New York Journal of Mathematics

New York J. Math. **6** (2000) 227–236.

The Three Point Pick Problem on the Bidisk

Jim Agler and John E. McCarthy

Abstract. We prove that a non-degenerate extremal 3 point Pick problem on the bidisk always has a unique solution.

Contents

0. Introduction

The original *Pick problem* is to determine, given N points $\lambda_1, \dots, \lambda_N$ in the unit disk $\mathbb{D}$ and N complex numbers $w_1, \dots, w_N$, whether there exists a function ϕ in the closed unit ball of $H^\infty(\mathbb{D})$ (the space of bounded analytic functions on $\mathbb{D}$) that maps each point λ_i to the corresponding value w_i. This problem was solved by G. Pick in 1916 [9], who showed that a necessary and sufficient condition is that the Pick matrix

$$\left(\frac{1 - \bar{w}_i w_j}{1 - \bar{\lambda}_i \lambda_j} \right)_{i,j=1}^{N}$$

be positive semi-definite.

It is well-known that if the problem is *extremal*, i.e., the problem can be solved with a function of norm one but not with a function of any smaller norm, then the Pick matrix is singular, and the corresponding solution is a unique Blaschke product, whose degree equals the rank of the Pick matrix [6, 5].

In [1], the first author extended Pick's theorem to the space $H^\infty(\mathbb{D}^2)$, the bounded analytic functions on the bidisk; see also [4, 3, 2]. It was shown in [2] that if the problem has a solution, then it has a solution that is a rational inner

Received March 23, 2000.

Mathematics Subject Classification. Primary: 32Q45, 32D15; Secondary: 46J15.

Key words and phrases. bidisk, H^∞, Pick interpolation, Nevanlinna-Pick.

The first author was partially supported by the National Science Foundation.

ISSN 1076-9803/00

function; however the qualitative properties of general solutions are not fully understood. The example $\lambda_1 = (0,0)$, $\lambda_2 = (\frac{1}{2}, \frac{1}{2})$, $w_1 = 0$, $w_2 = \frac{1}{2}$ shows that even extremal problems do not always have unique solutions.

The two point Pick problem on the bidisk is easily analyzed. It can be solved if and only if the Kobayashi distance between λ_1 and λ_2 is greater than or equal to the hyperbolic distance between w_1 and w_2. On the bidisk, the Kobayashi distance is just the maximum of the hyperbolic distance between the first coordinates, and the hyperbolic distance between the second coordinates. A pair of points in $\mathbb{D}^2$ is called *balanced* if the hyperbolic distance between their first coordinates equals the hyperbolic distance between their second coordinates.

The two point Pick problem has a unique solution if and only if the Kobayashi distance between λ_1 and λ_2 exactly equals the hyperbolic distance between w_1 and w_2, and moreover (λ_1, λ_2) is not balanced. In this case the solution is a Möbius map in the coordinate function in which the Kobayashi distance is attained. If the distance between λ_1 and λ_2 equals the distance between w_1 and w_2, but the pair (λ_1, λ_2) is balanced, then the function ϕ will be uniquely determined on the geodesic disk passing through λ_1 and λ_2, but will not be unique off this disk. (For the example $\lambda_1 = (0,0)$, $\lambda_2 = (\frac{1}{2}, \frac{1}{2})$, $w_1 = 0$, $w_2 = \frac{1}{2}$, on the diagonal $\{(z,z)\}$ we must have $\phi(z,z) = z$; but off the diagonal any convex combination of the two coordinate functions z^1 and z^2 will work).

It is the purpose of this article to examine the three point Pick problem on the bidisk. Our main result is the following:

Theorem 0.1. *The solution to an extremal non-degenerate three point problem on the bidisk is unique. The solution is given by a rational inner function of degree* 2. *There is a formula for the solution in terms of two uniquely determined rank one matrices.*

In the next section, we shall define precisely the terms "extremal" and "non-degenerate", but roughly it means that the problem is genuinely two-dimensional, is really a 3 point problem not a 2 point problem, and the minimal norm of a solution is 1.

