X^s_{3,1}\right) \equiv s\,v_1^p X^s_{3,1} t_1/v_2^{p+1} + \tbinom{s}{2} v_1^p v_3^{sp-1} t_1^2/v_2 \qquad \bmod v_1^{p+1}.$$

Collecting terms gives the result. □

For an integer n with $p \mid n+1$, we denote the integer $(n+1)/p$ by n', and the set of integers $\{s \in \mathbb{Z} \mid s = s'p - 1 \text{ with } p \nmid s'\}$ by $\mathbb{N}_0$. Then we introduce two elements $X_0(v_2^j, X^s_{3,r})$ and $X(v_2^j, X^s_{3,r})$ for $s \in \mathbb{N}_0$, each of which is congruent to $X^s_{3,r}/v_2^j \bmod (v_1)$.

Definition 4.1.4. When $s = s'p - 1 \in \mathbb{N}_0$, we define $X_0(v_2^j, X^s_{3,r})$ by

$$X_0(v_2^j, X^s_{3,r}) = X^s_{3,r}/v_2^j + j\,v_1 X^{s'}_{3,r+1}/s' v_2^{j+1+a_{3,r+1}}.$$

Then we have:

Lemma 4.1.5. *For odd r and $p \nmid j \le a_{3,r}$, $d_0(X_0(v_2^j, X^s_{3,r}))$ is expressed as*

$$(1-j)\,v_1 v_2^{a_{3,r}-j} v_3^{(sp-1)p^{r-1}} \zeta_2 \quad \bmod v_1^2$$

for $r \ge 3$ and $a_{3,r} - p \le j \le a_{3,r}$, and is expressed as

$$-\,j(j+1)\,v_1^2 X^{s'}_{3,r+1} t_1^p/s' v_2^{j+2+a_{3,r+1}} - \binom{j+1}{2} v_1^2 X^s_{3,r} t_1^{2p}/v_2^{j+2} + \varepsilon(r,j) \quad \bmod v_1^3$$

for $r = 1$ and $j \le p-2$; or for $r \ge 3$ and either $j = a_{3,r} - p + 2$ or $j \le a_{3,r} - p - 1$, where $\varepsilon(1,j) = 0$ and $\varepsilon(r,j) = v_1^2 v_2^{a_{3,r}-j-1} X^{sp-1}_{3,r-1}\,\{\,a(r)j\,w_{2,3} + b(r,j)\,t_1^p\zeta_2\,\}$ for $r \ge 3$ with integers $a(3) = 1$, $a(r) = 2$ for $r \ge 5$ and $b(r,j) = j^2 + (2 - a(r))\,j - 1$.

Proof. Here we prove only for $r \ge 3$ case (the $r = 1$ case is easier). We compute the differential on each term of $X_0(v_2^j, X^s_{3,r})$. For the first term, notice that $d_0(X^s_{3,r}/v_2^j) = d_0(1/v_2^j)X^s_{3,r} + \eta_R(1/v_2^j)d_0(X^s_{3,r})$. Easy calculation shows that

$$d_0(X^s_{3,r}) \equiv -v_2^{a_{3,r}} X^{sp-1}_{3,r-1}(t_1 - v_1 w_{2,2} + v_1^2 v_2^{-1} t_1^p w_{2,2}) \quad \bmod v_1^3, v_2^{2+a_{3,r}}.$$

Using this and Proposition 3.3.4, we see that $d_0(X^s_{3,r}/v_2^j)$ is congruent to

$$-j\,v_1 X^s_{3,r} t_1^p\,\{2\,v_2 - (j+1)\,v_1 t_1^p\}\,/2v_2^{j+2} + v_1 v_2^{a_{3,r}-j-1} X^{sp-1}_{3,r-1}\zeta_2\,\{v_2 - (j+1)\,v_1 t_1^p\}$$

mod (v_1^3). On the other hand, using Propositions 3.2.4 and 3.3.4, we observe that $d_0(j\,v_1 X^{s'}_{3,r+1}/s' v_2^{j+1+a_{3,r+1}})$ is congruent to

$$\begin{aligned}
&-\,j(j+1)v_1^2 X^{s'}_{3,r+1} t_1^p/s' v_2^{j+2+a_{3,r+1}} + j\,v_1 X^s_{3,r} t_1^p\,\{\,v_2 - (j+1)\,v_1 t_1^p\,\}\,/v_2^{j+2}\\
&\qquad +\; v_1 v_2^{a_{3,r}-j-1} X^{sp-1}_{3,r-1}\,\{-j\,v_2\zeta_2 + a(r)j\,v_1 w_{2,3} + j(\,j + 3 - a(r)\,)\,v_1 t_1^p \zeta_2\}
\end{aligned}$$

mod (v_1^3). Collecting two terms gives the result for $r \ge 3$. □

Definition 4.1.6. For $s=s'p-1\in\mathbb{N}_0$ and $j\le a_{3,r}$, we define $X(v_2^j,X_{3,r}^s)$ by

$$\begin{aligned}X(v_2^j,X_{3,r}^s)&=v_2^bX_{3,r-1}^{sp}/X_{2,1}^c-b\,v_1v_2^{b-1}X_{3,r}^{s'p}/s'X_{2,1}^{c+a_{3,r}}+b\,v_1^{p+1}X_{3,r-1}^{sp}u_{2,2}/v_2^{j+1}\\&\quad-\ (a(r)-1)b\,v_1^{p+1}v_2^{a_{3,r}-j-p}X_{3,r-2}^{(sp-1)p}u_{2,3},\end{aligned}$$

where $j=cp-b$ and $0\le b<p$. Moreover, we define $DX_{[m,n]}(v_1^{e_1},v_2^{e_2+1},X_{3,r}^s)\in v_3^{-1}BP_*/(p,v_2^\infty)\otimes_A\Gamma$ for $0\le m<p$, $0\le n<p$, $s=s'p-1\in\mathbb{N}_0$, and positive integers e_1 and e_2 by

$$\begin{aligned}DX_{[m,n]}(v_1^{e_1},v_2^{e_2+1},X_{3,r}^s)&=-(m+n)\,v_1^{e_1-1}X_{3,r}^st_1/v_2^{e_2+1}\\&\quad+n(m+n-2)v_1^{e_1}X_{3,r+1}^{s'}t_1/s'v_2^{e_2+2+a_{3,r+1}}\\&\quad+v_1^{e_1}X_{3,r}^s/v_2^{e_2+1}\cdot\{n\zeta_2+(m-n)w_{2,2}\}\end{aligned}$$

Then we obtain the following lemma.

Lemma 4.1.7. *Assume that $r\ge3$ is odd and $c\le a_{3,r-1}-2$. Then $d_0(X(v_2^j,X_{3,r}^s))$ is expressed as*

$$\begin{aligned}&DX_{[a,b]}(v_1^{p+1},v_2^{j+1},X_{3,r}^s)+\ A(v_2^j,X_{3,r}^s)\\&\qquad+\ a(r)b\,v_1v_2^{a_{3,r}-j-p-1}v_3^{(sp-1)p^{r-1}}(v_2^{p+2-b}\zeta_2^p\cdot\eta_R(v_2^{b-1})+v_1^p\theta_1(v_2^j))\end{aligned}$$

mod v_1^{p+2}. Here b and c are the integers in 4.1.6, $a\equiv c\ (p)$ with $0\le a<p$, and the element $A(v_2^j,X_{3,r}^s)$ is defined to be 0 for $0\le b\le1$ and to be

$$\frac{X_{3,r-1}^{sp}}{v_2^{j+b}}\{\sum_{i=2}^{b}(1-i)E(b,i)\}-b\frac{v_1X_{3,r}^{s'p}}{s'v_2^{j+b+a_{3,r+1}}}\{\sum_{i=1}^{b-1}E(b-1,i)\}$$

for $b\ge2$. Here $E(m,n)$ is as in Lemma 4.1.2 and

$$\theta_1(v_2^j)=(a-1)(v_3-v_2^pt_1)\zeta_2^p+\{(a+b-2)\,v_2^pt_1-v_2t_1^{p^2}\}u_{2,2}.$$

In particular, $v_2^{p^2-1}\theta_1(v_2^{a_{3,r}-p^2-p})=-\,v_3^{p+1}(t_1+v_2w_{3,2})/v_2^2$.

Proof. The differential on the first term of $X(v_2^j,X_{3,r}^s)$ has already been computed in Lemma 4.1.2. For the second term, notice that $-bv_1v_2^{b-1}X_{3,r}^{s'p}/s'X_{2,1}^{c+a_{3,r}}$ is congruent to

$$-bv_1X_{3,r}^{s'p}/s'v_2^{j+1+a_{3,r+1}}-b(a-1)v_1^{p+1}v_3X_{3,r}^{s'p}/s'v_2^{j+p+2+a_{3,r+1}}$$

mod (v_1^{p+2}). Using Propositions 3.2.3, 3.2.4 and 3.3.4, we obtain

$$\begin{aligned}&d_0(-bv_1v_2^{b-1}X_{3,r}^{s'p}/s'X_{2,1}^{c+a_{3,r}})\\&\quad\equiv-bv_1X_{3,r}^{s'p}/s'v_2^{j+b+a_{3,r+1}}\{(v_2+v_1t_1^p)^{b-1}-v_2^{b-1}\}\\&\qquad-bv_1X_{3,r-1}^{sp}t_1^p(v_2+v_1t_1^p)^{b-1}/v_2^{j+b}\\&\qquad+a(r)bv_1v_2^{a_{3,r}-j-b+1}X_{3,r-2}^{(sp-1)p}\zeta_2^p\cdot\eta_R(v_2^{b-1})\\&\qquad+(a+b-2)bv_1^{p+1}X_{3,r}^{s'p}t_1/s'v_2^{j+2+a_{3,r+1}}\\&\qquad+bv_1^{p+1}X_{3,r-1}^{sp}/v_2^{j+1}\\&\qquad\qquad\cdot\{w_{2,2}^p+(b-1)t_1^{p+1}/v_2+(a-1)(t_1^{p^2+p}-t_2^p)/v_2^p+(a-1)w_{2,2}\}\end{aligned}$$

$$
\begin{aligned}
&+ bv_1^{p+1}v_2^{a_{3,r}-j-p-1}X_{3,r-2}^{(sp-1)p}\\
&\qquad \cdot \{-v_2t_1^{p^2}\zeta_2^p + a(r)(a-1)\zeta_2^p(v_3 - v_2^pt_1) - (a(r)-1)v_2w_{2,3}^p\}
\end{aligned}
$$

mod (v_1^{p+2}). For the third term, notice that

$$d_0(bv_1^{p+1}X_{3,r-1}^{sp}u_{2,2}/v_2^{j+1}) \equiv bv_1^{p+1}d_0(X_{3,r-1}^{sp}u_{2,2})/v_2^{j+1}$$

mod (v_1^{p+2}). Propositions 3.1.3 and 3.2.3 give $d_0(X_{3,r-1}^{sp}u_{2,2})$, and we obtain

$$
\begin{aligned}
&d_0(bv_1^{p+1}X_{3,r-1}^{sp}u_{2,2}/v_2^{j+1})\\
&\qquad \equiv bv_1^{p+1}X_{3,r-1}^{sp}(\zeta_2 - \zeta_2^p)/v_2^{j+1} - bv_1^{p+1}v_2^{a_{3,r}-j-p}X_{3,r-2}^{(sp-1)p}t_1^{p^2}(u_{2,2} - \zeta_2^p)
\end{aligned}
$$

mod (v_1^{p+2}). For the last term, we may assume that $r \geq 5$ because $a(3) - 1 = 0$. Then it follows that $d_0(v_2^{a_{3,r}-j-p}X_{3,r-2}^{(sp-1)p}u_{2,3}) \equiv v_2^{a_{3,r}-j-p}X_{3,r-2}^{(sp-1)p}(u_{2,2}t_1^{p^2} - w_{2,3}^p)$ mod v_1 by Propositions 3.1.6 and 3.2.3, and hence

$$
\begin{aligned}
&d_0(-(a(r)-1)bv_1^{p+1}v_2^{a_{3,r}-j-p}X_{3,r-2}^{(sp-1)p}u_{2,3})\\
&\qquad\qquad \equiv -(a(r)-1)v_1^{p+1}v_2^{a_{3,r}-j-p}X_{3,r-2}^{(sp-1)p}(u_{2,2}t_1^{p^2} - w_{2,3}^p).
\end{aligned}
$$

Collecting four terms gives

$$
\begin{aligned}
&d_0(X(v_2^j, X_{3,r}^s))\\
&\quad \equiv X_{3,r-1}^{sp}\left\{(v_2 + v_1t_1^p)^b - v_2^b - bv_1t_1^p(v_2 + v_1t_1^p)^{b-1}\right\}/v_2^{j+b}\\
&\quad - bv_1X_{3,r}^{s'p}\left\{(v_2 + v_1t_1^p)^{b-1} - v_2^{b-1}\right\}/s'v_2^{j+b+a_{3,r+1}}\\
&\quad + a(r)bv_1v_2^{a_{3,r}-j-b+1}X_{3,r-2}^{(sp-1)p}\zeta_2^p \cdot \eta_R(v_2^{b-1})\\
&\quad + b(a+b-2)v_1^{p+1}X_{3,r}^{s'p}t_1/s'v_2^{j+2+a_{3,r+1}}\\
&\quad - (a+b)v_1^pX_{3,r-1}^{sp}t_1/v_2^{j+1} + v_1^{p+1}X_{3,r-1}^{sp}/v_2^{j+1} \cdot \{b\zeta_2 + (a-b)w_{2,2}\}\\
&\quad + a(r)bv_1^{p+1}v_2^{a_{3,r}-j-p-1}X_{3,r-2}^{(sp-1)p}\left\{(a-1)\zeta_2^p(v_3 - v_2^pt_1) - v_2t_1^{p^2}u_{2,2}\right\}.
\end{aligned}
$$

Then, apply the equation

$$(X+Y)^b - X^b - bY(X+Y)^{b-1} = \sum_{i=2}^{b}(1-i)\binom{b}{i}X^{b-i}Y^i$$

to the first term, and the congruence

$$X_{3,r}^{s'p} \equiv X_{3,r+1}^{s'} + a(r)s'v_2^{b(r+1)}X_{3,r-1}^{sp-1}u_{2,2} \qquad \text{mod } v_2^{b(r+1)} \quad \text{for odd } r \geq 3$$

to the fourth term. Modifying these terms completes the proof. $\square$

We also set $X(v_2^j, X_{3,1}^s)$ to $v_2^bX_{3,1}^s/X_{2,1} - bv_1v_2^{b-1}X_{3,2}^{s'}/s'X_{2,1}^{p+1}$ for $j = p-b$. Then we have:

Lemma 4.1.8. *$d_0(X(v_2^{p-1}, X_{3,1}^s))$ is expressed as $-3v_1^pX_{3,1}^st_1/v_2^p$ mod v_1^{p+1}, while $d_0(X(v_2^{p-2}, X_{3,1}^s))$ is expressed as*

$$
\begin{aligned}
&- 2v_1^2X_{3,2}^{s'}(t_1^p - 2v_1^{p-1}t_1)/s'v_2^{p^2+p}\\
&\quad - v_1^2X_{3,1}^s/v_2^{p-1} \cdot \{t_1^{2p}/v_2 + 4v_1^{p-2}t_1 - v_1^{p-1}(\zeta_2 + t_2/v_2 + C_1)\}\\
&\quad + v_1^{p+1}Z/v_2^{p-2}
\end{aligned}
$$

mod v_1^{p+2} *for some* $Z \in v_3^{-1}BP_*BP$, *where*

$$C_1 = (u_{3,0}t_1^{p^2+1} - w_{3,2})t_1^p + (u_{3,1} - w_{3,1}^p)t_2^p + u_{3,0}^p t_3^p.$$

Proof. Notice that $1/X_{2,1}^m \equiv 1/v_2^{mp} + mv_1^p v_3/v_2^{(m+1)p+1} \mod (v_1^{2p})$. So we may compute d_0 on

$$X_{3,1}^s/v_2^{p-b} + v_1^p v_3 X_{3,1}^s/v_2^{2p+1-b} - b\,v_1 X_{3,2}^{s'}/s'v_2^{p^2+p+1-b} - b\,v_1^{p+1}v_3X_{3,2}^{s'}/s'v_2^{p^2+2p+2-b}$$

instead of $d_0(X(v_2^{p-b}, X_{3,1}^s))$. For $b=1$ case, we consider the differential mod (v_1^{p+1}), so that the last term can be omitted. Because $X_{3,1}^s \equiv v_3^{sp} + v_2^p v_3^{(s-1)p-1} v_4$ mod v_2^{2p} and $X_{3,2}^{s'} \equiv X_{3,1}^{s'p}$ mod $v_2^{p^2+p}$, we may compute d_0 on

$$v_2(v_3^s/v_2)^p + v_1^p v_3^{sp+1}/v_2^{2p} + v_1^p v_3^{(s-1)p} v_4/v_2^p - v_1(X_{3,1}^{s'}/s'v_2^{p+1})^p.$$

Then, we obtain the desired result using (2.1.1) – (2.1.3), Proposition 3.2.3 and Proposition 3.3.4.

Similar but harder calculation shows the $b=2$ case. □

Next we define $X_+(v_2^{jp}, X_{3,r}^{sp})$ to modify the differential on $X_0(v_2^j, X_{3,r}^s)^p$. Let

$$\begin{aligned}
A_0 &= -\tbinom{j+1}{2} v_1^{2p} v_3^2 X_{3,r}^{sp}/v_2^{(j+2)p+2} + j(j+1) v_1^{2p} v_3 X_{3,r+1}^{s'p}/s'v_2^{(j+2+a_{3,r+1})p+1},\\
A_1 &= v_1^{2p} v_2^{(a_{3,r}-j-1)p-1} X_{3,r-1}^{(sp-1)p}\Big\{a(r)jv_2u_{2,3} - (j+1)v_3u_{2,2}\Big\},\\
A_2 &= v_1^{2p-1} v_3^{(sp-1)p+2}/v_2.
\end{aligned}$$

Using these elements, we define $X_+(v_2^{jp}, X_{3,r}^{sp})$ as follows. For $r=1$,

$$X_+(v_2^{jp}, X_{3,r}^{sp}) = \begin{cases} A_0 & \text{for } j \le p-3 \\ A_0 + A_2 & \text{for } j = p-2. \end{cases}$$

If $r \ge 3$ is odd and either $j = a_{3,r} - p + 2$ or $j \le a_{3,r} - p - 1$, then

$$X_+(v_2^{jp}, X_{3,r}^{sp}) = A_0 + A_1.$$

Then we obtain the following lemma.

Lemma 4.1.9. *Mod* (v_1^{2p+1}),

$$\begin{aligned}
&d_0(X_0(v_2^j, X_{3,r}^s)^p + X_+(v_2^{jp}, X_{3,r}^{sp}))\\
&\qquad\equiv -j(j+1)\frac{v_1^{2p}X_{3,r+2}^{s'}t_1}{s'v_2^{jp+2+a_{3,r+2}}} - \binom{j+1}{2}\frac{v_1^{2p}X_{3,r}^{sp}t_1^2}{v_2^{jp+2}}\\
&\qquad\quad - (j-1)v_1^{2p}v_2^{(a_{3,r}-j-1)p-1}X_{3,r-1}^{(sp-1)p}\theta_2(v_2^j),
\end{aligned}$$

where $\theta_2(v_2^j) = 0$ *for* $r=1$ *and*

$$\theta_2(v_2^j) = (j+1)(v_3 - v_2^p t_1)\zeta_2^p + \{\, (j+1)v_2^p t_1 - v_2 t_1^{p^2} \,\}u_{2,2}$$

for odd $r \ge 3$. *In particular,* $v_2^{p^2-1}\theta_2(v_2^{a_{3,r}-p-1}) = -v_3^{p+1}(t_1 + v_2 w_{3,2})/v_2^2$.

Proof. Here we prove only for $r \ge 3$ case. Other cases are similarly proved.

Notice that $d_0(A_0)$ is congruent to

$$-\binom{j+1}{2} v_1^{2p} d_0(v_3^2 X_{3,r}^{sp})/v_2^{(j+2)p+2} + j(j+1)v_1^{2p} d_0(v_3 X_{3,r+1}^{s'p})/s'v_2^{(j+2+a_{3,r+1})p+1}$$

mod (v_1^{2p+1}), so that it is sufficient to compute $d_0(v_3^2X_{3,r}^{sp})$ mod $(v_1, v_2^{(j+2)p+2})$ and $d_0(v_3X_{3,r+1}^{s'p})$ mod $(v_1, v_2^{(j+2+a_{3,r+1})p+1})$. Easy computations show that

$$\begin{aligned} d_0(v_3^2X_{3,r}^{sp}) &\equiv (v_2^2t_1^{2p^2} + v_2^{2p}t_1^2 + 2\,v_2v_3t_1^{p^2} - 2\,v_2^pv_3t_1 - 2\,v_2^{p+1}t_1^{p^2+1})X_{3,r}^{sp}, \\ d_0(v_3X_{3,r+1}^{s'p}) &\equiv (v_2t_1^{p^2} - v_2^pt_1)X_{3,r+1}^{s'p} + s'v_2^{pa_{3,r+1}}X_{3,r}^{sp}t_1^{p^2}(v_3 + v_2t_1^{p^2} - v_2^pt_1) \\ &\qquad -s'v_2^{b(r+2)}X_{3,r-1}^{(sp-1)p}\zeta_2^p(v_3 + v_2t_1^{p^2} - v_2^pt_1). \end{aligned}$$

Using these and the congruence

$$X_{3,r+2}^{s'} \equiv X_{3,r+1}^{s'p} - s'v_2^{a_{3,r+2}-p}v_3X_{3,r}^{sp} - s'v_2^{b(r+2)}X_{3,r}^{sp-1}u_{2,2},$$

we can show that $d_0(A_0)$ is congruent to

$$\binom{j+1}{2}\frac{v_1^{2p}X_{3,r}^{sp}(t_1^{2p^2} - v_2^{2p-2}t_1^2)}{v_2^{(j+2)p}} + j(j+1)\frac{v_1^{2p}(X_{3,r+1}^{s'p}t_1^{p^2} - v_2^{p-1}X_{3,r+2}^{s'}t_1)}{s'v_2^{(j+2+a_{3,r+1})p}}$$
$$- j(j+1)v_1^{2p}v_2^{(a_{3,r}-j-1)p-1}X_{3,r-1}^{(sp-1)p}\{\zeta_2^p(v_3 + v_2t_1^{p^2} - v_2^pt_1) + v_2^pt_1u_{2,2}\}.$$

On the other hand, $d_0(A_1)$ is congruent to

$$jv_1^{2p}v_2^{(a_{3,r}-j-1)p}d_0(a(r)\cdot X_{3,r-1}^{(sp-1)p}u_{2,3}) - (j+1)v_1^{2p}v_2^{(a_{3,r}-j-1)p-1}d_0(v_3X_{3,r-1}^{(sp-1)p}u_{2,2}).$$

Noticing that $(a_{3,r} - j - 1)p \geq (p-3)p$, we have

$$\begin{aligned} d_0(X_{3,r-1}^{(sp-1)p}u_{2,3}) &\equiv d_0(u_{2,3})X_{3,r-1}^{(sp-1)p} + \eta_R(u_{2,3})d_0(X_{3,r-1}^{(sp-1)p}) \\ &\equiv (u_{2,2}t_1^{p^2} - w_{2,3}^p)X_{3,r-1}^{(sp-1)p} + (u_{2,3} + u_{2,2}t_1^{p^2} - w_{2,3}^p)d_0(X_{3,r-1}^{(sp-1)p}) \end{aligned}$$

mod (v_1). Because $d_0(X_{3,r-1}^{(sp-1)p}) \equiv -v_2^{pa_{3,r-1}}X_{3,r-2}^{(sp^2-p-1)p}t_1^{p^2}$ mod $(v_1, v_2^{pa_{3,r-1}+1})$, the second term remains only for $r = 3$. Observe that $u_{2,3}$ and $w_{2,3}^p$ can be replaced with $-v_2^{-p^3}v_3^{p^2}u_{2,2}$ and $-v_2^{-p^3}v_3^{p^2}\zeta_2^p$ respectively. So we obtain

$$\begin{aligned} &d_0(a(r)\cdot X_{3,r-1}^{(sp-1)p}u_{2,3}) \\ &\quad\equiv a(r)\cdot X_{3,r-1}^{(sp-1)p}(u_{2,2}t_1^{p^2} - w_{2,3}^p) \\ &\qquad + a(r)\cdot v_2^{(a_{3,r-1}-p^2)p}v_3^{p^2}X_{3,r-2}^{(sp^2-p-1)p}t_1^{p^2}(u_{2,2} - \zeta_2^p) \\ &\quad\equiv a(r)\cdot X_{3,r-1}^{(sp-1)p}(u_{2,2}t_1^{p^2} - w_{2,3}^p) + (2 - a(r))\cdot X_{3,r-1}^{(sp-1)p}t_1^{p^2}(u_{2,2} - \zeta_2^p) \\ &\quad\equiv X_{3,r-1}^{(sp-1)p}\{\,2\,u_{2,2}t_1^{p^2} - a(r)\cdot w_{2,3}^p - (2 - a(r))\cdot t_1^{p^2}\zeta_2^p\,\}. \end{aligned}$$

Moreover, it is easy to see that

$$d_0(v_3X_{3,r-1}^{(sp-1)p}u_{2,2}) \equiv X_{3,r-1}^{(sp-1)p}\{v_2t_1^{p^2}u_{2,2} - v_2^pt_1u_{2,2} - \zeta_2^p(v_3 + v_2t_1^{p^2} - v_2^pt_1)\}.$$

Collecting terms gives $d_0(X_+(v_2^{jp}, X_{3,r}^{sp}))$ as

$$\binom{j+1}{2}\frac{v_1^{2p}X_{3,r}^{sp}(t_1^{2p^2} - v_2^{2p-2}t_1^2)}{v_2^{(j+2)p}} + j(j+1)\frac{v_1^{2p}(X_{3,r+1}^{s'p}t_1^{p^2} - v_2^{p-1}X_{3,r+2}^{s'}t_1)}{s'v_2^{(j+2+a_{3,r+1})p}}$$
$$- \varepsilon(r,j)^p - (j-1)v_1^{2p}v_2^{(a_{3,r}-j-1)p-1}X_{3,r-1}^{(sp-1)p}\theta_2(v_2^j).$$

This completes the proof for $r \geq 3$ case. $\square$

4.2. **Some inductive methods of constructing elements $x(sp^r/j;k)$.** Here we describe some inductive methods of constructing $x(k)=x(sp^r/j;k)$. Because $(x_{3,r}^s/v_2^j)^{p^k}=(X_{3,r}^s/v_2^j)^{p^k}$ for $j\le a_{3,r}$, we may start with $X_{3,r}^s/v_2^j$ $(p\nmid j\le a_{3,r})$. In general, each $x(k)$ $(k\ge 1)$ is constructed by adding some appropriate terms to $x(k-1)^p$ so that we can find an element of Coker δ (2.5.1) from the numerator of the leading term of $d_0(x(k))$. There are some patterns for adding terms and constructing $x(k)$ for a large enough k. We first observe the case with small j and $s\notin\mathbb{N}_0$, which is the simplest example of such patterns.

Proposition 4.2.1. *Assume that an element W_1 satisfies*

$$d_0(W_1)\equiv -jv_1^pX_{3,r}^{sp}t_1/v_2^{jp+1}\qquad \text{mod } v_1^{p+1}$$

for $1\le j\le a_{3,r}-2$, then W_k $(k\ge 2)$ defined inductively on k by

$$\begin{aligned} W_k &= W_1^p+jv_1^{a_{2,2}-p}v_2X_{3,r}^{sp^2}/X_{2,1}^{jp+1} && \textit{for } k=2,\\ &= W_{k-1}^p+2jv_1^{a_{2,k}-p}X_{3,r}^{sp^k}/v_2^{(jp+1)p^{k-1}-1} && \textit{for } k\ge 3\end{aligned}$$

satisfies

$$d_0(W_k)\equiv -2jv_1^{a_{2,k}}X_{3,r}^{sp^k}t_1/v_2^{(jp+1)p^{k-1}}\qquad \text{mod } v_1^{a_{2,k}+1}.$$

Proof. Using Propositions 3.2.3 and 3.3.4, for $k=2$, we have

$$d_0\left(jv_1^{a_{2,2}-p}v_2X_{3,r}^{sp^2}/X_{2,1}^{jp+1}\right)\equiv jv_1^{a_{2,2}-p+1}X_{3,r}^{sp^2}(t_1^p-2v_1^{p-1}t_1)/v_2^{(jp+1)p},$$

while for $k\ge 3$, we have

$$d_0\left(2jv_1^{a_{2,k}-p}X_{3,r}^{sp^k}/v_2^{(jp+1)p^{k-1}-1}\right)\equiv 2jv_1^{a_{2,k}-p+1}X_{3,r}^{sp^k}(t_1^p-v_1^{p-1}t_1)/v_2^{(jp+1)p^{k-1}}.$$

The result now follows by an easy calculation. □

We will see that each $x(sp^r/j;1)$ with $r\ge 1$, $1\le j\le a_{3,r}-2$ and $s\notin\mathbb{N}_0$ satisfies the condition for W_1 in the above proposition.

In general, we speculate on the existence of some rules in constructing elements $x(k)$ for large k as the above case. Next we observe another pattern, which occurs in most cases. To state this, we first introduce some elements.

Definition 4.2.2. For $e_1\ge 1$, $e_2\ge 1$, $s=s'p-1\in\mathbb{N}_0$ and odd $r\ge 3$, we define $Y_{[a,b]}(v_1^{e_1p},v_2^{(e_2+1)p},X_{3,r+1}^s)$ as follows: for $s'\notin\mathbb{N}_0$, it is given by

$$(a+b)\frac{v_1^{(e_1-1)p-1}v_2X_{3,r}^{sp}}{X_{2,1}^{e_2+1}}+a\frac{v_1^{e_1p}X_{3,r}^{sp}u_{2,2}}{v_2^{(e_2+1)p}}-(a+b)(b-2)\frac{v_1^{e_1p-1}X_{3,r+2}^{s'}}{s'v_2^{(e_2+1)p+a_{3,r+2}}};$$

for $s'\in\mathbb{N}_0$, it is given by

$$(\text{above terms})+2a\frac{v_1^{e_1p}X_{3,r+3}^{s''}}{s''v_2^{(e_2+1)p+b(r+3)}}.$$

Moreover, we define $Z(v_1^{e_1p^i}, v_2^{(e_2+1)p^i}, X_{3,r+i}^s)$ by

$$\begin{cases} -2\dfrac{v_1^{e_1p^i}v_3X_{3,r+i}^{s'p}}{s'v_2^{(e_2+1)p^i+p+1+a_{3,r+i+1}}} - \dfrac{v_1^{e_1p^i}X_{3,r+i-1}^{sp}u_{2,2}}{v_2^{(e_2+1)p^i}} & \text{for } s'\notin\mathbb{N}_0 \text{ and odd } r+i, \\ \dfrac{v_1^{e_1p^i}X_{3,r+i-1}^{sp}u_{2,2}}{v_2^{(e_2+1)p^i}} & \text{for } s'\notin\mathbb{N}_0 \text{ and even } r+i, \\ -\dfrac{v_1^{e_1p^i}X_{3,r+i}^{s}}{v_2^{(e_2+1)p^i}}(3u_{2,2}-2u_{3,3}) & \text{for} s'\in\mathbb{N}_0, \end{cases}$$

and $DZ(v_1^{e_1p^i}, v_2^{(e_2+1)p^i}, X_{3,r+i}^s) \in v_3^{-1}BP_*/(p, v_2^\infty)\otimes_A \Gamma$ by

$$\begin{cases} 2\dfrac{v_1^{e_1p^i}X_{3,r+i+1}^{s'}t_1}{s'v_2^{(e_2+1)p^i+1+a_{3,r+i+1}}} + \dfrac{v_1^{e_1p^i}X_{3,r+i}^{s}}{v_2^{(e_2+1)p^i}}(2w_{2,2}-\zeta_2) & \text{for } s'\notin\mathbb{N}_0 \text{ and odd } r+i, \\ 2\dfrac{v_1^{e_1p^i}X_{3,r+i+1}^{s'}t_1^p}{s'v_2^{(e_2+1)p^i+1+a_{3,r+i+1}}} + \dfrac{v_1^{e_1p^i}X_{3,r+i}^{s}}{v_2^{(e_2+1)p^i}}\left(2\dfrac{t_2}{v_2}-\zeta_2\right) & \text{for } s'\notin\mathbb{N}_0 \text{ and even } r+i, \\ -\dfrac{v_1^{e_1p^i}X_{3,r+i}^{s}}{v_2^{(e_2+1)p^i}}(3\zeta_2-2\zeta_3) & \text{for} s'\in\mathbb{N}_0. \end{cases}$$

Notice that each of these leading terms includes an element of Coker δ.

Proposition 4.2.3. *Assume that an element $x(k)$ satisfies*

$$d_0(x(k)) \equiv DX_{[a,b]}(v_1^{e_1}, v_2^{e_2+1}, X_{3,r}^s) + v_1^{e_1}Z_k/v_2^{e_2} \qquad \bmod v_1^{e_1+1}$$

for some $Z_k \in v_3^{-1}BP_*BP$, with $e_2 \equiv -b$ (p) and odd $r \geq 3$. Let

$$x(k+1) = x(k)^p + Y_{[a,b]}(v_1^{e_1p}, v_2^{(e_2+1)p}, X_{3,r+1}^s).$$

We compute $d_0(x(k+1))$. For $a \neq 0$, we get

$$d_0(x(k+1)) \equiv aDZ(v_1^{e_1p}, v_2^{(e_2+1)p}, X_{3,r+1}^s) + v_1^{e_1p}Z_{k+1}/v_2^{(e_2+1)p-1} \quad \bmod v_1^{e_1p+1}$$

for some $Z_{k+1} \in v_3^{-1}BP_*BP$. For $a = 0$, we get

$$d_0(x(k+1)) \equiv bv_1^{e_1p+1}X_{3,r+1}^st_1^p/v_2^{(e_2+1)p+1} + v_1^{e_1p}Z'_{k+1}/v_2^{(e_2+1)p-1} \quad \bmod v_1^{e_1p+2}$$

for some $Z'_{k+1} \in v_3^{-1}BP_*BP$.

Proof. Here we may ignore elements killed by $v_2^{(e_2+1)p-1}$. For the first term of $Y_{[a,b]}(v_1^{e_1p}, v_2^{(e_2+1)p}, X_{3,r+1}^s)$, notice that $(a+b)v_1^{(e_1-1)p-1}v_2X_{3,r}^{sp}/X_{2,1}^{e_2+1}$ is congruent to

$$(a+b)v_1^{(e_1-1)p-1}v_2X_{3,r}^{sp}\{1/v_2^{(e_2+1)p} - (b-1)v_1^pv_3/v_2^{(e_2+2)p+1}\}$$

mod $(p, v_1^{(e_1+1)p-1})$. Using (2.1.2), Propositions 3.2.4 and 3.3.4, we have

$$\begin{aligned} &d_0((a+b)v_1^{(e_1-1)p-1}v_2X_{3,r}^{sp}/X_{2,1}^{e_2+1}) \\ &\equiv (a+b)v_1^{(e_1-1)p}X_{3,r}^{sp} \\ &\quad\cdot\{v_2^pt_1^p + \underline{(b-2)v_1^{p-1}v_2^pt_1}_{(A)} + (b-1)v_1^p(\underline{t_1^{p^2+p}-t_2^p}_{(B)} + \underline{v_1v_2^{p-1}t_1^p}_{(C)})\}/v_2^{(e_2+2)p} \end{aligned}$$

mod $(v_1^{e_1p+2})$. For the second term, we can easily obtain

$$d_0(av_1^{e_1p}X_{3,r}^{sp}u_{2,2}/v_2^{(e_2+1)p}) \equiv \underline{av_1^{e_1p}X_{3,r}^{sp}(\zeta_2-\zeta_2^p)/v_2^{(e_2+1)p}}_{(B)}$$

mod $(v_1^{e_1p+1})$ by Proposition 3.1.3. For the third term, using the congruence

$$X_{3,r+2}^{s'} \equiv X_{3,r+1}^{s'p} - s'v_2^{a_{3,r+2}-p}v_3X_{3,r}^{sp} \mod v_2^{b(r+2)-p^2-p-1},$$

(2.1.2), Propositions 3.2.3 and 3.3.4, we have

$$\begin{aligned}
&d_0(-(a+b)(b-2)v_1^{e_1p-1}X_{3,r+2}^{s'}/s'v_2^{(e_2+1)p+a_{3,r+2}})\\
&\equiv -(a+b)(b-2)v_1^{e_1p}X_{3,r+1}^{s'p}t_1^p/s'v_2^{(e_2+1)p+1+a_{3,r+2}}\\
&\quad + (a+b)(b-2)v_1^{e_1p-1}X_{3,r}^{sp}\{\underline{-v_2^pt_1}_{(A)}\underline{-v_1(t_1^{p^2+p}-t_2^p)}_{(B)}\underline{-v_1^2v_2^{p-1}t_1^p}_{(C)}\}/v_2^{(e_2+2)p}
\end{aligned}$$

mod $(v_1^{e_1p+2})$. Collecting three terms gives

$$\begin{aligned}
(A) &= 0,\\
(B) &= v_1^{e_1p}X_{3,r}^{sp}/v_2^{(e_2+1)p}\{(a+b)(t_1^{p^2+p}-t_2^p)/v_2^p + a(\zeta_2-\zeta_2^p)\},\\
(C) &= (a+b)v_1^{e_1p+1}X_{3,r}^{sp}t_1^p/v_2^{(e_2+1)p+1}.
\end{aligned}$$

Thus $d_0(Y_{[a,b]}(v_1^{e_1p},v_2^{(e_2+1)p},X_{3,r+1}^s))$ is congruent to

$$\begin{aligned}
&-(a+b)(b-2)v_1^{e_1p}X_{3,r+1}^{s'p}t_1^p/s'v_2^{(e_2+1)p+1+a_{3,r+2}} + (a+b)v_1^{(e_1-1)p}X_{3,r}^{sp}t_1^p/v_2^{(e_2+1)p}\\
&\quad + v_1^{e_1p}X_{3,r}^{sp}/v_2^{(e_2+1)p}\cdot\{(a+b)(t_1^{p^2+p}-t_2^p)/v_2^p + a(\zeta_2-\zeta_2^p)\}
\end{aligned}$$

mod $v_1^{e_1p+1}$ for $a\neq 0$, and $d_0(Y_{[0,b]}(v_1^{e_1p},v_2^{(e_2+1)p},X_{3,r+1}^s))$ is congruent to

$$\begin{aligned}
&-b(b-2)v_1^{e_1p}X_{3,r+1}^{s'p}t_1^p/s'v_2^{(e_2+1)p+1+a_{3,r+2}}\\
&\quad + bv_1^{(e_1-1)p}X_{3,r}^{sp}\{v_2^pt_1^p + v_1^p(t_1^{p^2+p}-t_2^p) + v_1^{p+1}v_2^{p-1}t_1^p\}/v_2^{(e_2+2)p}
\end{aligned}$$

mod $v_1^{e_1p+2}$ for $a=0$. Now, recall definitions of ζ_2 and $w_{2,2}$ and notice the congruences

$$\begin{aligned}
X_{3,r+1}^{s'p} &\equiv X_{3,r+2}^{s'} + s'v_2^{a_{3,r+2}-p}v_3X_{3,r+1}^s && \mod v_2^{b(r+2)-p^2-p-1},\\
X_{3,r}^{sp} &\equiv X_{3,r+1}^s && \mod v_2^{b(r+1)-p^2-p-1}.
\end{aligned}$$

Modifying the differentials using the above two congruences completes the proof of this proposition for the case $s'\notin\mathbb{N}_0$.

For $s'\in\mathbb{N}_0$ case, the result is proved using the congruence

$$d_0(2av_1^{e_1p}X_{3,r+3}^{s''}/s''v_2^{(e_2+1)p+b(r+3)}) \equiv 2av_1^{e_1p}Y,$$

where

$$Y = X_{3,r+2}^{s'}t_1^p/v_2^{(e_2+1)p+1+a_{3,r+2}} - X_{3,r+1}^s(\zeta_2-\zeta_3+t_2/v_2)/v_2^{(e_2+1)p} + Z/v_2^{(e_2+1)p-1}$$

for some $Z\in v_3^{-1}BP_*BP$. □

We define $Y(v_1^{p^3+p^2-p},v_2^{p^2-p},X_{3,2}^{sp-1})$ to be

$$\begin{cases}
v_1^{p^3+p^2-2p-1}X_{3,2}^{sp-1}(4v_2+v_1^{p+1}u_{2,2})/X_{2,1}^{p-1} & \text{for } s\notin\mathbb{N}_0,\\
(\text{above terms}) + 4v_1^{p^3+p^2-p}X_{3,4}^{s'}/s'v_2^{p^3+p^2+a_{3,4}} & \text{for } s\in\mathbb{N}_0.
\end{cases}$$

By routine calculation, we have:

Lemma 4.2.4. *For $s \notin \mathbb{N}_0$, $d_0(Y(v_1^{p^3+p^2-p}, v_2^{p^2-p}, X_{3,2}^{sp-1}))$ is expressed as*

$$(v_1^{p^3+p^2-2p}X_{3,2}^{sp-1}/v_2^{p^2})\{4v_2^p t_1^p + v_1^p(4t_1^{p^2+p} - 5t_2^p) + v_1^p v_2^p(2\zeta_2 - \zeta_2^p + C_2 - w_{3,3})\}$$
$$+ v_1^{p^3+p^2-p}Z/v_2^{p^2-p-1}$$

mod $v_1^{p^3+p^2-p+1}$. For $s \in \mathbb{N}_0$, it is expressed as

$$(above\ terms) + 4v_1^{p^3+p^2-p}X_{3,3}^{s}t_1^p/v_2^{p^3+p^2}$$
$$- 4v_1^{p^3+p^2-p}X_{3,2}^{sp-1}(\zeta_2 - \zeta_3 + v_2^{-1}t_2)/v_2^{p^2-p} + v_1^{p^3+p^2-p}Z'/v_2^{p^2-p-1}$$

mod $v_1^{p^3+p^2-p+1}$. Here, $Z, Z' \in v_3^{-1}BP_*BP$, and

$$C_2 = (\sum_{i+j=2} u_{3,i}t_{2-j}^{p^i})t_1^{p^2} + u_{3,0}^p t_1^p(t_2^{p^2} - t_1^{p^3+p^2}).$$

Proof. By Proposition 3.3.4,

$$\begin{aligned} d_0(v_2X_{3,2}^{sp-1}/X_{2,1}^{p-1}) &= d_0(v_2X_{3,2}^{sp-1})/X_{2,1}^{p-1} + \eta_R(v_2X_{3,2}^{sp-1})d_0(1/X_{2,1}^{p-1}) \\ &\equiv d_0(v_2X_{3,2}^{sp-1})(1/v_2^{p^2-p} - v_1^p v_3/v_2^{p^2+1}) \\ &\quad + \eta_R(v_2X_{3,2}^{sp-1}) \cdot v_1^p(t_1 - v_1w_{2,2})/v_2^{p^2-p-1} \end{aligned}$$

mod (v_1^{p+2}). It is sufficient to compute $d_0(v_2X_{3,2}^{sp-1})$ mod $(v_1^2, v_2^{p^2+1})$ and mod $(v_1^{p+2}, v_2^{p^2-p})$. Notice that

$$\begin{aligned} d_0(v_2X_{3,2}^{sp-1}) &= d_0(v_2)X_{3,2}^{sp-1} + \eta_R(v_2)d_0(X_{3,2}^{sp-1}), \\ &\equiv d_0(v_2)v_3^{(sp-1)p^2} + \eta_R(v_2)d_0(v_3^{(sp-1)p^2}) \quad \text{mod } v_1^{p+2}, v_2^{p^2-p}. \end{aligned}$$

Using the results of 3.2, we see that $d_0(4\,v_1^{p^3+p^2-2p-1}v_2X_{3,2}^{sp-1}/X_{2,1}^{p-1})$ is congruent to

$$(4\,v_1^{p^3+p^2-2p}X_{3,2}^{sp-1}/v_2^{p^2-p})\{t_1^p + v_1^p(t_1^{p^2+p} - t_2^p)/v_2^p\}.$$

On the other hand, we have

$$d_0(v_1^{p^3+p^2-p}X_{3,2}^{sp-1}u_{2,2}/v_2^{p^2-p}) \equiv v_1^{p^3+p^2-p}d_0(X_{3,2}^{sp-1}u_{2,2})/v_2^{p^2-p}$$

mod $(v_1^{p^3+p^2-p+1})$, and

$$\begin{aligned} d_0(X_{3,2}^{sp-1}u_{2,2}) &= d_0(u_{2,2})X_{3,2}^{sp-1} + \eta_R(u_{2,2})d_0(X_{3,2}^{sp-1}) \\ &\equiv (\zeta_2 - \zeta_2^p)X_{3,2}^{sp-1} + (u_{2,2} + \zeta_2 - \zeta_2^p)d_0(X_{3,2}^{sp-1}) \end{aligned}$$

mod (v_1, v_2). According to Proposition 3.2.3,

$$d_0(X_{3,2}^{sp-1}) \equiv -v_2^{p^2}X_{3,1}^{sp^2-p-1}t_1^p + v_2^{p^2+p+1}v_3^{sp^3-p^2-p-1}(u_{2,0}t_2 - w_{3,3})$$

mod $(v_1, v_2^{p^2+p+2})$. Using this congruence and Definition 3.2.1, we obtain

$$d_0\left(\frac{v_1^{p^3+p^2-p}X_{3,2}^{sp-1}u_{2,2}}{v_2^{p^2-p}}\right) \equiv \frac{v_1^{p^3+p^2-2p}X_{3,2}^{sp-1}}{v_2^{p^2-p}}\left\{\left(2\zeta_2-\zeta_2^p-\frac{t_2^p}{v_2^p}\right)+C_2-w_{3,3}\right\}$$
$$+\frac{v_1^{p^3+p^2-p}Z}{v_2^{p^2-p-1}}$$

for some $Z\in v_3^{-1}BP_*BP$. Collecting terms gives the result for $s\notin\mathbb{N}_0$ case.

For $s\in\mathbb{N}_0$ case, it is easy to see that $d_0(4\,v_1^{p^3+p^2-p}X_{3,4}^{s'}/s'v_2^{p^3+p^2+a_{3,4}})$ is congruent to

$$4\,\frac{v_1^{p^3+p^2-p}X_{3,3}^{s}t_1^p}{v_2^{p^3+p^2}}-4\,\frac{v_1^{p^3+p^2-p}X_{3,2}^{sp-1}t_1^p}{v_2^{p^2-p}}(\zeta_2-\zeta_3+u_{2,0}t_2)+(\text{elements killed by } v_2^{p^2-p-1}).$$

□

Proposition 4.2.5. *Assume that an element $x(2)$ satisfies*

$$d_0(x(2))\equiv -2s(s-1)\frac{v_1^{p^2+p-2}X_{3,1}^{sp-1}t_1}{v_2^{p-1}}+2(s-1)\frac{v_1^{p^2+p-1}X_{3,2}^{s}t_1}{v_2^{p^2+p}}$$
$$+\binom{s}{2}\frac{v_1^{p^2+p-1}X_{3,1}^{sp-1}}{v_2^{p-1}}\left(\zeta_2+\frac{t_2}{v_2}+C_1\right)+\frac{v_1^{p^2+p-1}Z}{v_2^{p-2}}$$

mod $(v_1^{p^2+p})$ for some $Z\in v_3^{-1}BP_*BP$, where C_1 is as in Lemma 4.1.8. Then $x(3)=x(2)^p+\binom{s}{2}Y(v_1^{p^3+p^2-p},v_2^{p^2-p},X_{3,2}^{sp-1})$ satisfies

$$d_0(x(3))\equiv s(s-1)DZ(v_1^{p^3+p^2-p},v_2^{p^2-p},X_{3,2}^{sp-1})+v_1^{p^3+p^2-p}Z_3/v_2^{p^2-p-1}$$

mod $(v_1^{p^3+p^2-p+1})$ for some $Z_3\in v_3^{-1}BP_*BP$.

Proof. Collect the differentials on $x(2)^p$ and $\binom{s}{2}Y(v_1^{p^3+p^2-p},v_2^{p^2-p},X_{3,2}^{sp-1})$ using the congruences

$$C_1^p+C_2\equiv w_{3,3}\mod I_3,$$
$$X_{3,3}^s\equiv X_{3,2}^{sp}-sv_2^{p^3-1}v_3X_{3,2}^{sp-1}\mod v_2^{p^3+p^2}.$$

□

As the next step of Proposition 4.2.3 or 4.2.5, we show the following proposition. Here we state only the result.

Proposition 4.2.6. *Assume that an element $x(k+1)$ satisfies*

$$d_0(x(k+1))\equiv DZ(v_1^{e_1p},v_2^{(e_2+1)p},X_{3,r+1}^s)+v_1^{e_1p}Z_{k+1}/v_2^{(e_2+1)p-1}\mod v_1^{e_1p+1}$$

for some $Z_{k+1}\in v_3^{-1}BP_*BP$, where e_1 and e_2 are positive integers and $r\geq 1$. Then $x(k+i)$ defined inductively on $i\geq 2$ by

$$x(k+i)=x(k+i-1)^p+Z(v_1^{e_1p^i},v_2^{(e_2+1)p^i},X_{3,r+i}^s)$$

satisfies

$$d_0(x(k+i))\equiv DZ(v_1^{e_1p^i},v_2^{(e_2+1)p^i},X_{3,r+i}^s)+v_1^{e_1p^i}Z_{k+i}/v_2^{(e_2+1)p^i-1}\mod v_1^{e_1p^i+1}$$

for some $Z_{k+i} \in v_3^{-1}BP_*BP$. $\square$

Proposition 4.2.6 is regarded as the last step in this pattern. We will see in Section 5 that $x(k)$ is constructed in this way in many cases. Actually we have other two inductive ways of constructing $x(k)$, introduced in the following two propositions. Their proofs are relatively easy, so we leave the proofs to the reader.

Proposition 4.2.7. *Assume that an element* $x(k)$ *satisfies*

$$\begin{aligned} d_0(x(k)) &\equiv v_1^{e_1 p} v_2^{p^3-p} v_3^{(sp-1)p^{r+i-1}} \theta_i (v_2^{a_{3,r}-p^{3-i}-p^{2-i}})^p \mod v_1^{e_1+1} \\ &(= v_1^{e_1} v_3^{e_3 p} t_1^p / v_2^{2p} + v_1^{e_1} Z_k / v_2^p) \end{aligned}$$

for $\theta_i(v_2^{a_{3,r}-p^{3-i}-p^{2-i}})$ $(i = 1,2)$ *as in Lemma* 4.1.7 *or* 4.1.9, *and some* $Z_k \in v_3^{-1}BP_*BP$. *Then* $x(k+i)$ *defined inductively on* $i \geq 1$ *by*

$$\begin{aligned} x(k+i) &= x(k+i-1)^p - v_1^{e_1 p^i} v_3^{e_3 p^{i+1}+1} / v_2^{p^{i+2}+2p^{i+1}-pa_{3,i+1}+1} && \textit{for odd } i, \\ &= x(k+i-1)^p && \textit{for even } i \end{aligned}$$

satisfies

$$\begin{aligned} d_0(x(k+i)) \equiv\ & v_1^{e_1 p^i} v_3^{e_3 p^{i+1}} t_1^{p^{\delta(i)}} / v_2^{p^{i+2}+2p^{i+1}-a_{3,i+2}} \\ &+ v_1^{e_1 p^i} Z_{k+i} / v_2^{p^{i+1}+p^{i-1}} \mod v_1^{e_1 p^i + 1} \end{aligned}$$

for some $Z_{k+i} \in v_3^{-1}BP_*BP$, *where* $\delta(i) = 0$ *for odd* i *and* 1 *for even* i. $\square$

Proposition 4.2.8. *Assume that an element* $x(k)$ *satisfies*

$$d_0(x(k)) \equiv v_1^{e_1} v_2^{e_2} v_3^{e_3 p^r} w_{2,t} \mod v_1^{e_1+1}$$

with $Max\{p^{t-1} - p^r, 0\} + P(t-2) + 1 \leq e_2$. *Then* $x(k+i)$ *defined inductively on* $i \geq 1$ *by*

$$x(k+i) = x(k+i-1)^p + (-1)^{i-1} v_1^{e_1 p^i + P(i-1,1)} v_2^{e_2 p^i - P(i-1,1)} v_3^{e_3 p^{r+i}} u_{2,t+i-1}$$

satisfies

$$d_0(x(k+i)) \equiv (-1)^i v_1^{e_1 p^i + P(i-1)} v_2^{e_2 p^i - P(i-1)} v_3^{e_3 p^{r+i}} w_{2,t+i} \mod v_1^{e_1 p^i + P(i-1)+1},$$

where $P(i,j) = p^j(p^{i-j+1} - 1)/(p-1) = p^i + \cdots + p^j$ *for* $i \geq j$. *We will abbreviate* $P(i,0)$ *to* $P(i)$. $\square$

Our guide to constructing $x(k)$ is to add suitable elements to $x(k-1)^p$ so that the differential has one of the forms in the assumptions of Propositions 4.2.1, 4.2.6, 4.2.7 and 4.2.8. In fact, we will observe that each case follows one of the above four patterns by $k = 5$.

5. Proof of the main theorem

In this section we prove our main theorem by defining $x(k)$ $(= x(sp^r/j;k))$ for all cases using the preparatory computations displayed in Section 4. Notice that the smallest integer N with $d_0(x(k)) \not\equiv 0 \mod p, v_1^{N+1}$ gives the v_1-divisibility $N(k)$ $(= N(s,r,j;k))$ of $x(k)$.

5.1. **Definitions of $x(0)$ and the differentials.** Now we start with $X_{3,r}^s/v_2^j$ ($p\nmid j\le a_{3,r}$ and $p\nmid s\in\mathbb{Z}$). Using Propositions 3.2.4 and 3.3.4, we can easily calculate $d_0(X_{3,r}^s/v_2^j)$ mod v_1^2:

$$d_0(X_{3,0}^s/v_2^j)\equiv -v_1v_3^st_1^p/v_2^2+sv_1v_3^{s-1}(t_2^p-t_1^{p^2+p})/v_2,$$

and for $r=1$ or for even $r\ge 2$,

$$d_0(X_{3,r}^s/v_2^j)\equiv -jv_1X_{3,r}^st_1^p/v_2^{j+1}.$$

Finally, for odd $r\ge 3$,

$$d_0(X_{3,r}^s/v_2^j)\equiv -jv_1X_{3,r}^st_1^p/v_2^{j+1}-sv_1v_2^{a_{3,r}-j}X_{3,r-1}^{sp-1}\left(jt_1^{p+1}/v_2+w_{2,2}\right).$$

Moreover, we have

$$\begin{aligned} d_0(X_{3,0}^s/v_2) &\equiv v_1v_2^{p-1}v_3^{s-1}w_{2,2} && \text{mod } (v_1^2),\\ &\equiv v_1v_2^{p-2}v_3^{s-1}\zeta_2(v_2-v_1t_1^p) && \text{mod } (v_1^3)\end{aligned}$$

for $p\mid(s-1)$. Then the coboundary $\delta: H^0M_1^2\to H^1M_2^1$ on $X_{3,r}^s/v_1v_2^j$ is

$$\delta(X_{3,r}^s/v_1v_2^j)=-jv_3^{sp^r}t_1^p/v_2^{j+1}+\text{(elements killed by } v_2^j).$$

According to 2.5, its numerator belongs to Cokerδ, when $s\notin\mathbb{N}_0=\{s'p-1\mid p\nmid s'\}$ and $r\ge 1$ is odd or when $r\ge 0$ is even by (2.5.1). In this case, we can set $x(0)$ to $X_{3,r}^s/v_2^j$ with the v_1-divisibility $N(0)=1$. In another case, we need to modify $X_{3,r}^s/v_2^j$ so that its coboundary includes an element of Cokerδ.

5.1.1. For odd $r\ge 3$ and $s\in\mathbb{N}_0$.

(i) For $a_{3,r}-p\le j\le a_{3,r}$. We set $x(0)$ to $X_0(v_2^j,X_{3,r}^s)$. By Lemma 4.1.5, we have

$$d_0\left(x(0)\right)\equiv(1-j)v_1v_2^{a_{3,r}-j}v_3^{(sp-1)p^{r-1}}\zeta_2\quad\text{mod } v_1^2,$$

and so $N(0)=1$. Notice that this differential is trivial either for $j=a_{3,r}-p+2\equiv 1\ (p)$ or for $j\le a_{3,r}-p-1$.

(ii) Either for $j=a_{3,r}-p+2$ or for $j\le a_{3,r}-p-1$. We also set $x(0)$ to $X_0(v_2^j,X_{3,r}^s)$. Then, Lemma 4.1.5 shows that

$$d_0(x(0))\equiv -j(j+1)v_1^2X_{3,r+1}^{s'}t_1^p/s'v_2^{j+2+a_{3,r+1}}-\binom{j+1}{2}v_1^2X_{3,r}^st_1^{2p}/v_2^{j+2}+\varepsilon(r,j)$$

mod (v_1^3), and so $N(0)=2$. Note that all elements but $\varepsilon(r,j)$ are vanished when $p\mid(j+1)$, and $\varepsilon(r,j)=0$ unless $r\ge 3$ is odd and $a_{3,r}-p^2-p\le j\le a_{3,r}-2p$, in which case

$$\varepsilon(r,j)=-a(r)v_1^2v_2^{a_{3,r}-j-1}v_3^{(sp-1)p^{r-1}}w_{2,3}.$$

(iii) For $p\mid(j+1)$, $p^2\nmid(j+p+1)$ and $j\le a_{3,r}-p^2-p$. Set $x(0)=X(v_2^j,X_{3,r}^s)$. Notice that we replace $X_0(v_2^{a_{3,r}-p^2-p},X_{3,r}^s)$ with $X(v_2^{a_{3,r}-p^2-p},X_{3,r}^s)$ for $j=a_{3,r}-p^2-p$ although $\varepsilon(r,a_{3,r}-p^2-p)\ne 0$. Then Lemma 4.1.7 shows that

$$d_0(x(0))\equiv DX_{[j',1]}(v_1^{p+1},v_2^{j+1},X_{3,r}^s)+a(r)v_1^{p+1}v_2^{a_{3,r}-j-p-1}v_3^{(sp-1)p^{r-1}}\theta_1(v_2^j)$$

mod v_1^{p+2}. Note that $d_0(x(0)) \not\equiv 0 \bmod (v_1^{p+1})$, and so $N(0) = p$ when $p^2 \nmid (j+p+1)$.

(iv) For $p^2 \mid (j+p+1)$ and $j \le a_{3,r} - p^2 - 2p$. We first define $B(j)$ by

$$B(j) = X(v_2^j, X_{3,r}^s) \quad \text{for } j \le a_{3,r} - p^2 - 3p;$$
$$B(a_{3,r} - p^2 - 2p) = X(v_2^{a_{3,r}-p^2-2p}, X_{3,r}^s) - a(r) v_1^{p+1} v_3^{(sp-1)p^{r-1}+p+2} / 2v_2^2.$$

Then, let

$$x(0) = \begin{cases} B(j) & \text{for } s' \notin \mathbb{N}_0 \\ B(j) - 2\, v_1^{p+1} X_{3,r+2}^{s''} / s'' v_2^{j+1+b(r+2)} & \text{for } s' \in \mathbb{N}_0. \end{cases}$$

Easy calculation shows that $d_0(x(0)) \equiv -DZ(v_1^{p+1}, v_2^{j+1}, X_{3,r}^s) \bmod (v_1^{p+2})$, and so $N(0) = p+1$.

5.1.2. For $r = 1$ and $s \in \mathbb{N}_0$. For $j \le p-2$, we set $x(0)$ to $X_0(v_2^j, X_{3,1}^s)$. Then Lemma 4.1.5 shows that

$$d_0(x(0)) \equiv -j(j+1) v_1^2 X_{3,2}^{s'} t_1^p / s' v_2^{j+2+a_{3,2}} - \binom{j+1}{2} v_1^2 X_{3,1}^s t_1^{2p} / v_2^{j+2}$$

mod (v_1^3), and so $N(0) = 2$. For $j = p-1$, we set $x(0)$ to $X(v_2^{p-1}, X_{3,1}^s)$. By Lemma 4.1.8, we have $d_0(x(0)) \equiv -3 v_1^p X_{3,1}^s t_1 / v_2^p \bmod (v_1^{p+1})$, and so $N(0) = p$.

5.2. **Definitions of $x(k)$ $(k \ge 1)$ and the differentials.** We complete the proof of our main theorem by defining $x(k)$ and computing the differentials for all cases, based on the computations in $k = 0$ case.

5.2.1. For $r = 0$.

(i) For $p \nmid (s-1)$. We set $x(1)$ to $X_{3,1}^s / X_{2,1}$. Then Lemma 4.1.3 shows that

$$d_0(x(1)) \equiv (s-1) v_1^p X_{3,1}^s t_1 / v_2^{p+1} + \binom{s}{2} v_1^p v_3^{sp-1} t_1^2 / v_2$$

mod v_1^{p+1}, and so $N(1) = p$. For $k = 2$, we set

$$x(2) = x(1)^p + \binom{s}{2} v_1^{p^2-2} X(v_2^{p-2}, X_{3,1}^{sp-1}).$$

Then Lemma 4.1.8 implies

$$\begin{aligned} & d_0\left(\binom{s}{2} v_1^{p^2-2} X(v_2^{p-2}, X_{3,1}^{sp-1})\right) \\ & \equiv -(s-1) \frac{v_1^{p^2} X_{3,2}^s}{v_2^{p^2+p}} (t_1^p - 2 v_1^{p-1} t_1) \\ & \quad - \binom{s}{2} \frac{v_1^{p^2} X_{3,1}^{sp-1}}{v_2^{p-1}} \left\{ \frac{t_1^{2p}}{v_2} + 4 v_1^{p-2} t_1 - v_1^{p-1} \left(\zeta_2 + \frac{t_2}{v_2} + C_1 \right) \right\} + \frac{v_1^{p^2+p-1} Z_2}{v_2^{p-2}} \end{aligned}$$

mod $v_1^{p^2+p}$, and hence $d_0(x(2))$ is congruent to

$$-2s(s-1)\frac{v_1^{p^2+p-2}X_{3,1}^{sp-1}t_1}{v_2^{p-1}}+2(s-1)\frac{v_1^{p^2+p-1}X_{3,2}^{s}t_1}{v_2^{p^2+p}}$$
$$+\binom{s}{2}\frac{v_1^{p^2+p-1}X_{3,1}^{sp-1}}{v_2^{p-1}}\left(\zeta_2+\frac{t_2}{v_2}+C_1\right)+\frac{v_1^{p^2+p-1}Z_2}{v_2^{p-2}}$$

mod $v_1^{p^2+p}$. Notice that $d_0(x(2))\not\equiv 0 \bmod (v_1^{p^2+p-1})$, and so $N(2)=p^2+p-2$. For $k=3$, we set $x(3)$ to $x(2)^p+\binom{s}{2}Y(v_1^{p^3+p^2-p},v_2^{p^2-p},X_{3,2}^{sp-1})$. Then Lemma 4.2.5 shows that

$$d_0(x(3))\equiv s(s-1)DZ(v_1^{p^3+p^2-p},v_2^{p^2-p},X_{3,2}^{sp-1})+v_1^{p^3+p^2-p}Z_3/v_2^{p^2-p-1}$$

mod $(v_1^{p^3+p^2-p+1})$ for some $Z_3\in v_3^{-1}BP_*BP$, and so $N(3)=p^3+p^2-p$. Applying Proposition 4.2.6, we can define $x(k)$ inductively on $k\geq 4$ and obtain $N(k)=p^k+p^{k-1}-p^{k-2}$. □

Recall that $\mathbb{N}_1=\{(ap^2-p-1)p^r+1 \mid p\nmid a, r\geq 1:\text{odd}\}$.

(ii) For $p\mid(s-1)$ and $s\notin\mathbb{N}_1$. We have already computed $d_0(x(0))\equiv v_1v_2^{p-1}v_3^{s-1}w_{2,2} \bmod v_1^2$. Thus we can define $x(k)$ inductively on $k\geq 1$ using Proposition 4.2.8 and obtain $N(k)=P(k)$. □

(iii) For $s\in\mathbb{N}_1$. In this case s is expressed as $(s_0p-1)p^{r_0}+1$ ($s_0\in\mathbb{N}_0$ and odd $r_0\geq 1$). We set $x(1)$ to

$$x(0)^p-1/a(r_0+2)v_1^{p-1}X(v_2^{(a_{3,r_0+1}-p+2)p},X_{3,r_0+2}^{s_0})-v_1^{2p}v_2^{p^2-2p-1}v_3^{(s-1)p+1}u_{2,2}$$

with $a(3)=1$ and $a(r_0+2)=3$ for $r_0\geq 3$. Then the differential on the second term is given by Lemma 4.1.7. For the third term, we have

$$\begin{aligned}d_0(-v_1^{2p}v_2^{p^2-2p-1}v_3^{(s-1)p+1}u_{2,2})\\ \equiv v_1^{2p}v_2^{p^2-2p-1}v_3^{(s-1)p}\{v_3\zeta_2^p+(v_2^pt_1-v_2t_1^{p^2})(u_{2,2}-\zeta_2^p)\}\end{aligned}$$

mod (v_1^{2p+1}). Collecting three terms gives

$$d_0(x(1))\equiv -1/a(r_0+2)DX_{[2,1]}(v_1^{2p},v_2^{(a_{3,r_0+1}-p+2)p},X_{3,r_0+2}^{s_0})$$

mod v_1^{2p+1}, and so $N(1)=2p-1$. Applying Propositions 4.2.3 and 4.2.6, we can define $x(k)$ inductively on $k\geq 2$ and obtain $N(k)=2p^k$. □

5.2.2. For even $r\geq 2$. Define $x(1)$ by

$$\begin{aligned}x(1)&=X_{3,r}^{sp}/X_{2,1}^{a_{3,r}-1}+sv_1^pv_3^{(sp-1)p^r+2}/2v_2^2 &&\text{for } j=a_{3,r}-1,\\ &=X_{3,r}^{sp}/X_{2,1}^{j} &&\text{for } j\leq a_{3,r}-2.\end{aligned}$$

Then, in both cases we have

$$d_0(x(1))\equiv -jv_1^pX_{3,r+1}^s(t_1-v_1w_{2,2})/v_2^{jp+1}$$

mod (v_1^{p+2}). Notice that $d_0(x(1)) \not\equiv 0 \bmod (v_1^{p+1})$, and so $N(1) = p$ when $s \notin \mathbb{N}_0$. Then we can define $x(k)$ inductively on $k \geq 2$ by Proposition 4.2.1 and obtain $N(k) = a_{2,k}$. When $s \in \mathbb{N}_0$, notice that the above differential can be rewritten as

$$d_0(x(1)) \equiv DX_{[j,0]}(v_1^{p+1}, v_2^{jp+1}, X_{3,r+1}^s),$$

and so $N(1) = p$. Applying Propositions 4.2.3 and 4.2.6, we can define $x(k)$ inductively on $k \geq 2$ and obtain $N(k) = p^k + p^{k-1}$. □

5.2.3. FOR ODD $r \geq 1$ AND $s \notin \mathbb{N}_0$. Define $x(1)$ by

$$\begin{aligned} x(1) &= X_{3,1}^{sp}/X_{2,1}^j && \text{for } r = 1, \\ &= X_{3,r}^{sp}/X_{2,1}^j - sv_1^p v_2^{(a_{3,r}-j)p} v_3^{(sp-1)p^r} u_{2,2} && \text{for odd } r \geq 3. \end{aligned}$$

For odd $r \geq 3$ and $j \leq a_{3,r} - p - 1$, $d_0(X_{3,r}^{sp}/X_{2,1}^j)$ has already been computed in Lemma 4.1.2. We also obtain $d_0(X_{3,r}^{sp}/X_{2,1}^j)$ for other cases by easy computations. We first give the case $r = 1$:

$$d_0(X_{3,1}^{sp}/X_{2,1}^{p-1}) = v_1^p X_{3,1}^{sp} t_1/v_2^{(p-1)p+1} - sv_1^p v_3^{(sp-1)p+1} t_1^p/v_2,$$

and for $j \leq p-2$,

$$d_0(X_{3,1}^{sp}/X_{2,1}^j) = -jv_1^p X_{3,1}^{sp} t_1/v_2^{jp+1}.$$

Now let $r \geq 3$ be odd. Then

$$\begin{aligned} d_0(X_{3,r}^{sp}/X_{2,1}^{a_{3,r}}) &= v_1^p X_{3,r}^{sp} t_1/v_2^{pa_{3,r}+1} - sv_1^p v_3^{(sp-1)p^r}(\zeta_2^p - w_{2,2}), \\ d_0(X_{3,r}^{sp}/X_{2,1}^{a_{3,r}-1}) &= 2v_1^p X_{3,r}^{sp} t_1/v_2^{(a_{3,r}-1)p+1} - sv_1^p v_3^{(sp-1)p^r}(v_2^p \zeta_2^p + 2v_3 t_1^p/v_2), \end{aligned}$$

and for $j \leq a_{3,r} - 2$, we have

$$d_0(X_{3,r}^{sp}/X_{2,1}^j) = -jv_1^p X_{3,r}^{sp} t_1/v_2^{jp+1} - sv_1^p v_2^{(a_{3,r}-j)p} v_3^{(sp-1)p^r} \zeta_2^p.$$

Additionally, we consider the element $-sv_1^p v_2^{(a_{3,r}-j)p} v_3^{(sp-1)p^r} u_{2,2}$ for odd $r \geq 3$, on which the differential is congruent to

$$-sv_1^p v_2^{(a_{3,r}-j)p} v_3^{(sp-1)p^r}(\zeta_2 - \zeta_2^p)$$

mod v_1^{p+1}. Collecting terms shows that $d_0(x(1))$ is congruent to

$$\begin{array}{ll} v_1^p X_{3,r}^{sp} t_1/v_2^{pa_{3,r}+1} + sv_1^p v_3^{(sp-1)p^r}(t_1^{p+1} - t_2)/v_2 & \text{for odd } r \geq 3 \text{ and } j = a_{3,r}, \\ -jv_1^p X_{3,r}^{sp} t_1/v_2^{(a_{3,r}-1)p+1} - sv_1^p v_3^{(sp-1)p^r+1} t_1^p/v_2 & \text{for } j = a_{3,r} - 1, \\ -jv_1^p X_{3,r}^{sp} t_1/v_2^{jp+1} & \text{for } j \leq a_{3,r} - 2 \end{array}$$

mod v_1^{p+1}, and so $N(1) = p$.

(i) For $k \geq 2$ and $j \leq a_{3,r} - 2$. Using Proposition 4.2.1, we can define $x(k)$ inductively on k so that

$$d_0(x(k)) \equiv -2jv_1^{a_{2,k}} X_{3,r}^{sp^k} t_1/v_2^{(jp+1)p^{k-1}}$$

mod $(v_1^{1+a_{2,k}})$ and obtain $N(k) = a_{2,k}$. □

(ii) For $k\ge 2$ and $j=a_{3,r}-1$. Let

$$x(2)=x(1)^p+jv_1^{a_{2,2}-p}X_{3,r}^{sp^2}/v_2^{a_{3,r+2}-p^2}+sv_1^{p^2}v_2^{p^2}v_3^{(sp-1)p^{r+1}}u_{2,2}.$$

Then easy calculation shows that

$$d_0(x(2))\equiv -sv_1^{p^2+1}v_2^{p^2-1}v_3^{(sp-1)p^{r+1}}w_{2,3}$$

mod $(v_1^{p^2+2})$, and so $N(2)=p^2+1$. Applying Proposition 4.2.8, we can define $x(k)$ inductively on $k\ge 3$ and obtain $N(k)=P(k)-p^{k-1}$. □

(iii) For $k\ge 2$, odd $r\ge 3$ and $j=a_{3,r}$. Let $x(2)=x(1)^p-v_1^{p^2-1}X_{3,r+2}^s/v_2^{a_{3,r+2}}$, and $x(3)=x(2)^p+sv_1^{p^3+p}v_3^{(sp-1)p^{r+2}+1}/v_2^{p+1}$. Then easy computations show that

$$\begin{aligned}
d_0(x(2)) &\equiv -sv_1^{p^2+1}v_3^{(sp-1)p^{r+1}}t_1^p/v_2 \quad \mathrm{mod}\ v_1^{p^2+2},\\
d_0(x(3)) &\equiv -sv_1^{p^3+p}v_3^{(sp-1)p^{r+2}}(t_1-v_1w_{2,2})/v_2 \quad \mathrm{mod}\ v_1^{p^3+p+2}\\
&\quad (\equiv sDX_{[1,0]}(v_1^{p^3+p+1},v_2,X_{3,r+2}^{sp-1})),
\end{aligned}$$

and so $N(2)=p^2+1$ and $N(3)=p^3+p$. Applying Propositions 4.2.3 and 4.2.6, we can define $x(k)$ inductively on $k\ge 4$ and obtain $N(k)=p^k+p^{k-2}+p^{k-3}$. □

5.2.4. For $r=1$ and $s\in\mathbb{N}_0$.

(i) For $j=p-1$. First we define $\tilde{x}$ by $\tilde{x}=x(0)^p+3v_1^{a_{2,2}-p}X_{3,1}^{sp}/v_2^{p^2-1}$. Observe that $d_0(\tilde{x})\equiv -3v_1^{a_{2,2}}X_{3,1}^{sp}t_1/v_2^{p^2}$ mod $(v_1^{1+a_{2,2}})$. Then, let $x(1)=\tilde{x}+3v_1^{a_{2,2}}X_{3,3}^{s'}/s'v_2^{p^2+a_{3,3}}$, $x(2)=\tilde{x}^p+3v_1^{a_{2,3}-p}X_{3,2}^{sp}/v_2^{a_{3,3}-p}$, and $x(3)=x(2)^p+3v_1^{a_{2,4}-p}v_3^{sp^4}/v_2^{p^4-1}+3v_1^{a_{2,4}}X_{3,5}^{s'}/s'v_2^{p^4+a_{3,5}}$. Easy computations show that

$$\begin{aligned}
&d_0(x(1))\equiv -3v_1^{a_{2,2}}v_2v_3^{(sp-1)p}\zeta_2, \qquad d_0(x(2))\equiv -3v_1^{a_{2,3}}v_3^{sp^3}t_1/v_2^{p^3},\\
&d_0(x(3))\equiv 3v_1^{a_{2,4}}v_3^{(sp-1)p^3+p}t_1/v_2^p
\end{aligned}$$

mod $v_1^{1+a_{2,k+1}}$, and so $N(k)=a_{2,k+1}$ $(1\le k\le 3)$. For $k\ge 4$, we define $x(k)$ inductively on k by $x(k)=x(k-1)^p-3v_1^{a_{2,k+1}-p}v_3^{(sp-1)p^k+p^{k-2}}/v_2^{p^{k-2}-1}$. Parallel calculation to the proof of Proposition 4.2.1 shows that

$$d_0(x(k))\equiv 3v_1^{a_{2,k+1}}v_3^{(sp-1)p^k+p^{k-2}}t_1/v_2^{p^{k-2}}$$

mod $(v_1^{1+a_{2,k+1}})$, and so we obtain $N(k)=a_{2,k+1}$. □

(ii) For $j\le p-2$. We set $x(1)$ to $X_0(v_2^j,X_{3,1}^s)^p+X_+(v_2^{jp},X_{3,1}^{sp})$. As is already computed in Proposition 4.1.9,

$$d_0(x(1))\equiv -j(j+1)v_1^{2p}X_{3,3}^{s'}t_1/s'v_2^{jp+2+a_{3,3}}-\binom{j+1}{2}v_1^{2p}X_{3,2}^{s}t_1^2/v_2^{jp+2}$$

mod (v_1^{2p+1}), and so $N(1)=2p$. For $k\ge 2$, we can construct $x(k)$ as in 5.2.5 (ii) and obtain $N(k)=2p^k+p^{k-1}-p^{k-2}$. □

5.2.5. FOR ODD $r \geq 3$ AND $s \in \mathbb{N}_0$.

(i) For $a_{3,r} - p \leq j \leq a_{3,r}$ $(j \neq a_{3,r} - p + 2)$. We set

$$x(1) = x(0)^p + (1-j)v_1^p v_2^{(a_{3,r}-j)p} v_3^{(sp-1)p^r} u_{2,2}.$$

Easy calculation shows that for $a_{3,r} - 1 \leq j \leq a_{3,r}$,

$$d_0(x(1)) \equiv (1-j)v_1^p v_2^{(a_{3,r}-j)p} v_3^{(sp-1)p^r} \zeta_2 \quad \bmod v_1^{p+1},$$

and for $a_{3,r} - p \leq j \leq a_{3,r} - 2$,

$$d_0(x(1)) \equiv (j-1)v_1^{p+1} v_2^{(a_{3,r}-j)p-1} v_3^{(sp-1)p^r} w_{2,3} \quad \bmod v_1^{p+2}.$$

For $j = a_{3,r}$ and $k \geq 2$, we can define $x(k)$ as in 5.2.3 (iii). For $j = a_{3,r} - 1$, we set $x(2)$ to $x(1)^p - 3/2v_1^{p^2-1} X(v_2^{(a_{3,r+1}-p+1)p-1}, X_{3,r+2}^s)$. Then $d_0(x(2))$ is congruent to

$$3v_1^{p^2+p-1} X_{3,r+2}^s (2t_1 - v_1\zeta_2)/2v_2^{(a_{3,r+1}-p+1)p} + 3v_1^{p^2+p} v_2^{p^2-p} v_3^{(sp-1)p^{r+1}} t_1^{p^2} u_{2,2}$$
$$(= -3/2DX_{[1,1]}(v_1^{p^2+p}, v_2^{(a_{3,r+1}-p+1)p}, X_{3,r+2}^s) + 3v_1^{p^2+p} Z_2/v_2^{2p+1})$$

mod $(v_1^{p^2+p+1})$ for some $Z_2 \in v_3^{-1}BP_*BP$, and so $N(2) = p^2+p-1$. Applying Propositions 4.2.3 and 4.2.6, we can define $x(k)$ inductively on $k \geq 3$ and obtain $N(k) = p^k + p^{k-1}$. On the other hand, we can apply Proposition 4.2.8 to $a_{3,r} - p \leq j \leq a_{3,r} - 2$ case and obtain $N(k) = P(k)$. □

(ii) Either for $j = a_{3,r} - p + 2$, or for $j \leq a_{3,r} - p - 1$ and $p \nmid (j+1)$. For $j = a_{3,r} - p - 2$, we set

$$x(1) = X_0(v_2^{a_{3,r}-p-2}, X_{3,r}^s)^p + X_+(v_2^{(a_{3,r}-p-2)p}, X_{3,r}^{sp}) + 2v_1^{2p} v_3^{(sp-1)p^r+p+2}/v_2^2.$$

Otherwise, we set

$$x(1) = X_0(v_2^j, X_{3,r}^s)^p + X_+(v_2^{jp}, X_{3,r}^{sp}).$$

We now use Proposition 4.1.9, to compute $d_0(x(1))$ mod (v_1^{2p+1}). For $j = a_{3,r} - p - 1$,

$$d_0(x(1)) \equiv -2\frac{v_1^{2p} X_{r+2}^{s'} t_1}{s' v_2^{b(r+2)-p^2-p+1}} - \frac{v_1^{2p} X_{3,r+1}^s t_1^2}{v_2^{a_{3,r+1}-p^2-p+2}} + 3v_1^{2p} v_2^{p^2-1} X_{3,r-1}^{(sp-1)p} \theta_2(v_2^{a_{3,r}-p-1}).$$

Otherwise,

$$d_0(x(1)) \equiv -j(j+1)\frac{v_1^{2p} X_{r+2}^{s'} t_1}{s' v_2^{jp+2+a_{3,r+2}}} - \binom{j+1}{2}\frac{v_1^{2p} X_{3,r+1}^s t_1^2}{v_2^{jp+2}}.$$

So $N(1) = 2p$. For $k = 2$, we set

$$x(2) = x(1)^p - \binom{j+1}{2} v_1^{2p^2-2} X(v_2^{(jp+2)p-2}, X_{3,r+2}^s).$$

Using Proposition 4.1.7, we now compute $d_0(x(2))$, getting

$$\begin{array}{lll} 3v_1^{2p^2} v_2^{p^3-p} v_3^{(sp-1)p^{r+1}} \theta_2(v_2^{a_{3,r}-p-1})^p & \bmod v_1^{2p^2+1} & \text{for } j = a_{3,r} - p - 1, \\ -\binom{j+1}{2} DX_{[2,2]}(v_1^{2p^2+p-1}, v_2^{(jp+2)p-1}, X_{3,r+2}^s) & \bmod v_1^{2p^2+p} & \text{otherwise.} \end{array}$$

Thus, $N(2)=2p^2$ for $j=a_{3,r}-p-1$, and $N(2)=2p^2+p-2$ for other cases. In the former case, we can construct $x(k)$ inductively on $k\geq 3$ using Proposition 4.2.7 and obtain $N(k)=2p^k$. In the latter case, Propositions 4.2.3 and 4.2.6 work well, and so we obtain $N(k)=2p^k+p^{k-1}-p^{k-2}$. □

(iii) For $j\leq a_{3,r}-2p$ and $p\mid(j+1)$. For $a_{3,r}-p^2\leq j\leq a_{3,r}-2p$, we have already shown that

$$d_0(x(0))\equiv -a(r)v_1^2v_2^{a_{3,r}-j-1}v_3^{(sp-1)p^{r-1}}w_{2,3}$$

mod (v_1^3) in 5.1.1 (ii). Then we can define $x(k)$ inductively on $k\geq 1$ using Proposition 4.2.8 and obtain $N(k)=p^k+P(k)$. For $j\leq a_{3,r}-p^2-p$, 5.1.1 (iii) and (iv) also show how to compute the congruence class of $d_0(x(0))$ mod p,v_1^{p+2}. For $p^2\nmid(j+p+1)$, we get

$$DX_{[j',1]}(v_1^{p+1},v_2^{j+1},X_{3,r}^s)+a(r)v_1^{p+1}v_2^{a_{3,r}-j-p-1}v_3^{(sp-1)p^{r-1}}\theta_1(v_2^j).$$

For $p^2\mid(j+p+1)$, we get

$$-DZ(v_1^{p+1},v_2^{j+1},X_{3,r}^s).$$

When $p^2|(j+p+1)$, we can construct $x(k)$ inductively on $k\geq 1$ using Proposition 4.2.6 and obtain $N(k)=p^{k+1}+p^k$. On the other hand, we set $x(1)$ to $x(0)^p+Y_{[j',1]}(v_1^{p^2+p},v_2^{(j+1)p},X_{3,r+1}^s)$ when $p^2\nmid(j+p+1)$. Then $d_0(x(1))$ is expressed as follows. For $p^2\nmid(j+1)$,

$$d_0(x(1))\equiv j'DZ(v_1^{p^2+p},v_2^{(j+1)p},X_{3,r+1}^s)\qquad \bmod v_1^{p^2+p+1}.$$

For $j=a_{3,r}-p^2-p$,

$$d_0(x(1))\equiv a(r)v_1^{p^2+p}v_2^{p^3-p}v_3^{(sp-1)p^{r+1}}\theta_1(v_2^{a_{3,r}-p^2-p})^p\qquad \bmod v_1^{p^2+p+1}.$$

For $p^2\mid(j+1)$ and $j\leq a_{3,r}-2p^2-p$,

$$d_0(x(1))\equiv v_1^{p^2+p+1}X_{3,r+1}^st_1^p/v_2^{(j+1)p+1}\qquad \bmod v_1^{p^2+p+2}.$$

When $p^2\nmid(j+1)$, we construct $x(k)$ inductively on $k\geq 2$ using Proposition 4.2.6 and obtain $N(k)=p^{k+1}+p^k$. When $j=a_{3,r}-p^2-p$, we can construct $x(k)$ inductively on $k\geq 2$ as in $j=a_{3,r}-p-1$ case and $k\geq 3$ (discussed in 5.2.5 (ii)), and obtain $N(k)=p^{k+1}+p^k$. When $p^2\mid(j+1)$ and $j\leq a_{3,r}-2p^2-p$, we set $x(2)$ to $x(1)^p-v_1^{p^3+p^2+p}v_3X_{3,r+1}^{sp}/v_2^{(j+1)p^2+p+1}$. Since

$$\begin{aligned}d_0(&-v_1^{p^3+p^2+p}v_3X_{3,r+1}^{sp}/v_2^{(j+1)p^2+p+1})\\ &\equiv -v_1^{p^3+p^2+p}X_{3,r+2}^s/v_2^{(j+1)p^2+p}\left\{(t_1^{p^2}-v_2^{p-1}t_1)+v_1v_2^{p-1}w_{2,2}\right\}\end{aligned}$$

mod $(v_1^{p^3+p^2+p+2})$, we have

$$d_0(x(2))\equiv -DX_{[1,0]}(v_1^{p^3+p^2+p+1},v_2^{(j+1)p^2+1},X_{3,r+2}^s),$$

and so $N(2)=p^3+p^2+p$. Applying Propositions 4.2.3 and 4.2.6, we can define $x(k)$ inductively on $k\geq 3$ and obtain $N(k)=p^{k+1}+p^k+p^{k-1}+p^{k-2}$. Finally we have completed the computations for all cases. □

Now we can prove our main theorem.

Proof of Theorem. For each $\mathbb{F}_p$-basic element $(x_{3,r}^s/v_2^j)^{p^k}$ of Lemma 2.3.3, we have already constructed the element $x(k)$ $(= x(sp^r/j;k))$, which satisfies the congruence $x(sp^r/j;k) \equiv (x_{3,r}^s/v_2^j)^{p^k} \bmod (v_1)$, and determined the v_1-divisibility $N(k)$ $(= N(s,r,j;k))$ of $x(k)$ as the smallest integer with $d_0(x(k)) \not\equiv 0 \bmod (v_1^{N(k)+1})$. We also define $y(mp^r)$ by $y(mp^r) = 1/X_{2,r}^m$ and obtain $N(m;r) = a_{2,r}$ (Corollary 3.3.5). Using these results, let B^0 be the direct sum of the cyclic modules as in Proposition 2.4.1. The linear independence of the set

$$\{\delta(x(sp^r/j;k)/v_1^{N(s,r,j;k)})\} \cup \{\delta(1/v_1^{a_{2,r}}X_{2,r}^m)\}$$

is verified by checking with condition (2.5.2). This completes the proof. □

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