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Green's Functions for Elliptic and Parabolic Equations with Random Coefficients

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Abstract. This paper is concerned with linear uniformly elliptic and parabolic partial differential equations in divergence form. It is assumed that the coefficients of the equations are random variables, constant in time. The Green's functions for the equations are then random variables. Regularity properties for expectation values of Green's functions are obtained. In particular, it is shown that the expectation value is a continuously differentiable function whose derivatives are bounded by the corresponding derivatives of the heat equation. Similar results are obtained for the related finite difference equations.

Contents

1. Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\mathbf{a} : \Omega \to \mathbb{R}^{d(d+1)/2}$ be a bounded measurable function from Ω to the space of symmetric $d \times d$ matrices. We assume that there are positive constants Λ, λ such that

$$\lambda I_d \le \mathbf{a}(\omega) \le \Lambda I_d, \quad \omega \in \Omega, \tag{1.1}$$

in the sense of quadratic forms, where I_d is the identity matrix in d dimensions. We assume that $\mathbb{R}^d$ acts on Ω by translation operators $\tau_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$, which are measure preserving and satisfy the properties $\tau_x \tau_y = \tau_{x+y}$, $\tau_0 =$ identity, $x, y \in \mathbb{R}^d$. We assume also that the function from $\mathbb{R}^d \times \Omega$ to Ω defined by $(x, \omega) \to \tau_x \omega$,

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$x \in \mathbb{R}^d$, $\omega \in \Omega$, is measurable. It follows that with probability 1 the function $\mathbf{a}(x,\omega) = \mathbf{a}(\tau_x\omega)$, $x \in \mathbb{R}^d$, is a Lebesgue measurable function from $\mathbb{R}^d$ to $d \times d$ matrices.

Consider now for $\omega \in \Omega$ such that $\mathbf{a}(x,\omega)$ is a measurable function of $x \in \mathbb{R}^d$, the parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left[a_{i,j}(x,\omega)\frac{\partial}{\partial x_j}u(x,t,\omega)\right], \quad x \in \mathbb{R}^d, \quad t > 0, \tag{1.2}$$

$$u(x,0,\omega) = f(x,\omega), \quad x \in \mathbb{R}^d.$$

It is well known that the solution of this initial value problem can be written as

$$u(x,t,\omega) = \int_{\mathbb{R}^d} G_{\mathbf{a}}(x,y,t,\omega)f(y,\omega)dy \ ,$$

where $G_{\mathbf{a}}(x,y,t,\omega)$ is the Green's function, and $G_{\mathbf{a}}$ is measurable in (x,y,t,ω). Evidently $G_{\mathbf{a}}$ is a positive function which satisfies

$$\int_{\mathbb{R}^d} G_{\mathbf{a}}(x,y,t,\omega)dy = 1. \tag{1.3}$$

It also follows from the work of Aronson [1] (see also [5]) that there is a constant $C(d,\lambda,\Lambda)$ depending only on dimension d and the uniform ellipticity constants λ, Λ of (1.1) such that

$$0 \le G_{\mathbf{a}}(x,y,t,\omega) \le \frac{C(d,\lambda,\Lambda)}{t^{d/2}} \exp\left[\frac{-|x-y|^2}{C(d,\lambda,\Lambda)t}\right]. \tag{1.4}$$

In this paper we shall be concerned with the expectation value of $G_{\mathbf{a}}$ over Ω. Denoting expectation value on Ω by $\langle \ \rangle$ we define the function $G_{\mathbf{a}}(x,t)$, $x \in \mathbb{R}^d$, $t > 0$ by

$$\left\langle G_{\mathbf{a}}(x,0,t,\cdot)\right\rangle = G_{\mathbf{a}}(x,t) \ .$$

Using the fact that $\tau_x\tau_y = \tau_{x+y}$, $x,y \in \mathbb{R}^d$, we see from the uniqueness of solutions to (1.2) that

$$G_{\mathbf{a}}(x,y,t,\omega) = G_{\mathbf{a}}(x-y,0,t,\tau_y\omega),$$

whence the measure preserving property of the operator τ_y yields the identity,

$$\left\langle G_{\mathbf{a}}(x,y,t,\cdot)\right\rangle = G_{\mathbf{a}}(x-y,t) \ .$$

From (1.3), (1.4) we have

$$\int_{\mathbb{R}^d} G_{\mathbf{a}}(x,t)dx = 1, \quad t > 0,$$

$$0 \le G_{\mathbf{a}}(x,t) \le \frac{C(d,\lambda,\Lambda)}{t^{d/2}} \exp\left[\frac{-|x|^2}{C(d,\lambda,\Lambda)t}\right], \quad x \in \mathbb{R}^d, \ t > 0. \tag{1.5}$$

In general one cannot say anything about the smoothness properties of the function $G_{\mathbf{a}}(x,y,t,\omega)$. We shall, however, be able to prove here that $G_{\mathbf{a}}(x,t)$ is a C^1 function of (x,t), $x \in \mathbb{R}^d$, $t > 0$.

Theorem 1.1. *$G_{\mathbf{a}}(x,t)$ is a C^1 function of (x,t), $x \in \mathbb{R}^d$, $t > 0$. There is a constant $C(d,\lambda,\Lambda)$, depending only on d,λ,Λ such that*

$$\begin{aligned} \left| \frac{\partial G_{\mathbf{a}}}{\partial t}(x,t) \right| &\le \frac{C(d,\lambda,\Lambda)}{t^{d/2\,+\,1}} \exp\left[\frac{-|x|^2}{C(d,\lambda,\Lambda)t} \right], \\ \left| \frac{\partial G_{\mathbf{a}}}{\partial x_i}(x,t) \right| &\le \frac{C(d,\lambda,\Lambda)}{t^{d/2\,+\,1/2}} \exp\left[\frac{-|x|^2}{C(d,\lambda,\Lambda)t} \right]. \end{aligned}$$

The Aronson inequality (1.5) shows us that $G_{\mathbf{a}}(x,t)$ is bounded by the kernel of the heat equation. Theorem 1.1 proves that corresponding inequalities hold for the first derivatives of $G_{\mathbf{a}}(x,t)$. We cannot use our methods to prove existence of second derivatives of $G_{\mathbf{a}}(x,t)$ in the space variable x. In fact we are inclined to believe that second space derivatives do not in general exist in a pointwise sense.

As well as the parabolic problem (1.2) we also consider the corresponding elliptic problem,

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left[a_{i,j}(x,\omega)\frac{\partial u}{\partial x_j}(x,\omega) \right] = f(x,\omega), \quad x \in \mathbb{R}^d. \tag{1.6}$$

If $d \ge 3$ then the solution of (1.6) can be written as

$$u(x,\omega) = \int_{\mathbb{R}^d} G_{\mathbf{a}}(x,y,\omega) f(y,\omega)dy,$$

where $G_{\mathbf{a}}(x,y,\omega)$ is the Green's function and is measurable in (x,y,ω). It follows again by Aronson's work that there is a constant $C(d,\lambda,\Lambda)$, depending only on d,λ,Λ, such that

$$0 \le G_{\mathbf{a}}(x,y,\omega) \le C(d,\lambda,\Lambda)/|x-y|^{d-2}, \quad d \ge 3. \tag{1.7}$$

Again we consider the expectation of the Green's function, $G_{\mathbf{a}}(x)$, defined by

$$\Big\langle G_{\mathbf{a}}(x,y,\cdot) \Big\rangle = G_{\mathbf{a}}(x-y).$$

It follows from (1.7) that

$$0 \le G_{\mathbf{a}}(x) \le C(d,\lambda,\Lambda)/|x|^{d-2}, \quad d \ge 3.$$

Theorem 1.2. *Suppose $d \ge 3$. Then $G_{\mathbf{a}}(x)$ is a C^1 function of x for $x \ne 0$. There is a constant $C(d,\lambda,\Lambda)$ depending only on d,λ,Λ, such that*

$$\left| \frac{\partial G_{\mathbf{a}}}{\partial x_i}(x) \right| \le \frac{C(d,\lambda,\Lambda)}{|x|^{d-1}}, \quad x \ne 0.$$

We can also derive estimates on the Fourier transforms of $G_{\mathbf{a}}(x,t)$ and $G_{\mathbf{a}}(x)$. For a function $f : \mathbb{R}^d \to \mathbb{C}$ we define its Fourier transform $\hat{f}$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{ix\cdot\xi}dx, \quad \xi \in \mathbb{R}^d.$$

Evidently from the equation before (1.5) we have that $|\hat{G}_{\mathbf{a}}(\xi,t)| \le 1$

Theorem 1.3. *The function $\hat{G}_{\mathbf{a}}(\xi,t)$ is continuous for $\xi \in \mathbb{R}^d$, $t > 0$, and differentiable with respect to t. Let δ satisfy $0 \le \delta < 1$. Then there is a constant $C(\delta,\lambda,\Lambda)$ depending only on δ,λ,Λ, such that*

$$\begin{aligned} |\hat{G}_{\mathbf{a}}(\xi,t)| &\le \frac{C(\delta,\lambda,\Lambda)}{[1+|\xi|^2 t]^{\delta}} \ , \\ \left|\frac{\partial \hat{G}_{\mathbf{a}}}{\partial t}(\xi,t)\right| &\le \frac{C(\delta,\lambda,\Lambda)|\xi|^2}{[1+|\xi|^2 t]^{1+\delta}} \ , \end{aligned}$$

where $|\xi|$ denotes the Euclidean norm of $\xi \in \mathbb{R}^d$.

Remark 1.1. Note that the dimension d does not enter in the constant $C(\delta,\lambda,\Lambda)$. Also, our method of proof breaks down if we take $\delta \to 1$.

In this paper we shall be mostly concerned with a discrete version of the parabolic and elliptic problems (1.2), (1.6). Then Theorems 1.1, 1.2, 1.3 can be obtained as a continuum limit of our results on the discrete problem. In the discrete problem we assume $\mathbb{Z}^d$ acts on Ω by translation operators $\tau_x : \Omega \to \Omega$, $x \in \mathbb{Z}^d$, which are measure preserving and satisfy the properties $\tau_x\tau_y = \tau_{x+y}$, $\tau_0 =$ identity. For functions $g : \mathbb{Z}^d \to \mathbb{R}$ we define the discrete derivative $\nabla_i g$ of g in the i th direction to be

$$\nabla_i g(x) = g(x+\mathbf{e}_i) - g(x), \quad x \in \mathbb{Z}^d,$$

where $\mathbf{e}_i \in \mathbb{Z}^d$ is the element with entry 1 in the i th position and 0 in other positions. The formal adjoint of ∇_i is given by ∇_i^*, where

$$\nabla_i^* g(x) = g(x-\mathbf{e}_i) - g(x), \quad x \in \mathbb{Z}^d.$$

The discrete version of the problem (1.2) that we shall be interested in is given by

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\sum_{i,j=1}^{d} \nabla_i^* \left[a_{ij}(\tau_x\omega)\nabla_j u(x,t,\omega)\right], \quad x \in \mathbb{Z}^d, \quad t > 0, \\ u(x,0,\omega) &= f(x,\omega), \quad x \in \mathbb{Z}^d. \end{aligned} \tag{1.8}$$

The solution of (1.8) can be written as

$$u(x,t,\omega) = \sum_{y\in\mathbb{Z}^d} G_{\mathbf{a}}(x,y,t,\omega) f(y,\omega),$$

where $G_{\mathbf{a}}(x,y,t,\omega)$ is the discrete Green's function. As in the continuous case, $G_{\mathbf{a}}$ is a positive function which satisfies

$$\sum_{y\in\mathbb{Z}^d} G_{\mathbf{a}}(x,y,t,\omega) = 1.$$

It also follows from the work of Carlen et al [3] that there is a constant $C(d,\lambda,\Lambda)$ depending only on d,λ,Λ such that

$$0 \le G_{\mathbf{a}}(x,y,t,\omega) \le \frac{C(d,\lambda,\Lambda)}{1+t^{d/2}} \exp\left[-\frac{\min\{|x-y|,\ |x-y|^2/t\}}{C(d,\lambda,\Lambda)}\right].$$

Now let $G_{\mathbf{a}}(x,t)$, $x \in \mathbb{Z}^d$, $t > 0$, be the expectation of the Green's function,

$$\left\langle G_{\mathbf{a}}(x,y,t,\cdot)\right\rangle = G_{\mathbf{a}}(x-y,t). \tag{1.9}$$

Then we have

$$\sum_{x\in\mathbb{Z}^d} G_{\mathbf{a}}(x,t) = 1, \quad t>0\ ,$$

$$G_{\mathbf{a}}(x,t) \le \frac{C(d,\lambda,\Lambda)}{1+t^{d/2}} \exp\left[-\frac{\min\{|x|,\ |x|^2/t\}}{C(d,\lambda,\Lambda)}\right], \quad x\in\mathbb{Z}^d,\ t>0\ . \tag{1.10}$$

The discrete version of Theorem 1.1 which we shall prove is given by the following:

Theorem 1.4. *$G_{\mathbf{a}}(x,t)$, $x\in\mathbb{Z}^d$, $t>0$ is differentiable in t. There is a constant $C(d,\lambda,\Lambda)$, depending only on d,λ,Λ such that*

$$\begin{aligned}\left|\frac{\partial G_{\mathbf{a}}}{\partial t}(x,t)\right| &\le \frac{C(d,\lambda,\Lambda)}{1+t^{d/2\,+\,1}}\exp\left[-\frac{\min\{|x|,\ |x|^2/t\}}{C(d,\lambda,\Lambda)}\right],\\ |\nabla_i G_{\mathbf{a}}(x,t)| &\le \frac{C(d,\lambda,\Lambda)}{1+t^{d/2\,+\,1/2}}\exp\left[-\frac{\min\{|x|,|x|^2/t\}}{C(d,\lambda,\Lambda)}\right].\end{aligned}$$

Let δ satisfy $0\le\delta<1$. Then there is a constant $C(\delta,d,\lambda,\Lambda)$ depending only on δ,d,λ,Λ such that

$$|\nabla_i\nabla_j G_{\mathbf{a}}(x,t)| \le \frac{C(\delta,d,\lambda,\Lambda)}{1+t^{(d+1+\delta)/2}}\exp\left[-\frac{\min\{|x|,|x|^2/t\}}{C(d,\lambda,\Lambda)}\right]. \tag{1.11}$$

Remark 1.2. As in Theorem 1.1, Theorem 1.4 shows that first derivatives of $G_{\mathbf{a}}(x,t)$ are bounded by corresponding heat equation quantities. It also shows that second space derivatives are almost similarly bounded. We cannot put $\delta=1$ in (1.11) since the constant $C(\delta,d,\lambda,\Lambda)$ diverges as $\delta\to 1$.

The elliptic problem corresponding to (1.8) is given by

$$\sum_{i,j=1}^{d} \nabla_i^*\left[a_{i,j}(\tau_x\omega)\nabla_j u(x,\omega)\right] = f(x,\omega), \quad x\in\mathbb{Z}^d. \tag{1.12}$$

If $d\ge 3$ then the solution of (1.12) can be written as

$$u(x,\omega) = \sum_{y\in\mathbb{Z}^d} G_{\mathbf{a}}(x,y,\omega) f(y,\omega),$$

where $G_{\mathbf{a}}(x,y,\omega)$ is the discrete Green's function. It follows from Carlen et al [3] that there is a constant $C(d,\lambda,\Lambda)$ depending only on d,λ,Λ such that

$$0\le G_{\mathbf{a}}(x,y,\omega)\le C(d,\lambda,\Lambda)/[1+|x-y|^{d-2}], \quad d\ge 3. \tag{1.13}$$

Letting $G_{\mathbf{a}}(x)$ be the expectation of the Green's function,

$$\Big\langle G_{\mathbf{a}}(x,y,\cdot)\Big\rangle = G_{\mathbf{a}}(x-y),$$

it follows from (1.13) that

$$0\le G_{\mathbf{a}}(x)\le C(d,\lambda,\Lambda)/[1+|x|^{d-2}], \quad d\ge 3. \tag{1.14}$$

We shall prove a discrete version of Theorem 1.2 as follows:

Theorem 1.5. *Suppose $d\ge 3$. Then there is a constant $C(d,\lambda,\Lambda)$, depending only on d,λ,Λ such that*

$$|\nabla_i G_{\mathbf{a}}(x)| \le C(d,\lambda,\Lambda)/[1+|x|^{d-1}], \quad x\in\mathbb{Z}^d.$$

Let δ satisfy $0 \le \delta < 1$. Then there is a constant $C(\delta, d, \lambda, \Lambda)$ depending only on $\delta, d, \lambda, \Lambda$ such that

$$|\nabla_i \nabla_j G_{\mathbf{a}}(x)| \le C(\delta, d, \lambda, \Lambda)/[1 + |x|^{d-1+\delta}], \quad x \in \mathbb{Z}^d.$$

Remark 1.3. As in Theorem 1.4 our estimates on the second derivatives of $G_{\mathbf{a}}(x)$ diverge as $\delta \to 1$.

Next we turn to the discrete version of Theorem 1.3. For a function $f : \mathbb{Z}^d \to \mathbb{C}$ we define its Fourier transform $\hat{f}$ by

$$\hat{f}(\xi) = \sum_{x \in \mathbb{Z}^d} f(x) e^{ix \cdot \xi}, \quad \xi \in \mathbb{R}^d.$$

For $1 \le k \le d$, $\xi \in \mathbb{R}^d$, let $e_k(\xi) = 1 - e^{i\mathbf{e}_k \cdot \xi}$ and $e(\xi)$ be the vector $e(\xi) = (e_1(\xi), \dots, e_d(\xi))$. Let $\hat{G}_{\mathbf{a}}(\xi, t)$ be the Fourier transform of the function $G_{\mathbf{a}}(x, t)$, $x \in \mathbb{Z}^d$, $t > 0$, defined by (1.9). From the equation before (1.10) it is clear that $|\hat{G}_{\mathbf{a}}(\xi, t)| \le 1$.

Theorem 1.6. *The function $\hat{G}_{\mathbf{a}}(\xi, t)$ is continuous for $\xi \in \mathbb{R}^d$ and differentiable for $t > 0$. Let δ satisfy $0 \le \delta < 1$. Then there is a constant $C(\delta, \lambda, \Lambda)$ depending only on δ, λ, Λ, such that*

$$\begin{aligned} |\hat{G}_{\mathbf{a}}(\xi, t)| &\le \frac{C(\delta, \lambda, \Lambda)}{[1 + |e(\xi)|^2 t]^{\delta}}, \\ \left|\frac{\partial \hat{G}_{\mathbf{a}}}{\partial t}(\xi, t)\right| &\le \frac{C(\delta, \lambda, \Lambda)|e(\xi)|^2}{[1 + |e(\xi)|^2 t]^{1+\delta}}, \end{aligned}$$

where $|e(\xi)|$ denotes the Euclidean norm of $e(\xi) \in \mathbb{C}^d$.

In order to prove Theorems 1.1–1.6 we use a representation for the Fourier transform of the expectation of the Green's function for the elliptic problem (1.12), which was obtained in [4] . This in turn gives us a formula for the Laplace transform of the function $\hat{G}_{\mathbf{a}}(\xi, t)$ of Theorem 1.6. We can prove Theorem 1.6 then by estimating the inverse Laplace transform. In order to prove Theorems 1.4, 1.5 we need to use interpolation theory, in particular the Hunt Interpolation Theorem [10]. Thus we prove that $\hat{G}_{\mathbf{a}}(\xi, t)$ is in a weak L^p space which will then imply pointwise bounds on the Fourier inverse. We shall prove here Theorems 1.4–1.6 in detail. In the final section we shall show how to generalize the proof of Theorem 1.5 to prove Theorem 1.2. The proofs of Theorems 1.1 and 1.3 are left to the interested reader. We would like to thank Jana Björn and Vladimir Maz'ya for help with the proof of Lemma 2.6.

There is already a large body of literature on the problem of homogenization of solutions of elliptic and parabolic equations with random coefficients, [4] [6] [7] [8] [11]. These results prove in a certain sense that, asympotically, the lowest frequency components of the functions $G_{\mathbf{a}}(x)$ and $G_{\mathbf{a}}(x, t)$ are the same as the corresponding quantities for a constant coefficient equation. The constant depends on the random matrix $\mathbf{a}(\cdot)$. The problem of homogenization in a periodic medium has also been studied [2] [11], and similar results been obtained.

2. Proof of Theorem 1.6

Let $\hat{G}_{\mathbf{a}}(\xi,t),\ \xi\in\mathbb{R}^d,\ t>0$, be the function in Theorem 1.6. Our first goal will be to obtain a formula for the Laplace transform of $\hat{G}_{\mathbf{a}}(\xi,t)$, which we denote by $\hat{G}_{\mathbf{a}}(\xi,\eta)$,

$$\hat{G}_{\mathbf{a}}(\xi,\eta)=\int_0^\infty dt\ e^{-\eta t}\hat{G}_{\mathbf{a}}(\xi,t)\ ,\quad \mathrm{Re}(\eta)>0\ .$$

It is evident that $\hat{G}_{\mathbf{a}}(\xi,\eta)$ is the Fourier transform of the expectation of the Green's function for the elliptic problem,

$$\eta u(x,\omega)+\sum_{i,j=1}^d\nabla_i^*\left[a_{i,j}(\tau_x\omega)\nabla_j u(x,\omega)\right]=f(x,\omega)\ ,\quad x\in\mathbb{Z}^d\ . \tag{2.1}$$

In [4] we derived a formula for this. To do that we defined operators $\partial_i,\ 1\le i\le d$, on functions $\psi:\Omega\to\mathbb{C}$ by $\partial_i\psi(\omega)=\psi(\tau_{\mathbf{e}_i}\omega)-\psi(\omega)$, with corresponding adjoint operators $\partial_i^*,\ 1\le i\le d$, defined by $\partial_i^*\psi(\omega)=\psi(\tau_{-\mathbf{e}_i}\omega)-\psi(\omega)$. Hence for $\xi\in\mathbb{R}^d$ we may define an operator $\mathcal{L}_\xi$ on functions $\psi:\Omega\to\mathbb{C}$ by

$$\mathcal{L}_\xi\psi(\omega)=P\sum_{i,j=1}^d e^{i\xi\cdot(\mathbf{e}_i-\mathbf{e}_j)}\left[\partial_i^*+e_i(-\xi)\right]a_{i,j}(\omega)\left[\partial_j+e_j(\xi)\right]\psi(\omega)\ ,$$

where P is the projection orthogonal to the constant function and $e_j(\xi)$ is defined just before the statement of Theorem 1.6. Note that $\mathcal{L}_\xi$ takes a function ψ to a function $\mathcal{L}_\xi\psi$ satisfying $\langle\mathcal{L}_\xi\psi\rangle=0$. Now for $1\le k\le d,\ \xi\in\mathbb{R}^d,\ \mathrm{Re}(\eta)>0$, let $\psi_k(\xi,\eta,\omega)$ be the solution to the equation,

$$[\mathcal{L}_\xi+\eta]\psi_k(\xi,\eta,\omega)+\sum_{j=1}^d e^{i\mathbf{e}_j\cdot\xi}\left[\partial_j^*+e_j(-\xi)\right]\left[a_{k,j}(\omega)-\langle a_{k,j}(\cdot)\rangle\right]=0\ . \tag{2.2}$$

Then we may define a $d\times d$ matrix $q(\xi,\eta)$ by,

$$q_{k,k'}(\xi,\eta)=\left\langle a_{k,k'}(\cdot)+\sum_{j=1}^d a_{k,j}(\cdot)e^{-i\mathbf{e}_j\cdot\xi}\left[\partial_j+e_j(\xi)\right]\psi_{k'}(\xi,\eta,\cdot)\right\rangle\ . \tag{2.3}$$

The function $\hat{G}_{\mathbf{a}}(\xi,\eta)$ is then given by the formula,

$$\hat{G}_{\mathbf{a}}(\xi,\eta)=\frac{1}{\eta+e(\xi)q(\xi,\eta)e(-\xi)},\quad \xi\in\mathbb{R}^d,\quad \mathrm{Re}(\eta)>0\ . \tag{2.4}$$

We actually established the formula (2.4) in [4] when η is real and positive. In that case $q(\xi,\eta)$ is a $d\times d$ Hermitian matrix bounded below in the quadratic form sense by λI_d. It follows that $\hat{G}_{\mathbf{a}}(\xi,\eta)$ is finite for all positive η. We wish to establish this for all η satisfying $\mathrm{Re}(\eta)>0$. We can in fact argue this from (2.1). Suppose the function on the RHS of (2.1) is a function of x only, $f(x,\omega)=f(x)$. Then the Fourier transform $\hat{u}(\xi,\omega)$ of the solution to (2.1) satisfies the equation,

$$\langle\hat{u}(\xi,\cdot)\rangle=\hat{G}_{\mathbf{a}}(\xi,\eta)\hat{f}(\xi),\quad \xi\in\mathbb{R}^d.$$

If we multiply (2.1) by $\overline{u(x,\omega)}$, and sum with respect to x, we have by the Plancherel Theorem,

$$|\eta|^2 \int_{[-\pi,\pi]^d} |\hat{u}(\xi,\omega)|^2 d\xi \le \int_{[-\pi,\pi]^d} |\hat{f}(\xi)|^2 d\xi \ .$$

Since $\hat{f}(\xi)$ is an arbitrary function it follows that $|\hat{G}_{\mathbf{a}}(\xi,\eta)| \le 1/|\eta|$. We improve this inequality in the following:

Lemma 2.1. *Suppose* $\mathrm{Re}(\eta) > 0$ *and* $\xi \in \mathbb{R}^d$. *Let* $\rho = (\rho_1, \dots, \rho_d) \in \mathbb{C}^d$. *Then*

$$\mathrm{Re}[\eta + \bar{\rho} q(\xi,\eta)\rho] \ \ge \ \mathrm{Re}(\eta) + \lambda|\rho|^2, \tag{2.5}$$

$$\mathrm{Im}(\eta)\mathrm{Im}[\bar{\rho} q(\xi,\eta)\rho] \ \ge \ 0. \tag{2.6}$$

Proof. From (2.2), (2.3) we have that

$$\begin{aligned} q_{k,k'}(\xi,\eta) = \Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot) \left[\delta_{k,i} + e^{i\mathbf{e}_i\cdot\xi}[\partial_i + e_i(-\xi)]\psi_k(-\xi,\eta,\cdot)\right] & \\ \left[\delta_{k',j} + e^{-i\mathbf{e}_j\cdot\xi}[\partial_j + e_j(\xi)]\psi_{k'}(\xi,\eta,\cdot)\right] \Big\rangle + \eta\Big\langle \psi_k(-\xi,\eta,\cdot)\psi_{k'}(\xi,\eta,\cdot)\Big\rangle. \end{aligned}$$

Thus we have

$$\begin{aligned} \bar{\rho} q(\xi,\eta)\rho = \Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot) \Big[\bar{\rho}_i + e^{i\mathbf{e}_i\cdot\xi}[\partial_i + e_i(-\xi)]\overline{\varphi(\xi,\bar{\eta},\cdot)}\Big] & \\ \left[\rho_j + e^{-i\mathbf{e}_j\cdot\xi}[\partial_j + e_j(\xi)]\varphi(\xi,\eta,\cdot)\right] \Big\rangle + \eta\Big\langle \overline{\varphi(\xi,\bar{\eta},\cdot)}\varphi(\xi,\eta,\cdot)\Big\rangle, \end{aligned} \tag{2.7}$$

where

$$\varphi(\xi,\eta,\cdot) = \sum_{k=1}^d \rho_k \psi_k(\xi,\eta,\cdot) \ . \tag{2.8}$$

Evidently we have that

$$[\mathcal{L}_\xi + \eta]\varphi(\xi,\eta,\omega) + \sum_{k=1}^d \rho_k \sum_{j=1}^d e^{i\mathbf{e}_j\cdot\xi}[\partial_j^* + e_j(-\xi)]\left[a_{k,j}(\omega) - \langle a_{k,j}(\cdot)\rangle\right] = 0 \ . \tag{2.9}$$

It follows from the last equation that

$$[\mathcal{L}_\xi + \eta]\varphi(\xi,\eta,\cdot) = [\mathcal{L}_\xi + \bar{\eta}]\varphi(\xi,\bar{\eta},\cdot),$$

whence

$$[\mathcal{L}_\xi + \mathrm{Re}(\eta)][\varphi(\xi,\eta,\cdot) - \varphi(\xi,\bar{\eta},\cdot)] = -i\ \mathrm{Im}(\eta)[\varphi(\xi,\eta,\cdot) + \varphi(\xi,\bar{\eta},\cdot)] \ . \tag{2.10}$$

Hence

$$\begin{aligned} \Big\langle[\overline{\varphi(\xi,\eta,\cdot)} - \overline{\varphi(\xi,\bar{\eta},\cdot)}][\mathcal{L}_\xi + \mathrm{Re}(\eta)][\varphi(\xi,\eta,\cdot) - \varphi(\xi,\bar{\eta},\cdot)]\Big\rangle & \\ = -i\ \mathrm{Im}(\eta)\Big\langle[\overline{\varphi(\xi,\eta,\cdot)} - \overline{\varphi(\xi,\bar{\eta},\cdot)}][\varphi(\xi,\eta,\cdot) + \varphi(\xi,\bar{\eta},\cdot)]\Big\rangle \ . \end{aligned} \tag{2.11}$$

Observe that since the LHS of (2.11) is real, the quantity

$$\Big\langle[\overline{\varphi(\xi,\eta,\cdot)} - \overline{\varphi(\xi,\bar{\eta},\cdot)}][\varphi(\xi,\eta,\cdot) + \varphi(\xi,\bar{\eta},\cdot)]\Big\rangle$$

is pure imaginary.

Next for $1 \le j \le d$, let us put

$$A_j = \rho_j + e^{-i\mathbf{e}_j\cdot\xi}[\partial_j + e_j(\xi)]\frac{1}{2}\{\varphi(\xi,\eta,\cdot) + \varphi(\xi,\bar\eta,\cdot)\},$$
$$B_j = e^{-i\mathbf{e}_j\cdot\xi}[\partial_j + e_j(\xi)]\frac{1}{2}\{\varphi(\xi,\eta,\cdot) - \varphi(\xi,\bar\eta,\cdot)\}.$$

Then

$$\bar\rho q(\xi,\eta)\rho = \Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)[\bar A_i - \bar B_i][A_j + B_j]\Big\rangle + \eta\Big\langle\overline{\varphi(\xi,\bar\eta,\cdot)}\varphi(\xi,\eta,\cdot)\Big\rangle.$$

We can decompose this sum into real and imaginary parts. Thus

$$\Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)[\bar A_i - \bar B_i][A_j + B_j]\Big\rangle = \Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar A_i A_j\Big\rangle - \Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar B_i B_j\Big\rangle + 2i\ \mathrm{Im}\Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar A_i B_j\Big\rangle.$$

Evidently the first two terms on the RHS of the last equation are real while the third term is pure imaginary. We also have that

$$\begin{aligned}
&\Big\langle\overline{\varphi(\xi,\bar\eta,\cdot)}\varphi(\xi,\eta,\cdot)\Big\rangle \\
&\quad= \frac{1}{4}\Big\langle \Big\{[\overline{\varphi(\xi,\bar\eta,\cdot)} + \overline{\varphi(\xi,\eta,\cdot)}] + [\overline{\varphi(\xi,\bar\eta,\cdot)} - \overline{\varphi(\xi,\eta,\cdot)}]\Big\} \\
&\qquad\quad \{\varphi(\xi,\eta,\cdot) + \varphi(\xi,\bar\eta,\cdot)] + [\varphi(\xi,\eta,\cdot) - \varphi(\xi,\bar\eta,\cdot)]\}\Big\rangle \\
&\quad= \frac{1}{4}\Big\langle |\varphi(\xi,\eta,\cdot) + \varphi(\xi,\bar\eta,\cdot)|^2\Big\rangle - \frac{1}{4}\Big\langle |\varphi(\xi,\eta,\cdot) - \varphi(\xi,\bar\eta,\cdot)|^2\Big\rangle \\
&\qquad - \frac{i}{2\mathrm{Im}(\eta)}\Big\langle[\overline{\varphi(\xi,\eta,\cdot)} - \overline{\varphi(\xi,\bar\eta,\cdot)}][\mathcal{L}_\xi + \mathrm{Re}(\eta)][\varphi(\xi,\eta,\cdot) - \varphi(\xi,\bar\eta,\cdot)]\Big\rangle,
\end{aligned}$$

where we have used (2.11). Observe that the first two terms on the RHS of the last equation are real while the third term is pure imaginary. Hence

$$\begin{aligned}
&\eta\Big\langle\overline{\varphi(\xi,\bar\eta,\cdot)}\varphi(\xi,\eta,\cdot)\Big\rangle = \\
&\qquad \frac{\mathrm{Re}(\eta)}{4}\Big\langle|\varphi(\xi,\eta,\cdot) + \varphi(\xi,\bar\eta,\cdot)|^2\Big\rangle - \frac{\mathrm{Re}(\eta)}{4}\Big\langle|\varphi(\xi,\eta,\cdot) - \varphi(\xi,\bar\eta,\cdot)|^2\Big\rangle \\
&\qquad + \frac{1}{2}\Big\langle\Big[\overline{\varphi(\xi,\eta,\cdot)} - \overline{\varphi(\xi,\bar\eta,\cdot)}][\mathcal{L}_\xi + \mathrm{Re}(\eta)][\varphi(\xi,\eta,\cdot) - \varphi(\xi,\bar\eta,\cdot)]\Big]\Big\rangle \\
&\qquad + i\frac{\mathrm{Im}(\eta)}{4}\Big\langle|\varphi(\xi,\eta,\cdot) + \varphi(\xi,\bar\eta,\cdot)|^2\Big\rangle - i\frac{\mathrm{Im}(\eta)}{4}\Big\langle|\varphi(\xi,\eta,\cdot) - \varphi(\xi,\bar\eta,\cdot)|^2\Big\rangle \\
&\qquad - \frac{i\ \mathrm{Re}(\eta)}{2\ \mathrm{Im}(\eta)}\Big\langle\Big[[\overline{\varphi(\xi,\eta,\cdot)} - \overline{\varphi(\xi,\bar\eta,\cdot)}][\mathcal{L}_\xi + \mathrm{Re}(\eta)][\varphi(\xi,\eta,\cdot) - \varphi(\xi,\bar\eta,\cdot)]\Big]\Big\rangle.
\end{aligned}$$

We conclude then from the last four equations that

(2.12)

$$\begin{aligned}\mathrm{Re}[\,\bar\rho q(\xi,\eta)\rho] = &\Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar A_i A_j\Big\rangle - \Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar B_i B_j\Big\rangle \\ &+ \frac{\mathrm{Re}(\eta)}{4}\Big\langle|\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)|^2\Big\rangle - \frac{\mathrm{Re}(\eta)}{4}\Big\langle|\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)|^2\Big\rangle \\ &+\frac12\Big\langle\Big[[\overline{\varphi(\xi,\eta,\cdot)}-\overline{\varphi(\xi,\bar\eta,\cdot)}][\mathcal{L}_\xi+\mathrm{Re}(\eta)][\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\Big]\Big\rangle\ ,\end{aligned}$$

(2.13)

$$\begin{aligned}\mathrm{Im}[\bar\rho q(\xi,\eta)\rho] = &2\ \mathrm{Im}\ \Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar A_i B_j\Big\rangle + \frac{\mathrm{Im}(\eta)}{4}\Big\langle|\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)|^2\Big\rangle \\ &- \frac{\mathrm{Im}(\eta)}{4}\Big\langle|\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)|^2\Big\rangle - \frac{\mathrm{Re}(\eta)}{2\ \mathrm{Im}(\eta)}\cdot \\ &\Big\langle\Big[[\overline{\varphi(\xi,\eta,\cdot)}-\overline{\varphi(\xi,\bar\eta,\cdot)}][\mathcal{L}_\xi+\mathrm{Re}(\eta)][\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\Big]\Big\rangle.\end{aligned}$$

We can simplify the expression on the RHS of (2.12) by observing that

$$\Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar B_i B_j\Big\rangle = \frac14\Big\langle[\overline{\varphi(\xi,\eta,\cdot)}-\overline{\varphi(\xi,\bar\eta,\cdot)}]\mathcal{L}_\xi[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)]\Big\rangle.$$

Hence we obtain the expression,

$$\begin{aligned}(2.14)\quad \mathrm{Re}[\bar\rho q(\xi,\eta)\rho] = &\Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar A_i A_j\Big\rangle + \frac{\mathrm{Re}(\eta)}{4}\Big\langle|\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)|^2\Big\rangle \\ &+\frac14\Big\langle[\overline{\varphi(\xi,\eta,\cdot)}-\overline{\varphi(\xi,\bar\eta,\cdot)}][\mathcal{L}_\xi+\mathrm{Re}(\eta)][\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)]\Big\rangle.\end{aligned}$$

Now all the terms on the RHS of the last expression are nonnegative, and from Jensen's inequality,

$$\Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar A_i A_j\Big\rangle \ge \lambda\,|\rho|^2.$$

The inequality in (2.5) follows from this.

To prove the inequality (2.6) we need to rewrite the RHS of (2.13). We have now

$$\begin{aligned}\Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar A_i B_j\Big\rangle = &\Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar\rho_i B_j\Big\rangle \\ &+\frac14\Big\langle\ \Big[[\overline{\varphi(\xi,\eta,\cdot)}+\overline{\varphi(\xi,\bar\eta,\cdot)}]\mathcal{L}_\xi[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\Big]\ \Big\rangle.\end{aligned}$$

From (2.10) we have that

$$\begin{aligned}&\left\langle \left[\overline{\varphi(\xi,\eta,\cdot)}+\overline{\varphi(\xi,\bar\eta,\cdot)}\right]\mathcal{L}_\xi\left[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle\\
&\qquad= -i\,\mathrm{Im}(\eta)\left\langle|\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle\\
&\qquad\quad-\ \mathrm{Re}(\eta)\left\langle\left[\overline{\varphi(\xi,\eta,\cdot)}+\overline{\varphi(\xi,\bar\eta,\cdot)}\right]\left[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle\\
&\qquad= -i\,\mathrm{Im}(\eta)\left\langle|\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle\\
&\qquad\quad+i\frac{\mathrm{Re}(\eta)}{\mathrm{Im}(\eta)}\left\langle\left[\overline{\varphi(\xi,\eta,\cdot)}-\overline{\varphi(\xi,\bar\eta,\cdot)}\right]\left[\mathcal{L}_\xi+\mathrm{Re}(\eta)\right]\left[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle,\end{aligned}$$

where we have used (2.11). We also have that

$$\begin{aligned}&\left\langle\sum_{i,j=1}^d a_{i,j}(\cdot)\bar\rho_i B_j\right\rangle\\
&\qquad=\frac{1}{2}\left\langle\left\{\sum_{i,j=1}^d\overline{\rho_i e^{i\mathbf{e}_j\cdot\xi}\left[\partial_j^*+e_j(-\xi)\right]a_{i,j}(\cdot)}\right\}\left[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle\\
&\qquad=-\frac{1}{4}\left\langle\overline{\left\{\left[\mathcal{L}_\xi+\eta\right]\varphi(\xi,\cdot)+\left[\mathcal{L}_\xi+\bar\eta\right]\varphi(\xi,\bar\eta,\cdot)\right\}}\left[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle\\
&\qquad=-\frac{1}{4}\left\langle\overline{\left\{\left[\mathcal{L}_\xi+\ \mathrm{Re}(\eta)\right]\left[\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)\right]\right\}}\left[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle\\
&\qquad\quad+\frac{i}{4}\,\mathrm{Im}(\eta)\left\langle|\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle\\
&\qquad=-\frac{1}{4}\left\langle\left[\overline{\varphi(\xi,\eta,\cdot)}+\overline{\varphi(\xi,\bar\eta,\cdot)}\right]\left[\mathcal{L}_\xi+\ \mathrm{Re}(\eta)\right]\left[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle\\
&\qquad\quad+\frac{i}{4}\,\mathrm{Im}(\eta)\left\langle|\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle\\
&\qquad=\frac{i}{4}\,\mathrm{Im}(\eta)\left[\left\langle|\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle+\left\langle|\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle\right].\end{aligned}$$

It follows now from (2.13) and the last three identities that

$$\begin{aligned}\mathrm{Im}[\bar\rho q(\xi,\eta)\rho]&=\frac{1}{4}\,\mathrm{Im}(\eta)\left\langle|\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle\\
&\quad+\frac{1}{4}\,\mathrm{Im}(\eta)\left\langle|\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle.\end{aligned}\tag{2.15}$$

The inequality (2.6) follows. □

Let us denote by $\hat G_{\mathbf{a}}(\xi,t),\ t>0$ the inverse Laplace transform of $\hat G_{\mathbf{a}}(\xi,\eta)$. Thus from (2.4) we have

$$\hat G_{\mathbf{a}}(\xi,t)=\lim_{N\to\infty}\frac{1}{2\pi}\int_{-N}^{N}\frac{e^{\eta t}}{\eta+e(\xi)q(\xi,\eta)e(-\xi)}\,d[\mathrm{Im}(\eta)].\tag{2.16}$$

It follows from Lemma 2.1 that, provided $\mathrm{Re}(\eta)>0$, the integral in (2.16) over the finite interval $-N<\ \mathrm{Im}(\eta)<N$ exists for all N. We need then to show that the limit as $N\to\infty$ in (2.16) exists.

Lemma 2.2. *Suppose $Re(\eta)>0$ and $\xi\in\mathbb{R}^d$. Then the limit in* (2.16) *as $N\to\infty$ exists and is independent of $Re(\eta)>0$.*

Proof. We first note that for any $\rho \in \mathbb{C}^d$, $\bar\rho q(\xi,\eta)\rho$ and $\bar\rho q(\xi,\bar\eta)\rho$ are complex conjugates. This follows easily from (2.7). We conclude from this that

(2.17)
$$\frac{1}{2\pi}\int_{-N}^{N} \frac{e^{\eta t}}{\eta+e(\xi)q(\xi,\eta)e(-\xi)}\, d[\mathrm{Im}(\eta)] = \frac{1}{\pi}\int_0^N \mathrm{Re}\,\frac{e^{\eta t}}{\eta+e(\xi)q(\xi,\eta)e(-\xi)}\, d[\mathrm{Im}(\eta)]$$
$$= \frac{1}{\pi}\exp[\mathrm{Re}(\eta)t]\Big\{\int_0^N h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] + \int_0^N k(\xi,\eta)\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]\Big\},$$

where

$$\begin{aligned} h(\xi,\eta) &= \ \mathrm{Re}\,[\eta+e(\xi)q(\xi,\eta)e(-\xi)]/|\eta+e(\xi)q(\xi,\eta)e(-\xi)|^2,\\ k(\xi,\eta) &= \ \mathrm{Im}\,[\eta+e(\xi)q(\xi,\eta)e(-\xi)]/|\eta+e(\xi)q(\xi,\eta)e(-\xi)|^2. \end{aligned} \tag{2.18}$$

We show there is a constant $C_{\lambda,\Lambda}$, depending only on λ,Λ such that

$$\int_0^\infty |h(\xi,\eta)| d[\mathrm{Im}(\eta)] \le C_{\lambda,\Lambda}, \quad \mathrm{Re}(\eta)>0. \tag{2.19}$$

To see this, observe from (2.14), (2.15) that

$$|h(\xi,\eta)| \le \frac{\mathrm{Re}\ (\eta)+\Theta}{[\{\cdot\}^2 + \ (\mathrm{Im}\ \eta)^2]}, \quad \text{where} \tag{2.20}$$
$$\Theta = \frac{\left\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar A_i A_j\right\rangle + \frac14\left\langle\left[\overline{\varphi(\xi,\eta,\cdot)}-\overline{\varphi(\xi,\bar\eta,\cdot)}\right]\mathcal{L}_\xi\left[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle}{\left[1+\frac14\langle|\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)|^2\rangle + \frac14\langle|\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)|^2\rangle\right.},$$

and the quantity $\{\cdot\}$ in the first line of (2.20) is the same as the one in the second. It is easy to see that

(2.21)
$$\left\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar A_i A_j\right\rangle + \frac14\left\langle\left[\overline{\varphi(\xi,\eta,\cdot)}-\overline{\varphi(\xi,\bar\eta,\cdot)}\right]\mathcal{L}_\xi\left[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle \ge \lambda|e(\xi)|^2.$$

We can also obtain an upper bound using the fact that

$$\begin{aligned} &\left\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar A_i A_j\right\rangle + \frac14\left\langle\left[\overline{\varphi(\xi,\eta,\cdot)}-\overline{\varphi(\xi,\bar\eta,\cdot)}\right]\mathcal{L}_\xi\left[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle\\ &\quad\le 2\Big\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\overline{e_i(\xi)}e_j(\xi)\Big\rangle + \frac12\left\langle\left[\overline{\varphi(\xi,\eta,\cdot)}+\overline{\varphi(\xi,\bar\eta,\cdot)}\right]\mathcal{L}_\xi\left[\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle\\ &\qquad+\frac14\left\langle\left[\overline{\varphi(\xi,\eta,\cdot)}-\overline{\varphi(\xi,\bar\eta,\cdot)}\right]\mathcal{L}_\xi\left[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle\\ &\quad\le 2\Lambda|e(\xi)|^2 + \frac14\left\langle\left[\overline{\varphi(\xi,\eta,\cdot)}+\overline{\varphi(\xi,\bar\eta,\cdot)}\right]\mathcal{L}_\xi\left[\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle\\ &\qquad+\frac12\left\langle\overline{\varphi(\xi,\eta,\cdot)}\mathcal{L}_\xi\varphi(\xi,\eta,\cdot)\right\rangle + \frac12\left\langle\overline{\varphi(\xi,\bar\eta,\cdot)}\mathcal{L}_\xi\varphi(\xi,\bar\eta,\cdot)\right\rangle. \end{aligned}$$

We see from (2.9) that

$$\left\langle \overline{\varphi(\xi,\eta,\cdot)}\mathcal{L}_\xi \varphi(\xi,\eta,\cdot)\right\rangle \le \Lambda|e(\xi)|^2,$$
$$\left\langle \overline{\varphi(\xi,\bar\eta,\cdot)}\mathcal{L}_\xi \varphi(\xi,\bar\eta,\cdot)\right\rangle \le \Lambda|e(\xi)|^2,$$
$$\left\langle \left[\overline{\varphi(\xi,\eta,\cdot)}+\overline{\varphi(\xi,\bar\eta,\cdot)}\right]\mathcal{L}_\xi \left[\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle \le 4\Lambda|e(\xi)|^2.$$

Hence we obtain an upper bound

(2.22)
$$\left\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\bar A_i A_j\right\rangle + \frac14\left\langle \left[\overline{\varphi(\xi,\eta,\cdot)}-\overline{\varphi(\xi,\bar\eta,\cdot)}\right]\mathcal{L}_\xi \left[\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)\right]\right\rangle \le 4\Lambda|e(\xi)|^2.$$

Observe that the upper and lower bounds (2.21), (2.22) are comparable for all $\mathrm{Re}(\eta)>0$, $\xi\in\mathbb{R}^d$.

Next we need to find upper and lower bounds on the quantity,

$$\frac14\left\langle|\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle + \frac14\left\langle|\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle$$
$$= \frac12\left\langle|\varphi(\xi,\eta,\cdot)|^2\right\rangle + \frac12\left\langle|\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle.$$

Evidently zero is a trivial lower bound. To get an upper bound we use (2.9) again. We have from (2.9) that

$$|\eta|\left\langle|\varphi(\xi,\eta,\cdot)|^2\right\rangle \le \left\langle \overline{\varphi(\xi,\eta,\cdot)}\mathcal{L}_\xi\varphi(\xi,\eta,\cdot)\right\rangle^{1/2}$$
$$\left\langle \sum_{i,j=1}^d a_{i,j}(\cdot)\overline{e_i(\xi)}e_j(\xi)\right\rangle^{1/2} \le \Lambda|e(\xi)|^2,$$

whence

$$\left\langle|\varphi(\xi,\eta,\cdot)|^2\right\rangle \le \Lambda|e(\xi)|^2/|\eta|.$$

We conclude then that

$$\frac14\left\langle|\varphi(\xi,\eta,\cdot)+\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle + \frac14\left\langle|\varphi(\xi,\eta,\cdot)-\varphi(\xi,\bar\eta,\cdot)|^2\right\rangle \le \Lambda|e(\xi)|^2/|\eta|. \tag{2.23}$$

We use this last inequality together with (2.21), (2.22) to prove (2.19). First note from (2.5) that

$$|h(\xi,\eta)| \le 1/\left[\mathrm{Re}(\eta)+\lambda|e(\xi)|^2\right],\ \mathrm{Re}(\eta)>0,\quad \xi\in\mathbb{R}^d.$$

We also have using (2.20),(2.21), (2.22),(2.23) that

$$|h(\xi,\eta)| \le \left\{\mathrm{Re}(\eta)+4\Lambda|e(\xi)|^2\right\}/\left[\mathrm{Im}(\eta)^2+\left\{\mathrm{Re}(\eta)+\frac12\,\lambda|e(\xi)|^2\right\}^2\right], \tag{2.24}$$
$$\mathrm{Re}(\eta)>0,\quad |\eta|\ge\Lambda|e(\xi)|^2,\quad \xi\in\mathbb{R}^d.$$

We then have

$$\int_0^\infty |h(\xi,\eta)|d[\mathrm{Im}(\eta)] \le \int_0^{\Lambda|e(\xi)|^2}\frac{d[\mathrm{Im}(\eta)]}{\mathrm{Re}(\eta)+\lambda|e(\xi)|^2}$$
$$+\int_0^\infty \frac{\mathrm{Re}(\eta)+4\Lambda|e(\xi)|^2}{\mathrm{Im}(\eta)^2+[\mathrm{Re}(\eta)+\frac12\lambda|e(\xi)|^2]^2}d[\mathrm{Im}(\eta)] \le C_{\lambda,\Lambda},$$

where $C_{\lambda,\Lambda}$ depends only on λ, Λ, whence (2.19) follows.

Next we wish to show that the function $k(\xi,\eta)$ of (2.18) satisfies

$$|\partial k(\xi,\eta)/\partial[\mathrm{Im}(\eta)]| \le 1/|\mathrm{Im}(\eta)|^2,\ \mathrm{Re}(\eta) > 0,\ \xi \in \mathbb{R}^d. \tag{2.25}$$

In view of the analyticity in η of $q(\xi,\eta)$ we have

$$\begin{aligned}\partial k(\xi,\eta)/\partial[\mathrm{Im}(\eta)] &= -\mathrm{Re}\ \frac{\partial}{\partial\eta}\ \frac{1}{\eta + e(\xi)q(\xi,\eta)e(-\xi)} \\ &= \ \mathrm{Re}\ \frac{1 + e(\xi)[\partial q(\xi,\eta)/\partial\eta]e(-\xi)}{[\eta + e(\xi)q(\xi,\eta)e(-\xi)]^2}\ .\end{aligned}$$

Let us denote now by $\psi(\xi,\eta,\omega)$ the function $\varphi(\xi,\eta,\omega)$ of (2.8) with $\rho = e(-\xi)$. It is easy to see then from (2.9) that

$$\overline{\varphi(\xi,\bar{\eta},\omega)} = \psi(-\xi,\eta,\omega)\ . \tag{2.26}$$

It follows now from (2.7) and (2.9) that

$$\begin{aligned}e(\xi)[\partial q(\xi,\eta)/\partial\eta]e(-\xi) &= \big\langle\psi(-\xi,\eta,\cdot)\psi(\xi,\eta,\cdot)\big\rangle \\ &\quad + \eta\Big\langle\frac{\partial\psi}{\partial\eta}(-\xi,\eta,\cdot)\psi(\xi,\eta,\cdot)\Big\rangle + \eta\Big\langle\psi(-\xi,\eta,\cdot)\frac{\partial\psi}{\partial\eta}(\xi,\eta,\cdot)\Big\rangle \\ &\quad + \Big\langle\psi(-\xi,\eta,\cdot)\mathcal{L}_\xi\frac{\partial\psi}{\partial\eta}(\xi,\eta,\cdot)\Big\rangle + \Big\langle\frac{\partial\psi}{\partial\eta}(-\xi,\eta,\cdot)\mathcal{L}_\xi\psi(\xi,\eta,\cdot)\Big\rangle \\ &\quad - \Big\langle\frac{\partial\psi}{\partial\eta}(-\xi,\eta,\cdot)[\mathcal{L}_\xi + \eta]\psi(\xi,\eta,\cdot)\Big\rangle \\ &\quad - \Big\langle\psi(-\xi,\eta,\cdot)[\mathcal{L}_\xi + \eta]\frac{\partial\psi}{\partial\eta}(\xi,\eta,\cdot)\Big\rangle \\ &= \big\langle\psi(-\xi,\eta,\cdot)\psi(\xi,\eta,\cdot)\big\rangle\end{aligned}$$

Hence,

$$|\partial k(\xi,\eta)/\partial[\mathrm{Im}(\eta)]| \le \left|\frac{1 + \big\langle\psi(-\xi,\eta,\cdot)\psi(\xi,\eta,\cdot)\big\rangle}{[\eta + e(\xi)q(\xi,\eta)e(-\xi)]^2}\right| \le \frac{1}{\mathrm{Im}(\eta)^2}, \tag{2.27}$$

from (2.15).

We can use (2.19), (2.27) to show the limit in (2.16) exists. In fact from (2.19) it follows that

$$\lim_{N\to\infty}\int_0^N h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]$$

exists. We also have that

$$\begin{aligned}\int_0^N k(\xi,\eta)\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] &= -\frac{1}{t}\int_0^N \frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\Big\{1 - \cos[\mathrm{Im}(\eta)t]\Big\}d[\mathrm{Im}(\eta)] \\ &\quad + \frac{1}{t}\ k(\xi,\ \mathrm{Re}(\eta) + iN)\{1 - \cos[Nt]\}.\end{aligned}$$

From (2.6) we have that

$$|k(\xi,\ \mathrm{Re}(\eta) + iN)\{1 - \cos[Nt]\}| \le 2/N.$$

From (2.27) we have that

$$\frac{1}{t}\int_0^\infty \left|\frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\right| \Big\{1-\cos[\mathrm{Im}(\eta)t]\Big\} d[\mathrm{Im}(\eta)] \le C, \tag{2.28}$$

for some universal constant C. Hence the limit,

$$\lim_{N\to\infty}\int_0^N k(\xi,\eta)\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]$$

exists, whence the result holds. □

Lemma 2.3. *Let $\hat{G}_{\mathbf{a}}(\xi,t)$ be defined by* (2.16). *Then $\hat{G}_{\mathbf{a}}(\xi,t)$ is a continuous bounded function. Furthermore, for any δ, $0\le\delta<1$, there is a constant $C(\delta,\lambda,\Lambda)$, depending only on δ,λ,Λ, such that*

$$|\hat{G}_{\mathbf{a}}(\xi,t)| \le C(\delta,\lambda,\Lambda)\Big/[1+|e(\xi)|^2t]^\delta \ . \tag{2.29}$$

Proof. Consider the integral on the LHS of (2.28). We can obtain an improvement on the estimate of (2.28) by improving the estimate of (2.27). We have now from (2.27), (2.14), (2.15), and (2.23) that

$$|\partial k(\xi,\eta)/\partial[\mathrm{Im}(\eta)]| \le \frac{1+\Lambda|e(\xi)|^2/|\eta|}{\lambda^2|e(\xi)|^4+|\mathrm{Im}(\eta)|^2} \ . \tag{2.30}$$

Assume now that $|e(\xi)|^2t>2$ and write the integral on the LHS of (2.28) as a sum,

$$\frac{1}{t}\int_0^{1/t} + \frac{1}{t}\int_{1/t}^{|e(\xi)|^2} + \frac{1}{t}\int_{|e(\xi)|^2}^\infty .$$

We have now from (2.28), (2.30) that for $0\le\delta<1$,

$$\frac{1}{t}\int_0^{1/t} \le \frac{1}{t}\int_0^{1/t}\left[\frac{1+\Lambda}{\lambda^2|e(\xi)|^2\mathrm{Im}(\eta)}\right]^\delta \left[\frac{1-\cos[\mathrm{Im}(\eta)t]}{\mathrm{Im}(\eta)^2}\right]^{1-\delta} d[\mathrm{Im}(\eta)]$$

$$\le 10t^{1-2\delta}\int_0^{1/t}\left[\frac{1+\Lambda}{\lambda^2|e(\xi)|^2\mathrm{Im}(\eta)}\right]^\delta d[\mathrm{Im}(\eta)] \le C(\delta,\lambda,\Lambda)/[|e(\xi)|^2t]^\delta,$$

for some constant $C(\delta,\lambda,\Lambda)$ depending only on δ,λ,Λ. Next we have

$$\frac{1}{t}\int_{1/t}^{|e(\xi)|^2} \le \frac{1}{t}\int_{1/t}^{|e(\xi)|^2} \frac{1+\Lambda}{\lambda^2|e(\xi)|^2\mathrm{Im}(\eta)}\, d[\mathrm{Im}(\eta)]$$

$$= \frac{1+\Lambda}{\lambda^2}\ \frac{\log[|e(\xi)|^2\,t]}{|e(\xi)|^2\,t}.$$

Finally, we have

$$\frac{1}{t}\int_{|e(\xi)|^2}^\infty \le \frac{1}{t}\int_{|e(\xi)|^2}^\infty \frac{d[\mathrm{Im}(\eta)]}{\mathrm{Im}(\eta)^2} \le \frac{1}{|e(\xi)|^2\,t}.$$

We conclude therefore that the integral on the LHS of (2.28) is bounded by

$$\frac{C(\delta,\lambda,\Lambda)}{[1+|e(\xi)|^2\,t]^\delta}\ ,$$

for any $\delta,\ 0 \le \delta < 1$.

Next we need to estimate as $N \to \infty$ the integral

$$\int_0^N h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]\ d[\mathrm{Im}(\eta)] = -\frac{1}{t}\int_0^N \frac{\partial h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\sin[\mathrm{Im}(\eta)t]\ d[\mathrm{Im}(\eta)] + \frac{1}{t}\ h(\xi,\ \mathrm{Re}(\eta)+iN)\sin[Nt].$$

It is clear from (2.6) that $\lim_{N\to\infty} h(\xi,\ \mathrm{Re}(\eta)+iN) = 0$. We have now that

$$\frac{\partial h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]} = -\ \mathrm{Im}\ \frac{\partial}{\partial\eta}\ \frac{1}{\eta + e(\xi)q(\xi,\eta)e(-\xi)} = \mathrm{Im}\ \frac{1 + e(\xi)[\partial q(\xi,\eta)/\partial\eta]e(-\xi)}{[\eta + e(\xi)q(\xi,\eta)e(-\xi)]^2}. \tag{2.31}$$

Hence the estimates (2.27), (2.30) on the derivative $k(\xi,\eta)$ apply equally to the derivative of h. We therefore write the integral

$$\frac{1}{t}\int_0^\infty \left|\frac{\partial h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\right|\ |\sin[\mathrm{Im}(\eta)t]|d[\mathrm{Im}(\eta)] = \frac{1}{t}\int_0^{1/t} + \frac{1}{t}\int_{1/t}^{|e(\xi)|^2} + \frac{1}{t}\int_{|e(\xi)|^2}^\infty . \tag{2.32}$$

as a sum just as before. We have now from (2.30),

$$\frac{1}{t}\int_0^{1/t} \le \frac{1}{t}\int_0^{1/t}\left[\frac{1+\Lambda}{\lambda^2|e(\xi)|^2\ \mathrm{Im}(\eta)}\right]|\sin[\mathrm{Im}(\eta)t]|d[\mathrm{Im}(\eta)] \le C_{\lambda,\Lambda}/[|e(\xi)|^2 t],$$

where $C_{\lambda,\Lambda}$ depends only on λ,Λ. The other integrals on the RHS of (2.32) are estimated just as for the corresponding integrals in k. We conclude that

$$\left|\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]|d[\mathrm{Im}(\eta)]\right| \le C(\delta,\lambda,\Lambda)/[1+|e(\xi)|^2 t]^\delta,$$

for any $\delta,\ 0 \le \delta < 1$. It follows now from (2.17) that (2.29) holds for any $\delta,\ 0 \le \delta < 1$. $\square$

Lemma 2.4. *The function $\hat{G}_{\mathbf{a}}(\xi,t),\ \xi \in \mathbb{R}^d$ is t differentiable for $t > 0$. For any $\delta,\ 0 \le \delta < 1$, there is a constant $C(\delta,\lambda,\Lambda)$ depending only on δ,λ,Λ such that*

$$\left|\frac{\partial \hat{G}_{\mathbf{a}}(\xi,t)}{\partial t}\right| \le \frac{C(\delta,\lambda,\Lambda)}{t[1+|e(\xi)|^2 t]^\delta},\quad t > 0.$$

Proof. From Lemma 2.2 we have that

$$\pi\exp[-\mathrm{Re}(\eta)t]\hat{G}_{\mathbf{a}}(\xi,t) = \int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] - \frac{1}{t}\int_0^\infty \frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\{1-\cos[\mathrm{Im}(\eta)t]\}\ d[\mathrm{Im}(\eta)]. \tag{2.33}$$

We consider the first term on the RHS of (2.33). Evidently, for finite N, we have

(2.34)
$$\begin{aligned}\frac{\partial}{\partial t}\int_0^N h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] &= -\int_0^N h(\xi,\eta)\ \mathrm{Im}(\eta)\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]\\ &= \frac{1}{t}\int_0^N \frac{\partial}{\partial[\mathrm{Im}(\eta)]}\{h(\xi,\eta)\mathrm{Im}(\eta)\}\{1-\cos[\mathrm{Im}(\eta)t]\}\,d[\mathrm{Im}(\eta)]\\ &- \frac{1}{t}h(\xi,\ \mathrm{Re}(\eta)+iN)N\{1-\cos[Nt]\}.\end{aligned}$$

It is clear from the inequality (2.24) that

$$\lim_{N\to\infty}\ h(\xi,\ \mathrm{Re}(\eta)+iN)N = 0.$$

We have already seen from (2.24) that

$$\int_0^\infty |h(\xi,\eta)|d[\mathrm{Im}(\eta)] \le C_{\lambda,\Lambda}, \tag{2.35}$$

for some constant $C_{\lambda,\Lambda}$ depending only on λ,Λ. We shall show now that we also have

$$\int_0^\infty \left|\frac{\partial h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\right| |\mathrm{Im}(\eta)|d[\mathrm{Im}(\eta)] \le C_{\lambda,\Lambda}\ . \tag{2.36}$$

To see this we use (2.31) to obtain

$$\frac{\partial h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]} = \mathrm{Im}\ \frac{1+\langle\psi(\xi,\eta,\cdot)\psi(-\xi,\eta,\cdot)\rangle}{[\eta+e(\xi)q(\xi,\eta)e(-\xi)]^2}. \tag{2.37}$$

For any complex number $a+ib$ it is clear that

$$\mathrm{Im}\frac{1}{(a+ib)^2} = \frac{-2ab}{(a^2+b^2)^2},$$

whence

$$\left|\mathrm{Im}\frac{1}{(a+ib)^2}\right| \le \min\left[\frac{1}{a^2},\ \frac{2|a|}{|b|^3}\right].$$

We conclude then that

$$\left|\ \mathrm{Im}\ \frac{1}{[\eta+e(\xi)q(\xi,\eta)e(-\xi)]^2}\right| \le \min\left[\frac{1}{\lambda^2|e(\xi)|^4}\ ,\ \frac{8\Lambda|e(\xi)|^2}{|\mathrm{Im}(\eta)|^3}\right].$$

It follows that

(2.38)
$$\int_0^\infty \mathrm{Im}(\eta)\ \left|\mathrm{Im}\ \frac{1}{[\eta+e(\xi)q(\xi,\eta)e(-\xi)]^2}\right| d[\mathrm{Im}(\eta)] \le \int_0^{|e(\xi)|^2} + \int_{|e(\xi)|^2}^\infty \le C_{\lambda,\Lambda}.$$

We also have that

$$\left|\frac{\langle\psi(\xi,\eta,\cdot)\psi(-\xi,\eta,\cdot)\rangle}{[\eta+e(\xi)q(\xi,\eta)e(-\xi)]^2}\right| \le \min\left[\frac{\Lambda}{\lambda^2|e(\xi)|^2|\mathrm{Im}(\eta)|}\ ,\ \frac{\Lambda|e(\xi)|^2}{|\mathrm{Im}(\eta)|^3}\right].$$

We conclude that

$$\int_0^\infty \mathrm{Im}(\eta)\left|\frac{\langle\psi(\xi,\eta,\cdot)\psi(-\xi,\eta,\cdot)\rangle}{[\eta+e(\xi)q(\xi,\eta)e(-\xi)]^2}\right| d[\mathrm{Im}(\eta)] \le C_{\lambda,\Lambda}. \tag{2.39}$$

The inequality (2.36) follows from (2.38), (2.39). It follows from (2.34), (2.35), (2.36) that the first integral on the RHS of (2.33) is differentiable with respect to t for $t > 0$ and

$$(2.40)\quad \frac{\partial}{\partial t}\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)t] = \\ \frac{1}{t}\int_0^\infty \frac{\partial}{\partial[\mathrm{Im}(\eta)]}\{h(\xi,\eta)\ \mathrm{Im}(\eta)\}\Big[1-\cos[\mathrm{Im}(\eta)t]\Big]d[\mathrm{Im}(\eta)].$$

Furthermore, there is the inequality,

$$\left|\frac{\partial}{\partial t}\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]\right| \le C_{\lambda,\Lambda}/t.$$

Next we wish to improve this inequality to

$$(2.41)\qquad \left|\frac{\partial}{\partial t}\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]\right| \le \frac{C_{\lambda,\Lambda,\delta}}{t[1+|e(\xi)|^2t]^\delta}.$$

To do this we integrate by parts on the RHS of (2.40) to obtain

$$(2.42)\quad \frac{\partial}{\partial t}\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] = \\ \frac{1}{t^2}\int_0^\infty \left\{\frac{2\partial h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]} +\ \mathrm{Im}(\eta)\ \frac{\partial^2 h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2}\right\}\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)].$$

We have already seen in Lemma 2.3 that

$$\frac{1}{t}\int_0^\infty \left|\frac{\partial h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\right| |\sin[\mathrm{Im}(\eta)t]|d[\mathrm{Im}(\eta)] \le \frac{C_{\lambda,\Lambda,\delta}}{[|e(\xi)|^2t]^\delta}.$$

for any δ, $0 \le \delta < 1$, where we assume $|e(\xi)|^2\ t > 1$. The inequality (2.41) will follow therefore if we can show that

(2.43)

$$\frac{1}{t}\int_0^\infty \left|\frac{\partial^2 h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2}\right| |\mathrm{Im}(\eta)|\Big|\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] \le \frac{C_{\lambda,\Lambda,\delta}}{[|e(\xi)|^2t]^\delta},\quad |e(\xi)|^2t > 1, 0 \le \delta < 1.$$

To prove this we use the fact that

$$(2.44)\qquad \left|\frac{\partial^2 h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2}\right| \le \left|\frac{\partial^2}{\partial\eta^2}\frac{1}{\eta+e(\xi)q(\xi,\eta)e(-\xi)}\right|,$$

$$\begin{aligned}\frac{\partial^2}{\partial\eta^2}\frac{1}{\eta+e(\xi)q(\xi,\eta)e(-\xi)} &= -\frac{\partial}{\partial\eta}\ \frac{1+\langle\psi(-\xi,\eta,\cdot)\psi(\xi,\eta,\cdot)\rangle}{[\eta+e(\xi)q(\xi,\eta)e(-\xi)]^2} \\ &= \frac{2\{1+\langle\psi(-\xi,\eta,\cdot)\psi(\xi,\eta,\cdot)\rangle\}^2}{[\eta+e(\xi)q(\xi,\eta)e(-\xi)]^3} \\ &\quad - \frac{[\langle[\partial\psi(-\xi,\eta,\cdot)/\partial\eta]\psi(\xi,\eta,\cdot)\rangle+\langle\psi(-\xi,\eta,\cdot)[\partial\psi(\xi,\eta,\cdot)/\partial\eta]\rangle]}{[\eta+e(\xi)q(\xi,\eta)e(-\xi)]^2}.\end{aligned}$$

Observe now that similarly to (2.27), (2.30) we have that

$$(2.45)\quad \frac{|1+\langle\psi(-\xi,\eta,\cdot)\psi(\xi,\eta,\cdot)\rangle\,|^2}{|\eta+e(\xi)q(\xi,\eta)e(-\xi)|^3} \le \min\left[\frac{1}{|\mathrm{Im}(\eta)|^3}\ ,\frac{2}{\lambda^3|e(\xi)|^6}+\frac{2\Lambda^2}{\lambda^3|e(\xi)|^2|\eta|^2}\right].$$

We can conclude from this last inequality just like we argued in Lemma 2.3 that

$$\frac{1}{t}\int_0^\infty \frac{|1+\langle\psi(-\xi,\eta,\cdot)\psi(\xi,\eta,\cdot)\rangle|^2}{|\eta+e(\xi)q(\xi,\eta)e(-\xi)|^3}|\mathrm{Im}(\eta)||\sin[\mathrm{Im}(\eta)t]|d[\mathrm{Im}(\eta)] \le \frac{C_{\lambda,\Lambda,\delta}}{[|e(\xi)|^2t]^\delta},$$

for $0\le\delta<1,\ |e(\xi)|^2t>1$.

Next from (2.9) we see that $\partial\psi(\xi,\eta,\cdot)/\partial\eta$ satisfies the equation,

$$[\mathcal{L}_\xi+\eta]\frac{\partial\psi(\xi,\eta,\cdot)}{\partial\eta}+\psi(\xi,\eta,\cdot)=0. \tag{2.46}$$

From this equation and the Schwarz inequality we easily conclude that

$$\left\langle|\partial\psi(\xi,\eta,\cdot)/\partial\eta|^2\right\rangle \le |\eta|^{-2}\left\langle|\psi(\xi,\eta,\cdot)|^2\right\rangle.$$

It follows then that

$$\frac{|\langle\psi(-\xi,\eta,\cdot)[\partial\psi(\xi,\eta,\cdot)/\partial\eta]\rangle|^2}{|\eta+e(\xi)q(\xi,\eta)e(-\xi)|^2}\le\min\left[\frac{1}{|\mathrm{Im}(\eta)|^3}\,,\frac{\Lambda}{\lambda^2|e(\xi)|^2|\eta|^2}\right]. \tag{2.47}$$

Since this inequality is similar to (2.45) we conclude that (2.43) holds.

We have proved now that (2.41) holds. To complete the proof of the lemma we need to obtain a similar estimate for the second integral on the RHS of (2.33). To see this observe that we can readily conclude that the integral is differentiable in t and

$$\begin{aligned}\frac{\partial}{\partial t}&-\frac{1}{t}\int_0^\infty\frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\{1-\cos[\mathrm{Im}(\eta)t]\}\,d[\mathrm{Im}(\eta)]\\&=\frac{2}{t^2}\int_0^\infty\frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\{1-\cos[\mathrm{Im}(\eta)t]\}\,d[\mathrm{Im}(\eta)]\\&\quad+\frac{1}{t^2}\int_0^\infty\frac{\partial^2k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2}\,\mathrm{Im}(\eta)\,\{1-\cos[\mathrm{Im}(\eta)t]\}\,d[\mathrm{Im}(\eta)].\end{aligned} \tag{2.48}$$

We have already seen in Lemma 2.3 that

$$\frac{1}{t^2}\int_0^\infty\left|\frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\right|\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)]\le\frac{C_{\lambda,\Lambda,\delta}}{t[1+|e(\xi)|^2\,t]^\delta}\quad,$$

for any $\delta,\ 0\le\delta<1$. Hence we need to concern ourselves with the second integral on the RHS of (2.48). Now it is clear that $\partial^2k(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^2$ satisfies the same estimates we have just established for $\partial^2h(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^2$. It follows in particular that

$$\begin{aligned}\frac{1}{t^2}\int_0^\infty&\left|\frac{\partial^2k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2}\right||\mathrm{Im}(\eta)|\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)]\\&\le\frac{1}{t^2}\int_0^\infty\frac{1-\cos[\mathrm{Im}(\eta)t]}{[\mathrm{Im}(\eta)]^2}d[\mathrm{Im}(\eta)]\le\frac{C}{t},\end{aligned}$$

for some universal constant C. Arguing as in Lemma 2.3 we also have that

$$\begin{aligned}\frac{1}{t^2}\int_0^\infty&\left|\frac{\partial^2k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2}\right||\mathrm{Im}(\eta)|\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)]\\&\le\int_0^{1/t}+\int_{1/t}^{|e(\xi)|^2}+\int_{|e(\xi)|^2}^\infty\le\frac{C_{\lambda,\Lambda,\delta}}{t[|e(\xi)|^2t]^\delta},\end{aligned}$$

for $0 \le \delta < 1$, $|e(\xi)|^2\ t > 1$. The last three inequalities then give us the same estimate on the derivative of the second integral on the RHS of (2.33) as we have already obtained for the first integral. $\Box$

The estimate for $\partial \hat{G}_{\mathbf{a}}(\xi,t)/\partial t$ in Lemma 2.4 diverges as $t \to 0$. We rectify this in the following:

Lemma 2.5. *There is a constant $C_{\lambda,\Lambda}$ depending only on λ, Λ such that*

$$\left| \frac{\partial \hat{G}_{\mathbf{a}}(\xi,t)}{\partial t} \right| \le C_{\lambda,\Lambda}|e(\xi)|^2, \quad \xi \in \mathbb{R}^d, t > 0.$$

Proof. To bound the derivative of the first integral on the RHS of (2.33) it is sufficient to show that

$$\begin{aligned} \frac{1}{t}\int_0^\infty |h(\xi,\eta)|\,\{1-\cos[\mathrm{Im}(\eta)t]\}\,d[\mathrm{Im}(\eta)] \quad &\le \quad C_{\lambda,\Lambda}|e(\xi)|^2, \\ \frac{1}{t}\int_0^\infty \left|\frac{\partial h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\right| |\mathrm{Im}(\eta)|\,\{1-\cos[\mathrm{Im}(\eta)t]\}\,d[\mathrm{Im}(\eta)] \quad &\le \quad C_{\lambda,\Lambda}|e(\xi)|^2. \end{aligned} \tag{2.49}$$

We have now from (2.24) that

$$|h(\xi,\eta)| \le 4\Lambda|e(\xi)|^2/|\mathrm{Im}(\eta)|^2, \quad \mathrm{Re}(\eta) = 0,$$

and from the inequalities before (2.39) that

$$|\partial h(\xi,\eta)/\partial[\mathrm{Im}(\eta)]| \le 9\Lambda|e(\xi)|^2/|\mathrm{Im}(\eta)|^3, \quad \mathrm{Re}(\eta) = 0.$$

The inequalities (2.49) follow from these last two inequalities.

The derivative of the second integral is given by the RHS of (2.48). Using the identity

$$\begin{aligned} \frac{1}{t^2}\int_0^N &\left[\frac{2\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]} + \ \mathrm{Im}(\eta)\ \frac{\partial^2 k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2}\right]\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)] \\ &= \int_0^N k(\xi,\eta)\ \mathrm{Im}(\eta)\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)] \\ &\quad + \left\{k(\xi, Re(\eta)+iN) + N\frac{\partial k(\xi, Re(\eta)+iN)}{\partial[\mathrm{Im}(\eta)]}\right\}\frac{\{1-\cos[Nt]\}}{t^2} \\ &\quad - \frac{Nk(\xi, Re(\eta)+iN)\sin Nt}{t}, \end{aligned}$$

we see that the derivative of the second integral on the RHS of (2.33) is also given by the formula

$$\lim_{m\to\infty}\int_0^{\pi m/t} k(\xi,\eta)\ \mathrm{Im}(\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)], \tag{2.50}$$

where the limit in (2.50) is taken for integer $m \to \infty$. Writing $\eta + e(\xi)q(\xi,\eta)e(-\xi) = a(\eta) + ib(\eta)$ we see from (2.18) that

$$\mathrm{Im}(\eta)k(\xi,\eta) = \frac{\mathrm{Im}(\eta)b(\eta)}{a(\eta)^2+b(\eta)^2} = \frac{\mathrm{Im}(\eta)}{b(\eta)}\left[1 - \frac{a(\eta)^2}{a(\eta)^2+b(\eta)^2}\right].$$

Observe now that just as before we have

$$\begin{aligned}\int_0^\infty \frac{|\mathrm{Im}(\eta)|}{|b(\eta)|}\frac{a(\eta)^2}{a(\eta)^2+b(\eta)^2}d[\mathrm{Im}(\eta)] &\le \int_0^\infty \frac{a(\eta)^2}{a(\eta)^2+b(\eta)^2}d[\mathrm{Im}(\eta)]\\ &\le \int_0^{|e(\xi)|^2}+\int_{|e(\xi)|^2}^\infty \le C_{\lambda,\Lambda}|e(\xi)|^2,\end{aligned}$$

for a constant $C_{\lambda,\Lambda}$ depending only on λ,Λ. We also have from (2.15) that

$$\begin{aligned}\frac{\mathrm{Im}(\eta)}{b(\eta)} &= 1/\left[1+\frac{1}{2}\left\langle|\psi(\xi,\eta,\cdot)|^2\right\rangle+\frac{1}{2}\left\langle|\psi(-\xi,\eta,\cdot)|^2\right\rangle\right]\\ &= 1-\frac{\frac{1}{2}\left\langle|\psi(\xi,\eta,\cdot)|^2\right\rangle+\frac{1}{2}\left\langle|\psi(-\xi,\eta,\cdot)|^2\right\rangle}{1+\frac{1}{2}\left\langle|\psi(\xi,\eta,\cdot)|^2\right\rangle+\frac{1}{2}\left\langle|\psi(-\xi,\eta,\cdot)|^2\right\rangle}.\end{aligned}$$

It follows therefore from (2.50) that the result will be complete if we can show that

$$\int_0^\infty \left\langle|\psi(\xi,\eta,\cdot)|^2\right\rangle d[\mathrm{Im}(\eta)] \le C_{\lambda,\Lambda}|e(\xi)|^2, \tag{2.51}$$

with a similar inequality for $\psi(-\xi,\eta,\cdot)$. