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The C^*-Algebras of Row-Finite Graphs

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Abstract. We prove versions of the fundamental theorems about Cuntz-Krieger algebras for the C^*-algebras of row-finite graphs: directed graphs in which each vertex emits at most finitely many edges. Special cases of these results have previously been obtained using various powerful machines; our main point is that direct methods yield sharper results more easily.

Contents

In the last few years various authors have considered analogues of the Cuntz-Krieger algebras associated to infinite directed graphs. In [12] and [11] these *graph C^*-algebras* were studied using a groupoid model and the deep results of Renault on the ideal structure of groupoid C^*-algebras; in [16] and [10] they were viewed as the Cuntz-Pimsner algebras of appropriate Hilbert bimodules, as introduced in [15]. Because of the technical requirements of these general theories, it has usually been assumed that the graphs are locally finite, in the sense that every vertex receives and emits at most finitely many edges, and that the graphs do not have sinks. However, it was pointed out in [11] that to make sense of the Cuntz-Krieger relations in a C^*-algebra, one merely needs to insist that the graph is *row-finite*: each vertex emits at most finitely many edges.

Here we shall prove versions of the fundamental theorems about Cuntz-Krieger algebras for the C^*-algebras of row-finite graphs, and use them to give a new description of the primitive ideal spaces of graph C^*-algebras. We prove a uniqueness

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theorem like that of [7] whenever every loop has an exit [11], and find a parametrisation of the ideals like that of [6] and [9] whenever the graph satisfies Condition (K) of [12]: every vertex lies on either no loops or at least two loops. Both theorems apply to graphs with sinks; this new generality is important because it has been shown in [18] that the Cuntz-Krieger algebras of all infinite graphs and matrices can be approximated by the algebras of finite graphs with sinks. Our description of the primitive ideal space applies to any row-finite graph satisfying Condition (K).

To achieve the extra generality in the fundamental theorems, we use direct arguments rather than the machinery of groupoid or Cuntz-Pimsner algebras. Many of the techniques can be traced back to the original papers of Cuntz and Krieger [7, 6], but they have been reworked and refined many times since then, and we have been pleasantly surprised to discover how cleanly the arguments have emerged. Even those who are only interested in the Cuntz-Krieger algebras of finite $\{0,1\}$-matrices should find our arguments much easier than the original ones. To describe the primitive ideal spaces of graph algebras, on the other hand, we have had to develop new methods, because the arguments used in [9] depended heavily on finiteness of the vertex set. Once again, though, the result can be elegantly expressed in graph-theoretic terms.

We begin in §1 by recalling the basic definitions from [11] and setting up our notation, and prove a couple of technical lemmas which can be ignored by those interested only in finite graphs without sinks. The second of these lemmas shows how to reduce questions about graphs with sinks to graphs without sinks; it is curious to note that even for finite graphs with sinks, the reduction involves infinite graphs. Our approach to the general theory follows that of [9]. Thus the graph algebra $C^*(E)$ of a directed graph E is by definition universal for Cuntz-Krieger E-families, and the first main theorem says that this C^*-algebra is uniquely characterised by the existence of a canonical action of $\mathbb{T}$ called the gauge action (Theorem 2.1; compare [9, Theorem 2.3]). This *gauge-invariant uniqueness theorem* allows us to establish many of the basic properties of graph algebras without any extra hypotheses on the graph. That there is such a theorem will not be surprising to those familiar with crossed products $B \rtimes \mathbb{Z}$ and their generalisations: it is of interest here because for many years authors have assumed that their $\{0,1\}$-matrices A satisfied Condition (I) of [7] merely to ensure that the Cuntz-Krieger algebras $\mathcal{O}_A$ were well-defined, and now we can see that such hypotheses are required only if one needs uniqueness when there is no obvious gauge action. As an example of this, we use the gauge-invariant uniqueness theorem to show that the C^*-algebras of a graph and its dual are always canonically isomorphic, improving a result of [8].

We prove a generalisation of the full uniqueness theorem of Cuntz and Krieger in §3. While our result is slightly more general than [11, Theorem 3.7], and in potentially important ways, we believe the main interest lies in the clarity and directness of its proof. The same is true of the next section, in which we analyse the ideal structure of graph algebras. As in [9], we first use the gauge-invariant uniqueness theorem to analyse the gauge-invariant ideals in $C^*(E)$ without extra hypotheses on E; it is then relatively easy to deduce from the Cuntz-Krieger uniqueness theorem that these are all the ideals when E satisfies the analogue (K) of Cuntz's Condition (II) introduced in [12].

In §5, we characterise the graphs which have simple and purely infinite C^*-algebras. Our criterion for simplicity follows from the analysis of ideals in §4.

To prove infiniteness, we use arguments like those of §3 to plug into the standard program of, for example, [2] or [13]; in retrospect, our proof is similar to that of [10, §5], but is expressed in more elementary terms. We close in §6 with our description of the primitive ideal space of $C^*(E)$ when E satisfies (K), which is in Theorem 6.3 and Corollary 6.5.

1. The C^*-algebras of graphs

A directed graph $E = (E^0, E^1, r, s)$ consists of countable sets E^0 of vertices and E^1 of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. The graph is *row-finite* if each vertex emits at most finitely many edges. We write E^n for the set of paths $\mu = \mu_1\mu_2\cdots\mu_n$ of length $|\mu| := n$; that is, sequences of edges μ_i such that $r(\mu_i) = s(\mu_{i+1})$ for $1 \le i < n$. The maps r, s extend to $E^* := \bigcup_{n\ge 0} E^n$ in an obvious way, and s extends to the set E^∞ of infinite paths $\mu = \mu_1\mu_2\cdots$.

Let E be a row-finite (directed) graph. A *Cuntz-Krieger E-family* in a C^*-algebra B consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ satisfying the *Cuntz-Krieger relations*

$$s_e^* s_e = p_{r(e)} \text{ for } e \in E^1 \quad\text{and}\quad p_v = \sum_{\{e : s(e) = v\}} s_e s_e^* \text{ whenever } v \text{ is not a sink.}$$

We shall typically use small letters $\{s_e, p_v\}$ for Cuntz-Krieger families in a C^*-algebra and large letters $\{S_e, P_v\}$ for Cuntz-Krieger families of operators on Hilbert space.

It is proved in [11, Theorem 1.2] that there is a C^*-algebra $C^*(E)$ generated by a universal Cuntz-Krieger E-family $\{s_e, p_v\}$; in other words, for every Cuntz-Krieger E-family $\{t_e, q_v\}$ in a C^*-algebra B, there is a homomorphism $\pi = \pi_{t,q} : C^*(E) \to B$ such that $\pi(s_e) = t_e$ and $\pi(p_v) = q_v$ for all $e \in E^1$, $v \in E^0$. Since it is easy to construct families $\{S_e, P_v\}$ in which all the operators are non-zero, we have $p_v \ne 0$ for all $v \in E^0$; a product $s_\mu := s_{\mu_1}s_{\mu_2}\ldots s_{\mu_n}$ is non-zero precisely when $\mu = \mu_1\mu_2\cdots\mu_n$ is a path in E^n. Since the Cuntz-Krieger relations imply that the range projections $s_e s_e^*$ are also mutually orthogonal, we have $s_e^* s_f = 0$ unless $e = f$, and words in $\{s_e, s_f^*\}$ collapse to products of the form $s_\mu s_\nu^*$ for $\mu, \nu \in E^*$ satisfying $r(\mu) = r(\nu)$. (See [11, Lemma 1.1] for some specific formulas.) Indeed, because the family $\{s_\mu s_\nu^*\}$ is closed under multiplication and involution, we have

$$C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}. \tag{1.1}$$

We adopt the conventions that vertices are paths of length 0, that $s_v := p_v$ for $v \in E^0$, and that all paths μ, ν appearing in (1.1) are non-empty; we recover s_μ, for example, by taking $\nu = r(\mu)$, so that $s_\mu s_\nu^* = s_\mu p_{r(\mu)} = s_\mu$.

If $z \in \mathbb{T}$, then the family $\{z s_e, p_v\}$ is another Cuntz-Krieger E-family which generates $C^*(E)$, and the universal property gives a homomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(s_e) = z s_e$ and $\gamma_z(p_v) = p_v$. The homomorphism $\gamma_{\overline{z}}$ is an inverse for γ_z, so $\gamma_z \in \operatorname{Aut} C^*(E)$, and a routine $\epsilon/3$ argument using (1.1) shows that γ is a strongly continuous action of $\mathbb{T}$ on $C^*(E)$. It is called the *gauge action.* Because $\mathbb{T}$ is compact, averaging over γ with respect to normalised Haar measure gives an expectation Φ of $C^*(E)$ onto the fixed-point algebra $C^*(E)^\gamma$:

$$\Phi(a) := \int_{\mathbb{T}} \gamma_z(a)\,dz \quad \text{for} \quad a \in C^*(E).$$

The map Φ is positive, has norm 1, and is faithful in the sense that $\Phi(a^*a) = 0$ implies $a = 0$.

When we adapt arguments from finite graphs to infinite ones, formulas which involve sums of projections may contain infinite sums. To make sense of these, we use strict convergence in the multiplier algebra of $C^*(E)$:

Lemma 1.1. *Let E be a row-finite graph, let A be a C^*-algebra generated by a Cuntz-Krieger E-family $\{t_e, q_v\}$, and let $\{p_n\}$ be a sequence of projections in A. If $p_n t_\mu t_\nu^*$ converges for every $\mu, \nu \in E^*$, then $\{p_n\}$ converges strictly to a projection $p \in M(A)$.*

Proof. Since we can approximate any $a \in A = \pi_{t,q}(C^*(E))$ by a linear combination of $t_\mu t_\nu^*$, an $\epsilon/3$-argument shows that $\{p_n a\}$ is Cauchy for every $a \in A$. We define $p : A \to A$ by $p(a) := \lim_{n\to\infty} p_n a$. Since

$$b^* p(a) = \lim_{n\to\infty} b^* p_n a = \lim_{n\to\infty} (p_n b)^* a = p(b)^* a,$$

the map p is an adjointable operator on the Hilbert C^*-module A_A, and hence defines (left multiplication by) a multiplier p of A [17, Theorem 2.47]. Taking adjoints shows that $ap_n \to ap$ for all a, so $p_n \to p$ strictly. It is easy to check that $p^2 = p = p^*$. □

It will be important in applications that we allow our graphs to have sinks (see [18]), but it is technically easy to reduce to the case where there are no sinks. Notice, though, that even if we start with finite graphs, this reduction gives us infinite graphs.

By *adding a tail at a vertex w* we mean adding a graph of the form

(1.2)

$$\underset{w}{\bullet} \xrightarrow{e_1} \underset{v_1}{\bullet} \xrightarrow{e_2} \underset{v_2}{\bullet} \xrightarrow{e_3} \underset{v_3}{\bullet} \xrightarrow{e_4} \underset{v_4}{\bullet} \cdots$$

to E to form a new graph F; thus $F^0 := E^0 \cup \{v_i : 1 \le i < \infty\}$, $F^1 := E^1 \cup \{e_i : 1 \le i < \infty\}$, and r, s are extended to F^1 by $r(e_i) = v_i$, $s(e_i) = v_{i-1}$ and $s(e_1) = w$. When we add tails to sinks in E we have put exactly one edge out of each sink and new vertex, so it is easy to extend Cuntz-Krieger E-families to Cuntz-Krieger families for the larger graph F, and $C^*(E)$ embeds as a full corner in $C^*(F)$. The next Lemma makes this precise.

Lemma 1.2. *Let F be a directed graph obtained by adding a tail at each sink of a graph E.*

(a) *For each Cuntz-Krieger E-family $\{S_e, P_v\}$ on a Hilbert space $\mathcal{H}_E$, there is a Hilbert space $\mathcal{H}_F = \mathcal{H}_E \oplus \mathcal{H}_T$ and a Cuntz-Krieger F-family $\{T_e, Q_v\}$ such that $T_e = S_e$ for $e \in E^1$, $Q_v = P_v$ for $v \in E^0$, and $\sum_{v \notin E^0} Q_v$ is the projection on $\mathcal{H}_T$.*

(b) *If $\{T_e, Q_v\}$ is a Cuntz-Krieger F-family, then $\{T_e, Q_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family. If w is a sink in E such that $Q_w \neq 0$, then $Q_v \neq 0$ for every vertex v on the tail attached to w.*

(c) *If $\{t_e, q_v\}$ are the canonical generators of $C^*(F)$, then the homomorphism $\pi_{t,q}$ corresponding to the Cuntz-Krieger E-family $\{t_e, q_v : e \in E^1, v \in E^0\}$ is an isomorphism of $C^*(E)$ onto the C^*-subalgebra of $C^*(F)$ generated by $\{t_e, q_v : e \in E^1, v \in E^0\}$, which is the full corner in $C^*(F)$ determined by the projection $p := \sum_{v\in E^0} q_v$.*

Proof. To get the gist of the argument, we just add one tail; say we add (1.2) to a sink w. To extend $\{S_e, P_v\}$, we let $\mathcal{H}_T$ be the direct sum of infinitely many copies of $P_w\mathcal{H}_E$, define P_{v_i} to be the projection onto the ith summand, and let S_{e_i} be the identity map of the ith summand onto the $(i-1)$st, with S_{e_1} taking the first summand in $\mathcal{H}_T$ onto $P_w\mathcal{H} \subset \mathcal{H}_E$. This gives (a); because we have not changed $\{e : s(e) = v\}$ for any vertex v at which a Cuntz-Krieger E-relation for p_v applies, the extended family is a Cuntz-Krieger F-family. For the same reason, throwing away the extra elements of a Cuntz-Krieger F-family gives a Cuntz-Krieger E-family. The last statement in (b) holds because

$$\begin{aligned} S_{e_1}S_{e_1}^* = P_w \neq 0 &\Longrightarrow S_{e_2}S_{e_2}^* = P_{v_1} = S_{e_1}^*S_{e_1} \neq 0 \\ &\Longrightarrow S_{e_3}S_{e_3}^* = P_{v_2} = S_{e_2}^*S_{e_2} \neq 0 \Longrightarrow \cdots \end{aligned}$$

For the first part of (c), just use part (a) to see that every representation of $C^*(E)$ factors through a representation of $C^*(F)$.

We still have to show that the image of $C^*(E)$ is a full corner. We first claim that the series $\sum_{v\in E^0} q_v$ converges strictly in $M(C^*(F))$ to a projection p. To see this, order E^0, and set $p_n := \sum_{i=1}^n q_{v_i}$. Then for any $\mu, \nu \in F^*$ we have

$$p_n t_\mu t_\nu^* = \begin{cases} t_\mu t_\nu^* & \text{if } s(\mu) = v_i \text{ for some } i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

If $s(\mu) \in E^0$, then $s(\mu) = v_i$ for some i and $p_n t_\mu t_\nu^* = t_\mu t_\nu^*$ for $n \ge i$; if $s(\mu) \notin E^0$, then $p_n t_\mu t_\nu^* = 0$ for all n. Thus for fixed μ, ν the sequence $\{p_n t_\mu t_\nu^*\}$ is eventually constant, and Lemma 1.1 implies that $\{p_n\}$ converges strictly to a projection $p \in M(C^*(F))$ satisfying

$$p t_\mu t_\nu^* = \begin{cases} t_\mu t_\nu^* & \text{if } s(\mu) \in E^0, \\ 0 & \text{if } s(\mu) \notin E^0. \end{cases}$$

It follows from this formula that the corner $pC^*(F)p$ is precisely the image of $C^*(E)$.

To see that $pC^*(F)p$ is full, suppose J is an ideal in $C^*(F)$ containing $pC^*(F)p$. Then certainly J contains $\{q_v : v \in E^0\}$. If v is a vertex in the tail attached to w, then there is a unique path α with $s(\alpha) = w$ and $r(\alpha) = v$, and

$$q_w \in J \Longrightarrow t_\alpha = q_w t_\alpha \in J \Longrightarrow q_v = t_\alpha^* t_\alpha \in J.$$

Thus all the generators $\{t_e, q_v\}$ of $C^*(F)$ lie in J, $J = C^*(F)$, and $pC^*(F)p$ is full. □

2. The gauge-invariant uniqueness theorem

Theorem 2.1. *Let E be a row-finite directed graph, let $\{S_e, P_v\}$ be a Cuntz-Krieger E-family, and let $\pi = \pi_{S,P}$ be the representation of $C^*(E)$ such that $\pi(s_e) = S_e$ and $\pi(p_v) = P_v$. Suppose that each P_v is non-zero, and that there is a strongly continuous action β of $\mathbb{T}$ on $C^*(S_e, P_v)$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for $z \in \mathbb{T}$. Then π is faithful.*

To prove the theorem, we have to show that

(a) π is faithful on the fixed-point algebra $C^*(E)^\gamma$, and

(b) $\big\|\pi\big(\int_{\mathbb{T}}\gamma_z(a)\,dz\big)\big\| \leq \|\pi(a)\|$ for all $a \in C^*(E)$;

see [4, Lemma 2.2]. To establish (a), we need to analyse the structure of $C^*(E)^\gamma$; this analysis will be used again in §3. For each vertex v, we consider

$$\mathcal{F}_k(v) := \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu,\nu \in E^k, r(\mu) = r(\nu) = v\}.$$

When $|\nu| = |\alpha| = k$, we have

$$s_\nu^* s_\alpha = \begin{cases} p_{r(\nu)} & \text{if } \nu = \alpha \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Since $s_\mu p_v s_\beta^* = s_\mu s_\beta^*$ when $r(\mu) = r(\beta) = v$, it follows that the elements $s_\mu s_\nu^*$ are non-zero matrix units parametrised by pairs in $\{\mu \in E^k : r(\mu) = v\}$. Thus $\mathcal{F}_k(v)$ is isomorphic to the algebra $\mathcal{K}(\mathcal{H}_v)$ of compact operators on a possibly-infinite-dimensional Hilbert space $\mathcal{H}_v$. When the paths all have length k, we have $s_\mu s_\nu^* s_\alpha s_\beta^* = 0$ for $r(\nu) \neq r(\alpha)$, so the subalgebras $\{\mathcal{F}_k(v) : v \in E^0\}$ are mutually orthogonal, and

$$\mathcal{F}_k := \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu,\nu \in E^k\}$$

decomposes as a C^*-algebraic direct sum $\bigoplus_{v\in E^0}\mathcal{F}_k(v)$ of copies of the compact operators. If $r(\mu) = r(\nu) = v$ and v is not a sink, the Cuntz-Krieger relations give

$$s_\mu s_\nu^* = s_\mu p_v s_\nu^* = \sum_{\{e\in E^1 : s(e)=v\}} s_\mu(s_e s_e^*)s_\nu^* = \sum_{\{e\in E^1 : s(e)=v\}} s_{\mu e}s_{\nu e}^*,$$

so $\mathcal{F}_k \subset \mathcal{F}_{k+1}$.

Lemma 2.2. *When E does not have sinks, $C^*(E)^\gamma = \overline{\bigcup_{k\geq 0}\mathcal{F}_k}$.*

Proof. Since $\gamma_z(s_\mu s_\nu^*) = z^{|\mu|-|\nu|}s_\mu s_\nu^*$, we have $\mathcal{F}_k \subset C^*(E)^\gamma$ for all k. On the other hand, we can approximate any element a of $C^*(E)^\gamma$ by a finite sum $\sum_{\mu,\nu\in F}\lambda_{\mu,\nu}s_\mu s_\nu^*$. Now the continuity of $\Phi : b \mapsto \int \gamma_z(b)\,dz$ implies that

$$\begin{aligned} a = \Phi(a) \sim \Phi\Big(\sum_{\mu,\nu\in F}\lambda_{\mu,\nu}s_\mu s_\nu^*\Big) &= \sum_{\mu,\nu\in F}\lambda_{\mu,\nu}\Big(\int_{\mathbb{T}} z^{|\mu|-|\nu|}\,dz\Big)s_\mu s_\nu^* \\ &= \sum_{\mu,\nu\in F,\ |\mu|=|\nu|}\lambda_{\mu,\nu}s_\mu s_\nu^*, \end{aligned}$$

which belongs to $\mathcal{F}_k$ for $k = \max\{|\mu| : \mu \in F\}$. Thus $a \in \overline{\bigcup_{k\geq 0}\mathcal{F}_k}$, and $C^*(E)^\gamma \subset \overline{\bigcup_{k\geq 0}\mathcal{F}_k}$. □

Now suppose that E does have sinks. For each sink w and $k \in \mathbb{N}$, we still have a copy $\mathcal{F}_k(w)$ of the compact operators, but now there is no Cuntz-Krieger relation for p_w and $\mathcal{F}_k(w)$ does not embed in $\mathcal{F}_{k+1}$. However, $\mathcal{F}_k(w)$ is orthogonal to $\mathcal{F}_{k+1}(w)$ and to every other $\mathcal{F}_k(v)$ (this follows from the relations in [11, Lemma 1.1]). Hence we use instead of $\mathcal{F}_k$ the subalgebra

$$\mathcal{G}_k := \Big(\bigoplus_{v\text{ is not a sink}}\mathcal{F}_k(v)\Big) \oplus \Big(\bigoplus_{w\text{ is a sink}}\bigoplus_{i=0}^{k}\mathcal{F}_i(w)\Big).$$

The argument of Lemma 2.2 carries over to give:

Lemma 2.3. *For every row-finite graph, $C^*(E)^\gamma = \overline{\bigcup_{k\geq 0}\mathcal{G}_k}$.*

Corollary 2.4. *If E is a row-finite graph and $\{S_e, P_v\}$ is a Cuntz-Krieger E-family in which each P_v is non-zero, then the representation $\pi = \pi_{S,P}$ is faithful on $C^*(E)^\gamma$.*

Proof. For any ideal I in $C^*(E)^\gamma$, we have $I = \overline{\bigcup_{k\geq 0}(I \cap \mathcal{G}_k)}$ by, for example, [1, Lemma 1.3]; thus it is enough to prove that π is faithful on each $\mathcal{G}_k$. Each $\mathcal{G}_k$ is the direct sum of simple algebras of the form $\mathcal{F}_i(v)$, so it is enough to prove that each non-zero summand contains an element which is not mapped to zero under π. But if μ is any path with $r(\mu) = v$, then S_μ is a partial isometry with initial projection $S_\mu^* S_\mu = P_v \neq 0$, so $s_\mu s_\mu^* \in \mathcal{F}_{|\mu|}(v)$ satisfies $\pi(s_\mu s_\mu^*) = S_\mu S_\mu^* \neq 0$. □

Proof of Theorem 2.1. The Corollary gives (a), and (b) follows by averaging over β:

$$\|\pi(\Phi(a))\| \leq \int_{\mathbb{T}} \|\pi(\gamma_z(a))\|\,dz = \int_{\mathbb{T}} \|\beta_z(\pi(a))\|\,dz = \int_{\mathbb{T}} \|\pi(a)\|\,dz = \|\pi(a)\|.$$

Thus the result follows from [4, Lemma 2.2]. □

For our application, let $\widehat{E}$ be the dual graph of E defined by $\widehat{E}^0 = E^1$,

$$\widehat{E}^1 = \{(e,f) : e,f \in E^1 \text{ and } r(e) = s(f)\}$$

and $\widehat{r}(e,f) = f$, $\widehat{s}(e,f) = e$. It is trivial to check that $\widehat{E}$ is row-finite if E is. For finite graphs whose incidence matrices satisfy (I), the next result is in [8], and was later rediscovered in [14, Proposition 4.1]. There is an interesting generalisation in [3].

Corollary 2.5. *Let E be a row-finite directed graph with no sinks, and let $\{s_e, p_v\}$, $\{t_{e,f}, q_e\}$ be the canonical generating Cuntz-Krieger families for $C^*(E)$, $C^*(\widehat{E})$. Then there is an isomorphism ϕ of $C^*(\widehat{E})$ onto $C^*(E)$ such that*

$$\phi(t_{e,f}) = s_e s_f s_f^* \quad \textit{and} \quad \phi(q_e) = s_e s_e^*. \tag{2.2}$$

Proof. One can easily verify that $T_{e,f} := s_e s_f s_f^*$ and $Q_e := s_e s_e^*$ form a Cuntz-Krieger $\widehat{E}$-family in $C^*(E)$, and thus the universal property of $C^*(\widehat{E})$ gives a homomorphism $\phi = \pi_{T,Q} : C^*(\widehat{E}) \to C^*(E)$ satisfying (2.2). Because the gauge action γ^E on $C^*(E)$ satisfies $\gamma_z^E(T_{e,f}) = zT_{e,f}$ and $\gamma_z^E(Q_e) = Q_e$, the maps $\gamma_z^E \circ \phi$ and $\phi \circ \gamma_z^{\widehat{E}}$ agree on generators; since both are (automatically continuous) homomorphisms of C^*-algebras, they must agree on all of $C^*(\widehat{E})$. Thus Theorem 2.1 implies that ϕ is an isomorphism. □

3. The Cuntz-Krieger uniqueness theorem

Theorem 3.1. *Suppose that E is a row-finite directed graph in which every loop has an exit, and that $\{S_e, P_v\}$, $\{T_e, Q_v\}$ are two Cuntz-Krieger E-families in which all the projections P_v and Q_v are non-zero. Then there is an isomorphism ϕ of $C^*(S_e, P_v)$ onto $C^*(T_e, Q_v)$ such that $\phi(S_e) = T_e$ and $\phi(P_v) = Q_v$ for all $e \in E^1$ and $v \in E^0$.*

We first claim that we may as well assume that E has no sinks. For suppose it does have sinks, and that we have proved the theorem for graphs without sinks. Let F be the graph obtained by adding tails to each sink of E; since we have not added any loops, all loops in F have exits. By Lemma 1.2, we can extend $\{S_e, P_v\}$ and $\{T_e, Q_v\}$ to Cuntz-Krieger F-familes in which all the projections are non-zero. Applying the theorem to these families gives an isomorphism which in particular takes S_e to T_e and P_v to Q_v, and hence restricts to an isomorphism of $C^*(S_e, P_v)$ onto $C^*(T_e, Q_v)$. Thus we can suppose that E has no sinks.

We shall prove the theorem by showing that the representations $\pi_{S,P}$ and $\pi_{T,Q}$ of $C^*(E)$ are faithful; then $\phi := \pi_{T,Q} \circ \pi_{S,P}^{-1}$ is the required isomorphism. By symmetry, it is enough to show that $\pi_{S,P}$ is faithful. As in §2, it is enough by [4, Lemma 2.2] to show that

(a) π is faithful on $C^*(E)^\gamma$, and

(b) $\big\|\pi\big(\int_{\mathbb{T}} \gamma_z(a)\,dz\big)\big\| \le \|\pi(a)\|$ for $a \in C^*(E)$.

Since we are supposing that E has no sinks, we have already proved (a) in Corollary 2.4.

Before considering (b), we need a lemma.

Lemma 3.2. *Suppose E has no sinks and every loop in E has an exit. Then for every vertex v there is an infinite path λ in E such that $s(\lambda) = v$ and $\beta\lambda \neq \lambda$ for every finite path β.*

Proof. First suppose there is a finite path μ with $s(\mu) = v$ whose range vertex $r(\mu)$ is the starting point of distinct loops α and β. Then

$$\lambda := \mu\alpha\beta\alpha\alpha\beta\beta\alpha\alpha\alpha\beta\beta\beta\cdots$$

will do the job. If there is no such path μ, then we can construct a path λ which does not pass through the same vertex twice: we just take an exit from a loop whenever one is available, and we can never return. (See the proof of [11, Lemma 3.4] for more details.) □

Proof of Theorem 3.1. Recall that E now has no sinks, and that it is enough to prove (b) for a in the dense subspace $\operatorname{span}\{s_\mu s_\nu^*\}$. So suppose F is a finite subset of $E^* \times E^*$ and $a = \sum_{(\mu,\nu)\in F} \lambda_{\mu,\nu} s_\mu s_\nu^*$. The idea is to find a projection Q such that compressing by Q does not change the norm of $\pi(\Phi(a))$ but kills the terms in $\pi(a)$ for which $|\mu| \neq |\nu|$; we will then have

$$\|\pi(\Phi(a))\| = \|Q\pi(\Phi(a))Q\| = \|Q\pi(a)Q\| \le \|\pi(a)\|. \tag{3.1}$$

For $k := \max\{|\mu|, |\nu| : (\mu,\nu) \in F\}$, we have

$$\Phi(a) = \sum_{\{(\mu,\nu)\in F : |\mu| = |\nu|\}} \lambda_{\mu,\nu} s_\mu s_\nu^* \in \mathcal{F}_k;$$

since there are no sinks, we may suppose by applying the Cuntz-Krieger relations and changing F that $\min\{|\mu|, |\nu|\} = k$ for every pair $(\mu,\nu) \in F$ with $\lambda_{\mu,\nu} \neq 0$. (So that, if $\lambda_{\mu,\nu} \neq 0$ and $|\mu| = |\nu|$, then $|\mu| = |\nu| = k$.) Since $\mathcal{F}_k$ decomposes as a direct sum $\bigoplus_v \mathcal{F}_k(v)$, so does its image under π, and there is a vertex v such that

$$\begin{aligned}\|\pi(\Phi(a))\| &= \Big\|\sum_{\{(\mu,\nu)\in F:|\mu|=|\nu|,r(\mu)=v\}} \lambda_{\mu,\nu}\pi(s_\mu s_\nu^*)\Big\| \\ &= \Big\|\sum_{\{(\mu,\nu)\in F:|\mu|=|\nu|,r(\mu)=v\}} \lambda_{\mu,\nu} S_\mu S_\nu^*\Big\|.\end{aligned}$$

By Lemma 3.2 there is an infinite path λ^∞ such that $s(\lambda^\infty) = v$ and $\beta\lambda^\infty \neq \lambda^\infty$ for all finite paths β; since F is finite, we can truncate λ^∞ to obtain a finite path λ such that $\mu\lambda$ does not have the form $\lambda\alpha$ for any subpath μ of any path in F. With this choice of λ, the sum

$$Q := \sum_{\{\tau\in E^k:r(\tau)=v\}} S_{\tau\lambda}S_{\tau\lambda}^*$$

converges strictly to a projection Q in $M(C^*(S_e, P_v))$. (Because the partial sums are all projections, it is enough by Lemma 1.1 to notice that the partial sums of $\big(\sum S_{\tau\lambda}S_{\tau\lambda}^*\big)S_\alpha S_\beta^*$ are eventually constant for every $\alpha, \beta \in E^*$.) Observe that whenever $r(\tau) = v$, $S_{\tau\lambda}$ is a partial isometry with initial projection $P_{r(\lambda)}$, and hence is non-zero by hypothesis.

If $|\alpha| = |\beta| = k$ and $r(\alpha) = r(\beta) = v$, then

$$QS_\alpha S_\beta^* Q = S_\alpha S_\lambda S_\lambda^* P_{r(\alpha)} P_{r(\beta)} S_\lambda S_\lambda^* S_\beta = S_{\alpha\lambda}S_{\beta\lambda}^* \neq 0.$$

We verify using the identities $S_\gamma^* S_\delta = \delta_{\gamma,\delta}P_{r(\gamma)}$ for paths of equal length that

$$\{QS_\alpha S_\beta^* Q : |\alpha| = |\beta| = k \text{ and } r(\alpha) = r(\beta) = v\}$$

is a family of matrix units parametrised by pairs in $\{\alpha \in E^k : r(\alpha) = k\}$; since we just showed that all these matrix units are non-zero, we deduce that $b \mapsto Q\pi(b)Q$ is a faithful representation of $\mathcal{F}_k(v) \cong \mathcal{K}(\mathcal{H}_v)$. Since both π and $Q\pi Q$ are faithful on $\mathcal{F}_k(v)$, we have $\|\pi(b)\| = \|Q\pi(b)Q\|$ for all $b \in \mathcal{F}_k(v)$, and in particular for

$$b = \sum_{\{(\mu,\nu)\in F:|\mu|=|\nu|,r(\mu)=v\}} \lambda_{\mu,\nu} s_\mu s_\nu^*.$$

We conclude that $\|\pi(\Phi(a))\| = \|Q\pi(\Phi(a))Q\|$.

We next claim that $Q\pi(\Phi(a))Q = Q\pi(a)Q$. For this, we fix $(\mu,\nu) \in F$ such that $|\mu| \neq |\nu|$; notice that unless $r(\mu) = r(\nu)$, the product $s_\mu s_\nu^*$ is zero. If $r(\mu) = r(\nu) \neq v$, then $S_{\tau\lambda}S_{\tau\lambda}^* S_\mu = 0$ for every summand $S_{\tau\lambda}S_{\tau\lambda}^*$ of Q. So suppose $r(\mu) = r(\nu) = v$. One of μ, ν has length k and the other is longer; say $|\mu| = k$ and $|\nu| > k$. Then

$$S_{\tau\lambda}S_{\tau\lambda}^* S_\mu = \begin{cases} S_\mu S_\lambda S_\lambda^* & \text{if } \tau = \mu \\ 0 & \text{otherwise,}\end{cases}$$

so

$$QS_\mu S_\nu^* Q = \sum_{\{\tau\in E^k:r(\tau)=v\}} S_{\mu\lambda}S_{\nu\lambda}^* S_{\tau\lambda}S_{\tau\lambda}^*.$$

Since $|\nu| > |\tau|$, this can only have a non-zero summand if $\nu = \tau\nu'$ for some ν'. But then $S_{\nu\lambda}^* S_{\tau\lambda} = S_{\nu'\lambda}^* S_\lambda$ is only non-zero if $\nu'\lambda$ has the form $\lambda\alpha$, which is impossible by choice of λ. We deduce that $QS_\mu S_\nu^* Q = 0$ when $|\mu| \neq |\nu|$, or equivalently that $Q\pi(\Phi(a))Q = Q\pi(a)Q$.

Putting all this together shows that (3.1) holds, and we are done. $\square$

4. Ideals in graph algebras

Our description of the ideals in a graph algebra $C^*(E)$ is a direct generalisation of [12, Theorem 6.6]. It therefore differs slightly from the description in [6] and [9], where the ideals are completely determined by a preorder on the set of loops in E; in infinite graphs we have to take into account infinite tails as well as loops. So, as in [12], we phrase our results in terms of a preorder on the vertex set E^0.

Let E be a directed graph. Define a relation on E^0 by setting $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. This relation is transitive, but is not typically a partial order; for example, $v \geq w \geq v$ whenever v and w lie on the same loop. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set H is *saturated* if every vertex which feeds into H and only into H is again in H; that is, if

$$s^{-1}(v) \neq \emptyset \text{ and } \{r(e) : s(e) = v\} \subset H \implies v \in H.$$

The *saturation* of a hereditary set H is the smallest saturated subset $\overline{H}$ of E^0 containing H; the saturation $\overline{H}$ is itself hereditary.

Theorem 4.1. *Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph. For each subset H of E^0, let I_H be the ideal in $C^*(E)$ generated by $\{p_v : v \in H\}$.*

(a) *The map $H \mapsto I_H$ is an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of closed gauge-invariant ideals of $C^*(E)$.*

(b) *Suppose H is saturated and hereditary. If $F^0 := E^0 \setminus H$, $F^1 := \{e \in E^1 : r(e) \notin H\}$, and $F = F(E \setminus H) := (F^0, F^1, r, s)$, then $C^*(E)/I_H$ is canonically isomorphic to $C^*(F)$.*

(c) *If X is any hereditary subset of E^0, $G^1 := \{e \in E^1 : s(e) \in X\}$, and $G := (X, G^1, r, s)$, then $C^*(G)$ is canonically isomorphic to the subalgebra $C^*(s_e, p_v : e \in G^1, v \in X)$ of $C^*(E)$, and this subalgebra is a full corner in the ideal I_X.*

We are particularly pleased with our proof of Theorem 4.1, which avoids both the heavy machinery used in [12] and the subtle approximate identity arguments used in [6] and [9]. The key improvement occurs when we show that we can recover a saturated hereditary set H from the ideal I_H as $\{v : p_v \in I_H\}$: our short argument makes it very clear why we need to assume that H is saturated and hereditary. We begin with a couple of Lemmas.

Lemma 4.2. *Let I be an ideal in a graph C^*-algebra $C^*(E)$. Then $H := \{v \in E^0 : p_v \in I\}$ is a saturated hereditary subset of E^0.*

Proof. Suppose $v \in H$ and $v \geq w$, so that there is a path $\mu \in E^*$ such that $s(\mu) = v$ and $r(\mu) = w$. Then

$$p_v \in I \implies s_\mu = p_v s_\mu \in I \implies p_w = s_\mu^* s_\mu \in I,$$

so H is hereditary. If $w \in E^0$ satisfies $\{r(e) : s(e) = w\} \subset H$, then $\{s_e : s(e) = w\} \subset I$ and $p_w = \sum_{s(e)=w} s_e s_e^*$ belongs to I; thus H is saturated. □

Lemma 4.3. *If H is a hereditary subset of E^0, then*

$$I_H = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \in \overline{H}\}. \tag{4.1}$$

In particular, this implies that $I_H = I_{\overline{H}}$ and that I_H is gauge-invariant.

Proof. Following [9, Lemma 3.1], we first note that the Cuntz-Krieger relations imply that $\{v \in E^0 : p_v \in I_H\}$ is a saturated set, and which therefore contains $\overline{H}$. Thus the right-hand side J of (4.1) is contained in I_H. Any non-zero product of the form $(s_\mu s_\nu^*)(s_\alpha s_\beta^*)$ collapses to another of the form $s_\gamma s_\delta^*$; from an examination of the various possibilities for γ and δ, and the hereditary property of $\overline{H}$, we deduce that J is an ideal. Since J certainly contains the generators of I_H, we deduce that $J = I_H$. The last two remarks follow easily. □

Proof of Theorem 4.1. We begin by showing that $H \mapsto I_H$ is onto. Let I be a non-zero gauge-invariant ideal in $C^*(E)$, and set $H := \{v \in E^0 : p_v \in I\}$, which is saturated and hereditary by Lemma 4.2. Since $I_H \subset I$, $p_v \notin I$ implies $p_v \notin I_H$, and I and I_H contain exactly the same set of projections $\{p_v : v \in H\}$. Let $F = F(E \backslash H)$ be the graph of part (b), and let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger F-family generating $C^*(F)$. Both quotients $C^*(E)/I$ and $C^*(E)/I_H$ are generated by Cuntz-Krieger F-families in which all the projections are non-zero, and, since both I and I_H are gauge-invariant, both quotients carry gauge actions. Thus two applications of Theorem 2.1 show that there are isomorphisms $\phi : C^*(F) \to C^*(E)/I$ and $\psi : C^*(F) \to C^*(E)/I_H$ such that $\phi(\{t_e, q_v\}) = \{s_e + I, p_v + I\}$ and $\psi(\{t_e, q_v\}) = \{s_e + I_H, p_v + I_H\}$. But now $\phi \circ \psi^{-1}$ is an isomorphism of $C^*(E)/I_H$ onto $C^*(E)/I$ which agrees with the quotient map on generators; thus the quotient map is an isomorphism, and $I = I_H$.

To see that the map $H \mapsto I_H$ is injective, we have to show that if H is saturated and hereditary, then the corresponding set $\{v : p_v \in I_H\}$ is precisely H. We trivially have that $v \in H$ implies $p_v \in I_H$. For the converse, consider the graph $F = F(E \setminus H)$ of (b), and choose a Cuntz-Krieger F-family $\{S_e, P_v\}$ with all the projections P_v non-zero (for example, the canonical generating family for $C^*(F)$). Setting $P_v = 0$ for $v \in H$ and $S_e = 0$ when $r(e) \in H$ extends this to a Cuntz-Krieger E-family: to see this, we need to use that H is hereditary to get the Cuntz-Krieger relation at vertices in H, and that H is saturated to see that there are no vertices in $F^0 = E^0 \setminus H$ at which a new Cuntz-Krieger relation is being imposed (in other words, that all the sinks of F are also sinks in E). The universal property of $C^*(E)$ gives a homomorphism $\pi : C^*(E) \to C^*(S_e, P_v)$, which vanishes on I_H because it kills all the generators $\{p_v : v \in H\}$. But $\pi(p_v) = P_v \neq 0$ for $v \notin H$, so $v \notin H$ implies $p_v \notin I_H$. Thus $\{v : p_v \in I_H\} = H$, as required.

We have now shown that $H \mapsto I_H$ is bijective. Since it preserves containment, it is a lattice isomorphism, and we have proved (a). Since $H = \{v : p_v \in I_H\}$, the quotient $C^*(E)/I_H$ is generated by a Cuntz-Krieger F-family with all projections non-zero, which is isomorphic to $C^*(F)$ by Theorem 2.1.

For (c), we fix a hereditary subset X of E^0, and define $q_X := \sum_{v \in X} p_v$ using Lemma 1.1. We claim that $q_X I_{\overline{X}} q_X$ is generated by the Cuntz-Krieger G-family $\{s_e, p_v : s(e), v \in X\}$. Certainly this family lies in the corner; on the other hand, if $r(\alpha) = r(\beta) \in \overline{X}$, then $q_X(s_\alpha s_\beta^*)q_X = 0$ unless α and β both start in X. Thus the claim is verified, and Theorem 2.1 implies that $q_X I_{\overline{X}} q_X$ is isomorphic to $C^*(G)$. To see that the corner is full, suppose J is an ideal in $I_{\overline{X}}$ containing $q_X I_{\overline{X}} q_X$. Then Lemma 4.2 implies that $\{v : p_v \in J\}$ is a saturated set containing X, and hence containing $\overline{X}$; but this implies that J contains all the generators of $I_{\overline{X}}$, and hence is all of $I_{\overline{X}}$. □

To obtain a version of Theorem 4.1 which describes all the ideals of $C^*(E)$, we need to impose conditions on the graph E. Loosely speaking, we need to know that the uniqueness Theorem 3.1 is valid in every subgraph $F = F(E \setminus H)$ associated to the complement of a saturated hereditary subset H (cf. Theorem 4.1(b)). The appropriate condition was formulated in [12] as Condition (K). For $i = 0$, $i = 1$ and $i = 2$, let E^0_i denote the set of vertices v for which there are, respectively, no loops, precisely one loop, or at least two distinct loops based at v. Then E satisfies Condition (K) if $E^0 = E^0_0 \cup E^0_2$. Since the property "every loop based at v has an exit" is vacuously satisfied at vertices in E^0_0, and since every loop lies entirely within or without a hereditary set, Theorem 3.1 applies to every subgraph $F = F(E \setminus H)$.

If E satisfies (K) we can follow the first paragraph in the proof of Theorem 4.1 using Theorem 3.1 in place of Theorem 2.1, and deduce that every ideal I in $C^*(E)$ has the form I_H for some saturated hereditary subset H of E^0. Thus all the ideals in $C^*(E)$ are gauge-invariant, and Theorem 4.1 gives the following mild improvement on [12, Theorem 6.6].

Theorem 4.4. *Suppose E is a row-finite directed graph which satisfies Condition* (K). *Then $H \mapsto I_H$ is an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of ideals in $C^*(E)$.*

Remark 4.5. The hypothesis "every loop has an exit" was called Condition (L) in [11]; its relation to (K) is exactly the same as that of the Cuntz-Krieger condition (I) to (II). If E satisfies (K), so does each subgraph $F(E \setminus H)$ associated to a saturated hereditary set H. The weaker Condition (L), on the other hand, does not pass to subgraphs: a loop in E which misses H could have all its exits heading into H, and then the corresponding loop in F has no exit in F.

5. Simplicity and pure infiniteness

As in [12], we can use our classification of ideals to characterise the graphs whose C^*-algebras are simple. Recall from [12] that a graph is *cofinal* if every vertex v connects to every infinite path λ: there exists $n \geq 1$ such that $v \geq r(\lambda_n)$. (Unfortunately the proof of [12, Corollary 6.8] is incomplete: the same direction was proved twice. However, the missing direction is not difficult, as we shall see.)

Proposition 5.1. *Let E be a row-finite directed graph with no sinks. Then $C^*(E)$ is simple if and only if E is cofinal and every loop has an exit.*

Proof. First suppose E is cofinal and every loop has an exit. Suppose v is a vertex on a loop α. There is an exit e from α, and by applying cofinality to the path $\alpha\alpha\alpha\cdots$ we see that there must be a return path from $r(e)$ to α, which gives a second loop based at v. Thus E satisfies (K), and Theorem 4.4 applies.

We next claim that every saturated hereditary subset H is empty or all of E^0. Suppose there is a vertex v which is not in H. Because H is saturated, we can construct inductively an infinite path λ with $s(\lambda) = v$ and $r(\lambda_n) \notin H$ for all n. If $w \in H$, then the cofinality implies that w connects to some $r(\lambda_n)$, which is impossible because H is hereditary and $r(\lambda_n) \notin H$. Thus H must be empty, as claimed. Now Theorem 4.4 implies that the only non-zero ideal in $C^*(E)$ is $C^*(E)$ itself, and $C^*(E)$ is simple.

For the converse, we suppose that $C^*(E)$ is simple and prove first that E is cofinal. Let $\lambda \in E^\infty$ and $v \in E^0$. Then $H_\lambda := \{w : w \not\geq r(\lambda_n) \text{ for all } n\}$ is a saturated hereditary set, which is certainly not all of E^0 because $r(\lambda_n) \notin H_\lambda$. On the other hand, if H_λ were non-empty then I_{H_λ} would then be a proper ideal by Theorem 4.1; hence $H_\lambda = \emptyset$. In particular, v is not in H_λ, and hence connects to λ.

Next we suppose that $C^*(E)$ is simple and prove that every loop in E has an exit. Suppose α is a loop with no exit. Then the vertices on α form a hereditary set H, whose saturation $\overline{H}$ must be all of E^0 (or $I_{\overline{H}}$ would be a proper ideal). Thus if we set $G^1 := \{e \in E^1 : s(e) \in H\}$ and $G := (H, G^1, r, s)$, then Theorem 4.1(c) implies that $C^*(G)$ is a full corner in $C^*(E)$. But since α has no exit, G is a simple loop, and [9, Lemma 2.4] implies that $C^*(G) \cong C(\mathbb{T}, M_{|H|}(\mathbb{C}))$, which is impossible since $C^*(E)$ and hence also $C^*(G)$ are simple. Thus α must have an exit. □

Remark 5.2. When E has sinks, the concept of cofinality is inappropriate. Since simplicity is preserved by passing to full corners, one can test for simplicity by adding tails and applying Proposition 5.1 to the enlarged graph F. Notice, though, that $C^*(E)$ cannot be simple if E has more than one sink: one sink in E is not connected in F to the tail attached to another, and hence F is not cofinal.

Proposition 5.3. *Suppose E is a row-finite directed graph in which every vertex connects to a loop and every loop has an exit. Then $C^*(E)$ is purely infinite.*

For the proof we need a simple lemma.

Lemma 5.4. *Let $w \in E^0$ and let t be a positive element of $\mathcal{F}_k(w)$. Then there is a projection r in the C^*-subalgebra of $\mathcal{F}_k(w)$ generated by t such that $rtr = \|t\|r$.*

Proof. We know from §2 that $\mathcal{F}_k(w)$ is spanned by the matrix units $\{s_\mu s_\nu^*\}$ where μ, ν run through the set $S := \{\mu \in E^k : r(\mu) = w\}$, and hence the map $\pi : \sum c_{\mu\nu} s_\mu s_\nu^* \mapsto (c_{\mu\nu})$ is an isomorphism of $\mathcal{F}_k(w)$ onto $\mathcal{K}(\ell^2(S))$. Since $\pi(t)$ is a positive compact operator it has an eigenvector with eigenvalue $\|\pi(t)\| = \|t\|$ (by [5, Lemma 5.9], for example), and we can take r to be the element $\pi^{-1}(R)$ corresponding to the projection R onto the span of this eigenvector. □

Proof of Proposition 5.3. We have to show that every hereditary subalgebra A of $C^*(E)$ contains an infinite projection; we shall produce one which is dominated by a fixed positive element $a \in A$ whose average $\Phi(a) \in C^*(E)^\gamma$ has norm 1. Choose a finite sum $b = \sum_i c_i s_{\mu^i} s_{\nu^i}^*$ in $C^*(E)$ such that $b \geq 0$ and $\|a - b\| < 1/4$. Then $b_0 := \Phi(b)$ satisfies $\|b_0\| \geq 3/4$ and $b_0 \geq 0$. We may suppose by applying the Cuntz-Krieger relations a few times that there is a fixed $k \in \mathbb{N}$ such that $\min(|\mu^i|, |\nu^i|) = k$ for all i, and then $b_0 \in \bigoplus_{\{w : w = r(\mu^i)\}} \mathcal{F}_k(w)$. In fact $\|b_0\|$ must be attained in some summand $\mathcal{F}_k(w)$; let b_1 be the component of b_0 in $\mathcal{F}_k(w)$, and note that $b_1 \geq 0$ and $\|b_1\| = \|b_0\|$. By Lemma 5.4 there is a projection $r \in C^*(b_1) \subset \mathcal{F}_k(w)$ such that $rb_1r = \|b_1\|r$. Since b_1 is a finite sum of $s_{\mu^i} s_{\nu^i}^*$ and $r \in C^*(b_1)$, we can write r as a sum $\sum c_{\mu\nu} s_\mu s_\nu^*$ over all pairs of paths in

$$S = \{\mu \in E^k : \mu = \mu^i \text{ or } \nu^i \text{ for some } i, \text{ and } r(\mu) = w\};$$

notice that the $S \times S$-matrix $(c_{\mu\nu})$ is also a projection.

Now let λ^∞ be an infinite path with $s(\lambda^\infty) = w$ and $\lambda^\infty \neq \beta\lambda^\infty$ for any finite path β (see Lemma 3.2). Since there are only finitely many summands in b, we can truncate λ^∞ to obtain a finite path λ with $s(\lambda) = w$ such that λ is not the initial segment of $\beta\lambda$ for any finite segment β of any μ^i or ν^i. Then because $\{s_{\mu\lambda} s_{\nu\lambda}^*\}$ is also a family of nonzero matrix units parametrised by $S\times S$, $q := \sum_{\mu,\nu\in S} c_{\mu\nu}s_{\mu\lambda}s^*_{\nu\lambda}$ is a projection, and

$$r=\sum c_{\mu\nu}s_\mu s^*_\nu=\sum c_{\mu\nu}s_\mu(s_\lambda s^*_\lambda+(p_w-s_\lambda s^*_\lambda))s^*_\nu\ge q.$$

Our choice of λ ensures that $qs_{\mu^i}s^*_{\nu^i}q=0$ unless $r(\mu^i)=r(\nu^i)=w$ and $|\mu^i|=|\nu^i|=k$. Since $q\le r$, we have

$$qbq=qb_0q=qb_1q=qrb_1rq=\|b_1\|rq=\|b_0\|q\ge\tfrac{3}{4}q.$$

Because $\|a-b\|\le\frac14$, we have $qaq\ge qbq-\frac14 q\ge\frac12 q$, so qaq is invertible in $qC^*(E)q$. Let c denote its inverse, and put $v=c^{1/2}qa^{1/2}$. Then $vv^*=c^{1/2}qaqc^{1/2}=q$, and $v^*v=a^{1/2}qcqa^{1/2}\le\|c\|a$, so v^*v belongs to the hereditary subalgebra A.

To finish off, we show that v^*v is an infinite projection. By hypothesis, there is a path β such that $s(\beta)=r(\lambda)$ and $v:=r(\beta)$ lies on a loop α; we may as well suppose that α has an exit e with $s(e)=v$ (otherwise replace v by the source of an exiting edge). Then

$$p_v=s^*_\alpha s_\alpha\sim s_\alpha s^*_\alpha\le s_{\alpha_1}s^*_{\alpha_1}<s_{\alpha_1}s^*_{\alpha_1}+s_es^*_e\le p_v,$$

so p_v is infinite. But if μ is any path with $|\mu|=k$ and $r(\mu)=w=s(\lambda)$, then $\mu\lambda\beta\alpha$ is a path with range v, so

$$p_v=s^*_{\mu\lambda\beta\alpha}s_{\mu\lambda\beta\alpha}\sim s_{\mu\lambda\beta\alpha}s^*_{\mu\lambda\beta\alpha}\le s_{\mu\lambda}s^*_{\mu\lambda},$$

which is a minimal projection in the matrix algebra $\operatorname{span}\{s_{\mu\lambda}s^*_{\nu\lambda}:\mu,\nu\in S\}$, and hence is equivalent to a subprojection of q. Thus q is infinite too. Since $q=vv^*\sim v^*v$, this completes the proof. □

Remark 5.5. The converse of Proposition 5.3 is also true: if $C^*(E)$ is purely infinite, then every vertex connects to a loop and every loop has an exit. The argument in the third and fourth paragraphs of [11, page 172] works for row-finite graphs and is elementary.

Remark 5.6. One can deduce from Propositions 5.1 and 5.3 a more general version of the dichotomy of [11, Corollary 3.10]: if $C^*(E)$ is simple, then it is either AF or purely infinite. For if E has no loops, Theorem 2.4 of [11] says that $C^*(E)$ is AF. (Note that the proof of [11, Theorem 2.4] is elementary.) If E does have loops, Proposition 5.1 says they all have exits (we can apply this argument to the larger graph F if E has sinks), and that E is cofinal; thus every vertex connects to every loop and Proposition 5.3 applies.

6. The primitive ideal space

In this section we describe the primitive ideal space of the C^*-algebra of a graph E which satisfies Condition (K). Our description will necessarily look quite different from its analogue in [9] for finite graphs, because new phenomena arise in infinite graphs: in particular, they need not contain any loops or sinks. We shall indicate at the end how [9, Proposition 4.1] may be deduced from our analysis.

We know from Theorem 4.4 that the ideals all have the form I_H for some saturated hereditary subset H of E^0, so our first problem is to determine the sets H for which I_H is primitive (or equivalently, for which I_H is prime). Interestingly, it is easier to describe the complements of these sets. To begin with, we shall assume that E has no sinks, and later extend our results using Lemma 1.2.

Proposition 6.1. *Let E be a row-finite graph with no sinks which satisfies* (K), *and suppose $H \subset E^0$. Then H is a saturated hereditary subset of E^0 such that I_H is primitive if and only if $\gamma := E^0 \setminus H$ is non-empty and satisfies*

(a) *for every $v_1, v_2 \in \gamma$ there exists $z \in \gamma$ such that $v_1 \geq z$ and $v_2 \geq z$;*

(b) *for every $v \in \gamma$ there is an edge e with $s(e) = v$ and $r(e) \in \gamma$; and*

(c) *$v \geq w$ and $w \in \gamma$ imply $v \in \gamma$.*

The proof needs a lemma which allows us to get our hands on elements of saturations.

Lemma 6.2. *Suppose F is a directed graph with no sinks and $v \in F^0$. If $y \in \overline{\{x \in F^0 : v \geq x\}}$, then there exists $z \in F^0$ such that $v \geq z$ and $y \geq z$.*

Proof. First note that $L_v := \{x \in F^0 : v \geq x\}$ is hereditary, so its saturation is by definition the smallest saturated set containing L_v. Suppose K is any saturated set containing L_v. Then $K_1 := \{w \in K : w \geq x \text{ for some } x \in L_v\}$ contains L_v; we claim that it is saturated. For suppose $z \in F^0$ and $r(e) \in K_1$ for all edges e with $s(e) = z$. Then $z \in K$ because K is saturated. Since there is at least one edge e with $s(e) = z$, and since we then have $r(e) \geq x$ for some $x \in L_v$ because $r(e) \in K_1$, we have $z \geq x$ for some $x \in L_v$. Thus $z \in K_1$, and K_1 is saturated, as claimed. Thus if K is the *smallest* saturated set containing L_v, then $K = \{w \in K : w \geq x \text{ for some } x \in L_v\}$. □

Proof of Proposition 6.1. First suppose that $\gamma \subset E^0$ satisfies (a), (b) and (c). From (c) we see immediately that $H := E^0 \setminus \gamma$ is hereditary, and from (b) that $H = E^0 \setminus \gamma$ is saturated. To see that I_H is prime, suppose I_1, I_2 are ideals in $C^*(E)$ such that $I_1 \cap I_2 \subset I_H$. Theorem 4.4 implies that there are saturated sets H_i such that $I_i = I_{H_i}$, and that $I_{H_1 \cap H_2} = I_{H_1} \cap I_{H_2}$. Thus $I_1 \cap I_2 \subset I_H$ implies $H_1 \cap H_2 \subset H$. If $H_1 \not\subset H$ and $H_2 \not\subset H$, there are vertices $v_i \in H_i \setminus H$. By (a), there exists $v \in \gamma$ such that $v_1 \geq v$ and $v_2 \geq v$. Then $v \in H_1 \cap H_2$ because the H_i are hereditary, and $v \notin H$ because $\gamma = E^0 \setminus H$; this contradicts $H_1 \cap H_2 \subset H$. Thus either $H_1 \subset H$ or $H_2 \subset H$ and $I_1 = I_{H_1} \subset I_H$ or $I_2 = I_{H_2} \subset I_H$. This shows that I_H is prime, and hence primitive.

Next we suppose that H is saturated and hereditary, and I_H is primitive. The complement of any saturated set satisfies (c) and, because E has no sinks, $\gamma := E^0 \setminus H$ also satisfies (b). We prove (a) by passing to the quotient $C^*(E)/I_H$, which by Theorem 4.1 is isomorphic to $C^*(F(E \setminus H))$. Because I_H is primitive in $C^*(E)$, $\{0\}$ is primitive in $C^*(F(E \setminus H))$. Suppose $v_1, v_2 \in E^0 \setminus H$. Then $H_i := \{x \in E^0 \setminus H : v_i \geq x\}$ are non-empty hereditary subsets of $E^0 \setminus H = F(E \setminus H)^0$. Since $\{0\}$ is prime in $C^*(F(E \setminus H))$, we must have $I_{\overline{H_1}} \cap I_{\overline{H_2}} \neq \{0\}$, and Theorem 4.1 implies that $\overline{H_1} \cap \overline{H_2} \neq \emptyset$. Say $y \in \overline{H_1} \cap \overline{H_2}$. Applying the Lemma to $F(E \setminus H)$ and v_1 shows there exists $x \in E^0 \setminus H$ such that $y \geq x$ and $v_1 \geq x$ in $F(E \setminus H)$. Since $y \in \overline{H_2}$ and $\overline{H_2}$ is hereditary, we have $x \in \overline{H_2}$, and another application of the Lemma gives $z \in E^0 \setminus H$ satisfying $x \geq z$ and $v_2 \geq z$. We now have $v_1 \geq x \geq z$ and $v_2 \geq z$ in $F(E \setminus H)$. Thus we have proved that $E^0 \setminus H$ satisfies (a). □

We shall call a subset γ of E^0 satisfying Conditions (a), (b) and (c) of Proposition 6.1 a *maximal tail*; the word "tail" is meant to convey the sense of Conditions (a) and (b), and "maximal" that of Condition (c). We denote by χ_E the set of maximal tails in E (whether or not E has sinks).

For subsets K, L of E^0, we write $K \geq L$ to mean that for each $v \in K$, there exists $w \in L$ such that $v \geq w$. Thus Condition (c) of Proposition 6.1 says that "$v \geq \gamma \Longrightarrow v \in \gamma$". In view of (c), we can describe the saturated hereditary set H_γ corresponding to $\gamma \in \chi_E$ as either $H_\gamma = E^0 \setminus \gamma$ or $H_\gamma = \{v : v \not\geq \gamma\}$; this second description makes our parametrisation of $\operatorname{Prim} C^*(E)$ look more like that of [9, Proposition 4.1].

Theorem 6.3. *Let E be a row-finite directed graph which satisfies Condition* (K) *and has no sinks. Then there is a topology on the set χ_E of maximal tails in E such that*

$$\overline{S} = \big\{\delta \in \chi_E : \delta \geq \textstyle\bigcup_{\gamma\in S}\gamma\big\}$$

for $S \subset \chi_E$, and then $\gamma \mapsto I_{H_\gamma}$ is a homeomorphism of χ_E onto $\operatorname{Prim} C^*(E)$.

Proof. We verify that the operation $S \mapsto \overline{S}$ satisfies Kuratowski's closure axioms. The axiom $\overline{\emptyset} = \emptyset$ is trivially true. That $S \subset \overline{S}$ is trivial. We then have $\overline{S} \subset \overline{\overline{S}}$. If $\delta \in \overline{\overline{S}}$, then for every vertex $v \in \delta$ there exist $\epsilon \in \overline{S}$ and $w \in \epsilon$ such that $v \geq w$. But $\epsilon \in \overline{S}$, so there exist $\gamma \in S$ and $z \in \gamma$ such that $w \geq z$, and then $v \geq z$. Thus $v \geq \bigcup_{\gamma \in S}\gamma$ for all $v \in \delta$, and we have $\delta \in \overline{S}$.

For $S, T \subset \chi_E$, we trivially have $\overline{S} \subset \overline{S \cup T}$, $\overline{T} \subset \overline{S \cup T}$ and $\overline{S} \cup \overline{T} \subset \overline{S \cup T}$, so to see that $\overline{S} \cup \overline{T} = \overline{S \cup T}$ it suffices to prove $\overline{S \cup T} \subset \overline{S} \cup \overline{T}$. Let $\delta \in \overline{S \cup T}$, and set

$$\delta_S := \{v \in \delta : v \geq \textstyle\bigcup_{\gamma\in S}\gamma\}, \quad \delta_T := \{v \in \delta : v \geq \bigcup_{\gamma\in T}\gamma\}.$$

Then $\delta = \delta_S \cup \delta_T$; we claim that δ is either δ_S or δ_T. If not, there exist $w \in \delta_S \setminus \delta_T$ and $v \in \delta_T \setminus \delta_S$. Because δ is a tail, there is a vertex $z \in \delta$ such that $w \geq z$ and $v \geq z$. Then $z \in \delta_S$ or $z \in \delta_T$, and either leads to a contradiction; for example, if $z \in \delta_S$, then $v \geq z$ implies $v \in \delta_S$. Thus δ must be either δ_S or δ_T, as claimed, and this is just a convoluted way of saying that $\delta \in \overline{S}$ or $\delta \in \overline{T}$.

We have now verified that the closure operation $S \mapsto \overline{S}$ does define a topology on χ_E. Theorem 4.4 and Proposition 6.1 imply that $I : \gamma \mapsto I_{H_\gamma}$ is a bijection of χ_E onto $\operatorname{Prim} C^*(E)$. To see that I is a homeomorphism, we let S be a subset of χ_E, and show that $I(\overline{S}) = \overline{I(S)}$. Because all tails in χ_E are maximal, $\delta \geq \bigcup_{\gamma\in S}\gamma$ if and only if $\delta \subset \bigcup_{\gamma\in S}\gamma$, and hence

$$\begin{aligned} I(\overline{S}) &= \{I_{H_\delta} : \delta \subset \textstyle\bigcup_{\gamma\in S}\gamma\} \\ &= \{I_{H_\delta} : H_\delta \supset \textstyle\bigcap_{\gamma\in S} H_\gamma\} \\ &= \{I_{H_\delta} : I_{H_\delta} \supset I_{\cap_{\gamma\in S} H_\gamma}\}. \end{aligned}$$

Now because $H \mapsto I_H$ is order-preserving and bijective, general nonsense shows that $I_{\cap H_\gamma} = \bigcap I_{H_\gamma}$; thus

$$I(\overline{S}) = \{I_{H_\delta} : I_{H_\delta} \supset \textstyle\bigcap_{\gamma\in S} I_{H_\gamma}\} = \overline{I(S)},$$

and I is a homeomorphism. □

Remark 6.4. Finding maximal tails in E is easy: just take the vertices on any infinite path and toss in the vertices which connect to the path. In other words, let $x \in E^\infty$ and take

$$\gamma := \{v \in E^0 : v \geq r(x_n) \text{ for some } n \geq 1\}.$$

Two paths x and y give the same maximal tail if and only if for every $n \geq 1$ there exist j, k such that $r(x_n) \geq r(y_j)$ and $r(y_n) \geq r(x_k)$.

To describe $\operatorname{Prim} C^*(E)$ when E has sinks, we apply Theorem 6.3 to the graph F obtained by adding a tail T_v at every sink v, as in Lemma 1.2. Each sink v gives a maximal tail

$$\gamma_v := T_v \cup \{w \in E^0 : w \geq v\},$$

in χ_F, and $\chi_F = \chi_E \cup \{\gamma_v : v \text{ is a sink in } E\}$. Since the full corner $pC^*(F)p = C^*(E)$ is Morita equivalent to $C^*(F)$ via the imprimitivity bimodule $X := pC^*(F)$, it follows from Theorem 6.3 and [17, Corollary 3.33] that the map $\gamma \mapsto X\text{–}\operatorname{Ind} I_{H_\gamma}$ is a homeomorphism of χ_F onto $\operatorname{Prim} C^*(E)$.

To get a more concrete description of this homeomorphism, we first note that if pAp is a full corner in a C^*-algebra A and I is an ideal in A, then by [17, Proposition 3.24] we have

$$pA\text{–}\operatorname{Ind} I = \overline{\operatorname{span}}\{{}_{pAp}\langle pA \cdot I, pA\rangle\} = p(AIA)p = pIp.$$

Applying this to I_{H_γ} and using the description of I_{H_γ} in Lemma 4.3 gives

$$pC^*(F)\text{–}\operatorname{Ind} I_{H_\gamma} = \overline{\operatorname{span}}\{ps_\alpha s_\beta^* p : r(\alpha) = r(\beta) \not\geq \gamma\}.$$

Now $ps_\alpha s_\beta^* p = 0$ unless $s(\alpha)$ and $s(\beta)$ are in E^0 and $r(\alpha) = r(\beta)$; if $r(\alpha) \in F^0 \setminus E^0$, say $r(\alpha) \in T_v$, then we can write $\alpha = \alpha'\alpha''$ with $\alpha' \in E^*$ and $r(\alpha') = v$, and $r(\alpha'') = r(\beta'')$ forces $\alpha'' = \beta''$, $s_{\alpha''}s_{\beta''}^* = p_v$, and $ps_\alpha s_\beta^* p = s_{\alpha'}s_{\beta'}^*$. Truncating α at v does not affect whether or not $r(\alpha) \not\geq \gamma$, so

$$pC^*(F)\text{–}\operatorname{Ind} I_{H_\gamma} = \overline{\operatorname{span}}\{s_{\alpha'}s_{\beta'}^* : \alpha', \beta' \in E^* \text{ and } r(\alpha') = r(\beta') \not\geq E^0 \cap \gamma\}.$$

Thus if we let $\lambda_v := \{w \in E^0 : w \geq v\}$ and set

$$\Lambda_E := \chi_E \cup \{\lambda_v : v \text{ is a sink in } E\},$$

then

$$\lambda \mapsto I_{H_\lambda} := \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \not\geq \lambda\}$$

is a bijection of Λ_E onto $\operatorname{Prim} C^*(E)$. To sum up:

Corollary 6.5. *Suppose E is a row-finite graph which satisfies Condition* (K). *Then there is a topology on Λ_E such that*

$$\overline{S} = \left\{\delta \in \Lambda_E : \delta \geq \textstyle\bigcup_{\lambda \in S}\lambda\right\}$$

for $S \subset \Lambda_E$, and then the map $\lambda \mapsto I_{H_\lambda}$ is a homeomorphism of Λ_E onto $\operatorname{Prim} C^*(E)$.

Remark 6.6. If E is a finite graph with no sinks and α is an equivalence class in the set Γ_E described in [9], then $\gamma_\alpha := \{v \in E^0 : v \geq \alpha\}$ belongs to χ_E. We claim that $\alpha \mapsto \gamma_\alpha$ is a homeomorphism of Γ_E onto χ_E. To see that it is injective, note that $\alpha \subset \gamma_\alpha$, and hence $\gamma_\alpha \subset \gamma_\beta$ if and only if $\alpha \geq \beta$. To see that it is surjective, let $\gamma \in \chi_E$, and note that a class $\beta \in \Gamma_E$ is either contained in γ or entirely misses γ. Let α be a minimal element of $\{\beta \in \Gamma_E : \beta \subset \gamma\}$; in fact, there is a unique such α because γ is a tail, and we have $\gamma = \gamma_\alpha$. The map $\alpha \mapsto \gamma_\alpha$ is easily seen to preserve the closure operation, and hence is a homeomorphism, as claimed. Thus we recover [9, Proposition 4.1] from Theorem 6.3.

At this stage, however, we have been unable to find a satisfactory extension of [9, Theorem 4.7] to describe the primitive ideal space of the C^*-algebra of an arbitrary row-finite graph.

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