1. Notation and preliminaries

We wish to consider the N point Pick interpolation problem

$$\begin{aligned} \phi(\lambda_i) &= w_i, \quad i = 1, \ldots, N, \quad \text{and} \\ \|\phi\|_{H^\infty(\mathbb{D}^2)} &\leq 1. \end{aligned} \tag{1.1}$$

We shall say that a solution ϕ to (1.1) is an *extremal solution* if $\|\phi\| = 1$, and no solution has a smaller norm.

For a point λ in $\mathbb{D}^2$, we shall use superscripts to denote coordinates:

$$\lambda \;=\; (\lambda^1, \lambda^2).$$

Let W, Λ^1 and Λ^2 denote the N-by-N matrices

$$\begin{aligned} W &= (1 - \bar{w}_i w_j)_{i,j=1}^N \\ \Lambda^1 &= \left(1 - \bar{\lambda}_i^1 \lambda_j^1\right)_{i,j=1}^N \\ \Lambda^2 &= \left(1 - \bar{\lambda}_i^2 \lambda_j^2\right)_{i,j=1}^N . \end{aligned}$$

A pair Γ, Δ of N-by-N positive semi-definite matrices is called *permissible* if

$$W = \Lambda^1 \cdot \Gamma \ + \ \Lambda^2 \cdot \Delta. \tag{1.2}$$

Here $\cdot$ denotes the Schur or entrywise product:

$$(A \cdot B)_{ij} \ := \ A_{ij} B_{ij}.$$

The main result of [1] is that the problem (1.1) has a solution if and only if there is a pair Γ, Δ of permissible matrices.

A *kernel* K on $\{\lambda_1, \dots, \lambda_N\} \times \{\lambda_1, \dots, \lambda_N\}$ is a positive definite N-by-N matrix

$$K_{ij} = K(\lambda_i, \lambda_j).$$

We shall call the kernel K *admissible* if

$$\begin{aligned} \Lambda^1 \cdot K &\geq 0 \quad \text{and} \\ \Lambda^2 \cdot K &\geq 0. \end{aligned}$$

If the problem (1.1) has a solution and K is an admissible kernel, then (1.2) implies that $K \cdot W \geq 0$. We shall call the kernel K *active* if it is admissible and $K \cdot W$ has a non-trivial null-space. Notice that all extremal problems have an active kernel.

If one can find a pair of permissible matrices one of which is 0, then the Pick problem is really a one-dimensional problem because one can find a solution ϕ that depends only on one of the coordinate functions. If this occurs, we shall call the problem *degenerate*; otherwise we shall call it *non-degenerate*.

2. The three point problem

We wish to analyze extremal solutions to three point Pick problems. Fix three points $\lambda_1, \lambda_2, \lambda_3$ in $\mathbb{D}^2$, and three numbers w_1, w_2, w_3. Let notation be as in the previous section. We shall make the following assumptions throughout this section:

(a) The function ϕ is an extremal solution to the Pick problem of interpolating λ_i to w_i, where i ranges from 1 to 3.
(b) The function ϕ is not an extremal solution to any of the three two point Pick problems mapping two of the λ_i's to the corresponding w_i's.
(c) The three point problem is non-degenerate.

Lemma 2.1. *If K is admissible, then* $\operatorname{rank}(K \cdot W) > 1$.

Proof. Suppose (Γ, Δ) is permissible. By (1.2), we have

$$K \cdot W = K \cdot \Lambda^1 \cdot \Gamma + K \cdot \Lambda^2 \cdot \Delta.$$

If $\operatorname{rank}(K \cdot W) = 1$, then either $\Gamma = 0$ (which violates (c)), or there exists $t > 0$ such that

$$K \cdot \Lambda^1 \cdot \Gamma = tK \cdot W.$$

But then $(\frac{1}{t}\Gamma, 0)$ is permissible, violating assumption (c). ☐

Lemma 2.2. *If K is an admissible kernel with a non-vanishing column, then* $\operatorname{rank}(K \cdot \Lambda^1) \geq 2$ *and* $\operatorname{rank}(K \cdot \Lambda^2) \geq 2$.

Proof. Suppose that $\operatorname{rank}(K \cdot \Lambda^1) = 1$. As no entry of Λ^1 can be 0, and some column of K is non-vanishing, there is a column of $K \cdot \Lambda^1$ that is non-vanishing. As $K \cdot \Lambda^1$ is self-adjoint and rank one and has non-zero diagonal entries, the other two columns of $K \cdot \Lambda^1$ must be non-zero multiples of this non-vanishing column. So $Q := K \cdot \Lambda^1$ is a positive rank one matrix with no zero entries, and K has no zero entries.

So

$$\begin{aligned}\Lambda^2 &= \left(\frac{1}{K}\right) \cdot \left(K \cdot \Lambda^2\right) \\ &= \left(\frac{1}{Q} \cdot \Lambda^1\right) \cdot \left(K \cdot \Lambda^2\right) \\ &= \left(\frac{1}{Q} \cdot K \cdot \Lambda^2\right) \cdot \Lambda^1,\end{aligned}$$

where by $\frac{1}{K}$ and $\frac{1}{Q}$ is meant the entrywise reciprocal. Now $K \cdot \Lambda^2$ is positive by hypothesis, and $\frac{1}{Q}$ is positive because Q is rank one and non-vanishing; moreover the Schur product of two positive matrices is positive [8, Thm 5.2.1]. Therefore $(\Gamma + \Delta \cdot \frac{1}{Q} \cdot K \cdot \Lambda^2, 0)$ is permissible, which violates assumption (c). ☐

Lemma 2.3. *If K is an active kernel, it has a non-vanishing column.*

Proof. By assumption (b), we cannot have both $K(\lambda_1, \lambda_2) = 0$ and $K(\lambda_1, \lambda_3) = 0$; for then K restricted to $\{\lambda_2, \lambda_3\} \times \{\lambda_2, \lambda_3\}$ would be an active kernel for the two point problem on λ_2, λ_3, and so any solution to the two point problem would have norm at least one, so ϕ would be an extremal solution to the two point problem.

If neither of $K(\lambda_1, \lambda_2)$ or $K(\lambda_1, \lambda_3)$ are 0, we are done. So assume without loss of generality that the first is non-zero and the second equals zero. But then $K(\lambda_2, \lambda_3)$ cannot equal zero, for then K restricted to $\{\lambda_1, \lambda_2\} \times \{\lambda_1, \lambda_2\}$ would be active, violating assumption (b). Thus we can conclude that the second column of K is non-vanishing. ☐

Lemma 2.4. *If (Γ, Δ) is a permissible pair, then* $\operatorname{rank}(\Gamma) = 1 = \operatorname{rank}(\Delta)$.

Proof. Let K be an active kernel. Then $K \cdot W$ is rank 2, and annihilates some vector

$$\vec{\gamma} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}.$$

Moreover, by assumption (b), none of the entries of $\vec{\gamma}$ are 0.

Suppose $\operatorname{rank}(\Gamma) > 1$. We have

$$K \cdot W = K \cdot \Lambda^1 \cdot \Gamma + K \cdot \Lambda^2 \cdot \Delta.$$

As $K \cdot \Lambda^1$ has non-zero diagonal terms, Oppenheim's Theorem [7, Thm 7.8.6] guarantees that $\operatorname{rank}(K \cdot \Lambda^1 \cdot \Gamma) \geq \operatorname{rank}(\Gamma)$. As $K \cdot W$ has rank 2, and $K \cdot \Lambda^2 \cdot \Delta \geq 0$, we must have $\operatorname{rank}(\Gamma) = 2$. Write

$$\Gamma = \vec{u} \otimes \vec{u} + \vec{v} \otimes \vec{v},$$

where $\vec{u}$ and $\vec{v}$ are not collinear; if

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

then $\vec{u} \otimes \vec{u}$ denotes the matrix

$$(\vec{u} \otimes \vec{u})_{ij} \;=\; u_i \bar{u}_j.$$

Let

$$K \cdot \Lambda^1 = \vec{w} \otimes \vec{w} + \vec{x} \otimes \vec{x}$$

if $K \cdot \Lambda^1$ is rank two, and

$$K \cdot \Lambda^1 = \vec{w} \otimes \vec{w} + \vec{x} \otimes \vec{x} + \vec{y} \otimes \vec{y}$$

if it is rank three.

Notice that $\left(K \cdot \Lambda^1 \cdot \Gamma\right) \vec{\gamma} = 0$, because $K \cdot \Lambda^1 \cdot \Gamma$ is positive and

$$\begin{aligned} \langle \left(K \cdot \Lambda^1 \cdot \Gamma\right) \vec{\gamma}, \vec{\gamma} \rangle &= -\langle \left(K \cdot \Lambda^2 \cdot \Delta\right) \vec{\gamma}, \vec{\gamma} \rangle \\ &\leq 0. \end{aligned}$$

Therefore all four of $(\vec{u}\otimes\vec{u})\cdot(\vec{w}\otimes\vec{w})$, $(\vec{u}\otimes\vec{u})\cdot(\vec{x}\otimes\vec{x})$, $(\vec{v}\otimes\vec{v})\cdot(\vec{w}\otimes\vec{w})$, $(\vec{v}\otimes\vec{v})\cdot(\vec{x}\otimes\vec{x})$ annihilate $\vec{\gamma}$. Therefore

$$\begin{aligned} \sum_{j=1}^{3} \bar{u}_j \bar{w}_j \gamma_j = 0 & \\ &= \sum_{j=1}^{3} \bar{u}_j \bar{x}_j \gamma_j \\ &= \sum_{j=1}^{3} \bar{v}_j \bar{w}_j \gamma_j \\ &= \sum_{j=1}^{3} \bar{v}_j \bar{x}_j \gamma_j \end{aligned}$$

Therefore the vectors

$$\begin{pmatrix} \overline{w_1}\gamma_1 \\ \overline{w_2}\gamma_2 \\ \overline{w_3}\gamma_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \overline{x_1}\gamma_1 \\ \overline{x_2}\gamma_2 \\ \overline{x_3}\gamma_3 \end{pmatrix}$$

are both orthogonal to both $\vec{u}$ and $\vec{v}$, and therefore are collinear (since $\vec{u}$ and $\vec{v}$ span a two-dimensional subspace of $\mathbb{C}^3$). As none of the entries of $\vec{\gamma}$ are 0, it follows that $\vec{w}$ and $\vec{x}$ are collinear. Therefore $\text{rank}(K \cdot \Lambda^1) = 1$, contradicting Lemmata (2.2) and (2.3). $\square$

Lemma 2.5. *The matrices* Γ *and* Δ *are unique.*

Proof. If both (Γ_1, Δ_1) and (Γ_2, Δ_2) were permissible, then $(\frac{1}{2}(\Gamma_1 + \Gamma_2), \frac{1}{2}(\Delta_1 + \Delta_2))$ would also be permissible. As all permissible matrices are rank one by Lemma 2.4, it follows that Γ_1 and Γ_2 are constant multiples of each other, and so are Δ_1 and Δ_2.

So suppose

$$W = \Lambda^1 \cdot \Gamma + \Lambda^2 \cdot \Delta \quad \text{and}$$
$$W = \Lambda^1 \cdot t_1\Gamma + \Lambda^2 \cdot t_2\Delta,$$

where both t_1, t_2 are positive, one is less than 1, and the other is bigger than 1. Then

$$(1-t_1)\Lambda^1 \cdot \Gamma + (1-t_2)\Lambda^2 \cdot \Delta = 0.$$

Assume without loss of generality that $t_1 < 1 < t_2$. Then $(\frac{t_2 - t_1}{1 - t_1}\Gamma, 0)$ is permissible, which contradicts Assumption (c). $\square$

Theorem 2.6. *The solution to an extremal non-degenerate three point problem satisfying Assumptions* (a)-(c) *is unique. It is given by a rational inner function of degree* 2*, and there is a formula in terms of* Γ *and* Δ*.*

Proof. We have

$$W = \Lambda^1 \cdot \Gamma + \Lambda^2 \cdot \Delta. \tag{2.7}$$

Choose vectors $\vec{a}$ and $\vec{b}$ so that $\Gamma = \vec{a} \otimes \vec{a}$ and $\Delta = \vec{b} \otimes \vec{b}$.

Choose some point λ_4 in $\mathbb{D}^2$, distinct from the first three points. Let w_4 be the value attained at λ_4 by some solution ϕ of the three point problem (1.1). Then the four point problem, interpolating λ_i to w_i for $i = 1, \ldots, 4$ has a solution, so we can find a pair of 4-by-4 permissible matrices $\tilde{\Gamma}$ and $\tilde{\Delta}$ satisfying (1.2). As the restriction of these matrices to the first three points satisfy (2.7), and Γ and Δ are unique by Lemma 2.5, we get that $\tilde{\Gamma}$ and $\tilde{\Delta}$ are extensions of Γ and Δ. Therefore we have

$$(*) \quad \left(\begin{array}{ccc|c} & & & 1-\bar{w}_1 w_4 \\ & W & & 1-\bar{w}_2 w_4 \\ & & & 1-\bar{w}_3 w_4 \\ \hline * & * & * & 1-|w_4|^2 \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} & & & g_1 \\ & \Gamma & & g_2 \\ & & & g_3 \\ \hline \bar{g}_1 & \bar{g}_2 & \bar{g}_3 & g_4 \end{array}\right) \cdot \left(\begin{array}{ccc|c} & & & 1-\bar{\lambda}_1^1 \lambda_4^1 \\ & \Lambda^1 & & 1-\bar{\lambda}_2^1 \lambda_4^1 \\ & & & 1-\bar{\lambda}_3^1 \lambda_4^1 \\ \hline * & * & * & 1-|\lambda_4^1|^2 \end{array}\right)$$
$$+ \left(\begin{array}{ccc|c} & & & d_1 \\ & \Delta & & d_2 \\ & & & d_3 \\ \hline \bar{d}_1 & \bar{d}_2 & \bar{d}_3 & d_4 \end{array}\right) \cdot \left(\begin{array}{ccc|c} & & & 1-\bar{\lambda}_1^2 \lambda_4^2 \\ & \Lambda^2 & & 1-\bar{\lambda}_2^2 \lambda_4^2 \\ & & & 1-\bar{\lambda}_3^2 \lambda_4^2 \\ \hline * & * & * & 1-|\lambda_4^2|^2 \end{array}\right)$$

As $\tilde{\Gamma}$ is positive, it must be that $\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$ is in the range of Γ, so

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = s\vec{a} = s\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

for some constant s. Similarly,

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = t\vec{b} = t \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

for some t.

Let

$$\vec{v_1} = \begin{pmatrix} (1-\bar{\lambda}_1^1\lambda_4^1)a_1 \\ (1-\bar{\lambda}_2^1\lambda_4^1)a_2 \\ (1-\bar{\lambda}_3^1\lambda_4^1)a_3 \end{pmatrix}$$
$$\vec{v_2} = \begin{pmatrix} (1-\bar{\lambda}_1^2\lambda_4^2)b_1 \\ (1-\bar{\lambda}_2^2\lambda_4^2)b_2 \\ (1-\bar{\lambda}_3^2\lambda_4^2)b_3 \end{pmatrix}$$
$$\vec{v_3} = \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \bar{w}_3 \end{pmatrix}.$$

Looking at the first three entries of the last column of Equation $(*)$, we get

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = s\vec{v_1} + t\vec{v_2} + w_4\vec{v_3}. \tag{2.8}$$

Equation (2.8) has a unique solution for s, t and w_4 unless

$$\det \begin{pmatrix} (1-\bar{\lambda}_1^1\lambda_4^1)a_1 & (1-\bar{\lambda}_1^2\lambda_4^2)b_1 & \bar{w}_1 \\ (1-\bar{\lambda}_2^1\lambda_4^1)a_2 & (1-\bar{\lambda}_2^2\lambda_4^2)b_2 & \bar{w}_2 \\ (1-\bar{\lambda}_3^1\lambda_4^1)a_3 & (1-\bar{\lambda}_3^2\lambda_4^2)b_3 & \bar{w}_3 \end{pmatrix} = 0. \tag{2.9}$$

Notice that the determinant in (2.9) is analytic in λ_4. So if there is a single point λ_4 for which the determinant does not vanish, there is an open neighborhood of this point for which the determinant doesn't vanish. Consequently w_4 (and hence ϕ) would be determined uniquely on this open set, and hence on all of $\mathbb{D}^2$.

Suppose the determinant in (2.9) vanished identically. Then there is a set of uniqueness of λ_4's on which Equation (2.8) can be solved with either s or t equal to 0. (If both s and t were uniquely determined, then $\vec{v_3}$ would be 0, violating Assumption (a)). Without loss of generality, take $t = 0$. Moreover, we can also assume without loss of generality that w_1 and w_2 do not both vanish.

Then one can use the first component of Equation (2.8) to solve for s, and the second one to get

$$w_4 = \frac{(1-\bar{\lambda}_1^1\lambda_4^1)a_1 + (1-\bar{\lambda}_2^1\lambda_4^1)a_2}{(1-\bar{\lambda}_1^1\lambda_4^1)a_1\bar{w}_2 + (1-\bar{\lambda}_2^1\lambda_4^1)a_2\bar{w}_1}$$

Then w_4 is given uniquely as a rational function of degree 1 of λ_4^1, violating both Assumptions (b) and (c).

Therefore we can assume that there is an open set on which Equation (2.8) has a unique solution, so by Cramer's rule we get

$$\phi(\lambda_4) = w_4 = \frac{\det\begin{pmatrix} (1-\bar\lambda_1^1\lambda_4^1)a_1 & (1-\bar\lambda_1^2\lambda_4^2)b_1 & 1 \\ (1-\bar\lambda_2^1\lambda_4^1)a_2 & (1-\bar\lambda_2^2\lambda_4^2)b_2 & 1 \\ (1-\bar\lambda_3^1\lambda_4^1)a_3 & (1-\bar\lambda_3^2\lambda_4^2)b_3 & 1 \end{pmatrix}}{\det\begin{pmatrix} (1-\bar\lambda_1^1\lambda_4^1)a_1 & (1-\bar\lambda_1^2\lambda_4^2)b_1 & \bar w_1 \\ (1-\bar\lambda_2^1\lambda_4^1)a_2 & (1-\bar\lambda_2^2\lambda_4^2)b_2 & \bar w_2 \\ (1-\bar\lambda_3^1\lambda_4^1)a_3 & (1-\bar\lambda_3^2\lambda_4^2)b_3 & \bar w_3 \end{pmatrix}}. \tag{2.10}$$

Equation (2.10) gives a formula for ϕ that shows that ϕ is a rational function of degree at most 2, whose second order terms only involve the mixed product $\lambda_4^1\lambda_4^2$.

To show that ϕ is inner, we follow [2]. We can rewrite (1.2) as

$$1 + \bar\lambda_i^1\lambda_j^1 a_i\bar a_j + \bar\lambda_i^2\lambda_j^2 b_i\bar b_j \;=\; \bar w_i w_j + a_i\bar a_j + b_i\bar b_j. \tag{2.11}$$

Realizing both sides of (2.11) as Grammians, we get that there exists a 3-by-3 unitary U such that, for $j = 1, 2, 3$,

$$U\begin{pmatrix} 1 \\ \lambda_j^1\bar a_j \\ \lambda_j^2\bar b_j \end{pmatrix} = \begin{pmatrix} w_j \\ \bar a_j \\ \bar b_j \end{pmatrix}. \tag{2.12}$$

Writing

$$U = \begin{array}{c c} & \begin{array}{cc} \mathbb{C} & \mathbb{C}^2 \end{array} \\ \begin{array}{c} \mathbb{C} \\ \mathbb{C}^2 \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array},$$

and letting

$$E_\lambda = \begin{pmatrix} \lambda^1 & 0 \\ 0 & \lambda^2 \end{pmatrix},$$

we can solve (2.12) to get

$$w_j = A + BE_{\lambda_j}(1 - DE_{\lambda_j})^{-1}C.$$

So the function

$$\psi(\lambda) = A + BE_\lambda(1 - DE_\lambda)^{-1}C$$

interpolates the original data. Moreover ψ is inner, because a calculation shows that

$$1 - \overline{\psi(\lambda)}\psi(\lambda) = ((1 - DE_\lambda)^{-1}C)^*(1 - E_\lambda^*E_\lambda)((1 - DE_\lambda)^{-1}C),$$

so $|\psi|$ is less than 1 on $\mathbb{D}^2$ and equals 1 on the distinguished boundary. By uniqueness, we must have $\psi = \phi$, and hence ϕ is inner.

Finally, we must show that the degree of ϕ is exactly two. This is because an easy calculation shows that a rational function of degree one

$$\frac{c_1 + c_2z^1 + c_3z^2}{c_4 + c_5z^1 + c_6z^2}$$

is inner only if it is a function of either just z^1 or just z^2, i.e., either both c_2 and c_5 or both c_3 and c_6 can be chosen to be zero. This would violate Assumption (c). □

3. Finding Γ and Δ

Formula (2.10) works fine, provided one knows Γ and Δ (or, equivalently, a_1, a_2, a_3 and b_1, b_2, b_3). Lemma 2.5 assures us that Γ and Δ are unique; how does one find them?

First, let us make a simplifying normalization. One can pre-compose ϕ with an automorphism of $\mathbb{D}^2$, and post-compose it with an automorphism of $\mathbb{D}$; so one can assume that $\lambda_1 = (0,0)$ and $w_1 = 0$. Write $\lambda_2 = (\alpha_2, \beta_2)$ and $\lambda_3 = (\alpha_3, \beta_3)$. Moreover, as $\Gamma_{ij} = a_i \bar{a}_j$ and $\Delta_{ij} = b_i \bar{b}_j$, we can choose $a_1 \geq 0$ and $b_1 \geq 0$; again without loss of generality we can assume that $b_1 > 0$. Thus we have

$$(**) \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1-|w_2|^2 & 1-\bar{w}_2 w_3 \\ 1 & 1-w_2\bar{w}_3 & 1-|w_3|^2 \end{pmatrix}$$
$$= \begin{pmatrix} a_1^2 & a_1\bar{a}_2 & a_1\bar{a}_3 \\ a_1 a_2 & |a_2|^2 & a_2\bar{a}_3 \\ a_1 a_3 & \bar{a}_2 a_3 & |a_3|^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1-|\alpha_2|^2 & 1-\bar{\alpha}_2\alpha_3 \\ 1 & 1-\alpha_2\bar{\alpha}_3 & 1-|\alpha_3|^2 \end{pmatrix}$$
$$+ \begin{pmatrix} b_1^2 & b_1\bar{b}_2 & b_1\bar{b}_3 \\ b_1 b_2 & |b_2|^2 & b_2\bar{b}_3 \\ b_1 b_3 & \bar{b}_2 b_3 & |b_3|^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1-|\beta_2|^2 & 1-\bar{\beta}_2\beta_3 \\ 1 & 1-\beta_2\bar{\beta}_3 & 1-|\beta_3|^2 \end{pmatrix}$$

Looking at the first column of $(**)$ we get

$$b_1 = \frac{1-a_1^2}{\sqrt{1-a_1^2}} \qquad b_2 = \frac{1-a_1a_2}{\sqrt{1-a_1^2}} \qquad b_3 = \frac{1-a_1a_3}{\sqrt{1-a_1^2}}$$

Thus we have three equations that *uniquely* determine a_1, a_2, a_3:

$$(3.1) \qquad (1-a_1^2)(1-|w_2|^2) = (1-a_1^2)|a_2|^2(1-|\alpha_2|^2) + |1-a_1a_2|^2(1-|\beta_2|^2)$$

$$(3.2) \qquad (1-a_1^2)(1-\bar{w}_2w_3) = (1-a_1^2)a_2\bar{a}_3(1-\bar{\alpha}_2\alpha_3) + (1-a_1a_2)(1-a_1\bar{a}_3)(1-\bar{\beta}_2\beta_3)$$

$$(3.3) \qquad (1-a_1^2)(1-|w_3|^2) = (1-a_1^2)|a_3|^2(1-|\alpha_3|^2) + |1-a_1a_3|^2(1-|\beta_3|^2)$$

Equation (3.2) can be used to solve for a_3 as a rational function of a_1 and $\bar{a}_2$; then one is left with two real algebraic equations in three real variables, $a_1, \Re(a_2)$ and $\Im(a_3)$. Provided the original data is really extremal, this system of two equations will have a unique solution with $a_1 \geq 0$. If the original data is not extremal, multiply w_2 and w_3 by a positive real number t, and choose the largest t for which equations (3.1)–(3.3) can be solved. This will produce an inner function ϕ via (2.10); then the function $\frac{1}{t}\phi$ will be the unique function of minimal norm solving the original problem.

Example 3.4. Let us consider a very symmetric special case. Let $\lambda_1 = (0,0)$, $\lambda_2 = (r,0)$, $\lambda_3 = (0,r)$, $w_1 = 0$, $w_2 = t$ and $w_3 = t$, where t is to be chosen as large as possible and r is a fixed *positive* number. Then by symmetry, we can assume that $a_1 = b_1 = \frac{1}{\sqrt{2}}$ and $a_2 = b_3 = \bar{a}_2$.

Equations (3.1)–(3.3) then reduce to:

$$\frac{1}{2}(1-t^2) = \frac{1}{2}a_2^2(1-r^2) + (1 - \frac{1}{\sqrt{2}}a_2)^2$$

$$\frac{1}{2}(1-t^2) = a_2(\sqrt{2} - a_2)$$

Solving, one gets two solutions. One solution is

$$t = \frac{r}{2-r}, \quad a_2 = \frac{\sqrt{2}}{2-r};$$

the other is

$$t = \frac{r}{2+r}, \quad a_2 = \frac{\sqrt{2}}{2+r}.$$

The first of these is clearly the extremal solution, and formula (2.10) then gives

$$\phi(z) = \frac{z^1 + z^2 - 2z^1z^2}{2 - z^1 - z^2}$$

as the extremal solution.

The second solution also corresponds to a pair of rank one matrices Γ and Δ that satisfy (2.7), even though the problem is non-extremal. If one plugs in to (2.10) one gets the inner function

$$\phi_2(z) = \frac{z^1 + z^2 + 2z^1z^2}{2 + z^1 + z^2}.$$

References

[1] J. Agler, *Some interpolation theorems of Nevanlinna-Pick type*, Preprint, 1988.

[2] J. Agler and J. E. McCarthy, *Nevanlinna-Pick interpolation on the bidisk*, J. Reine Angew. Math. **506** (1999), 191–204, MR 2000a:47034, Zbl 919.32003.

[3] J. A. Ball and T. T. Trent, *Unitary colligations, reproducing kernel Hilbert spaces, and Nevanlinna-Pick interpolation in several variables*, J. Funct. Anal. **197** (1998), 1–61, MR 2000b:47028, Zbl 914.47020.

[4] B. J. Cole and J. Wermer, *Pick interpolation, von Neumann inequalities, and hyperconvex sets*, Complex Potential Theory (P. M. Gauthier, ed.), NATO ASI Ser., Ser. C, Math. Phys. Sci., no. 439, Kluwer Acad. Publ., Dordrecht, 1994, pp. 89–129, MR 96m:46092, Zbl 811.30024.

[5] C. Foiaş and A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Operator Theory: Advances and Applications, no. 44, Birkhäuser, Basel, 1990, MR 92k:47033, Zbl 718.47010.

[6] John B. Garnett, *Bounded Analytic Functions*, Pure and Applied Mathematics, no. 96, Academic Press, New York, 1981, MR 83g:30037, Zbl 469.30024.

[7] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985, MR 87e:15001, Zbl 576.15001.

[8] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991, MR 92e:15003, Zbl 729.15001.

[9] G. Pick, *Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden*, Math. Ann. **77** (1916), 7–23.

U.C. San Diego, La Jolla, California 92093
jagler@math.ucsd.edu

Washington University, St. Louis, Missouri 63130
mccarthy@math.wustl.edu http://www.math.wustl.edu/~mccarthy/

This paper is available via http://nyjm.albany.edu:8000/j/2000/6-11.html.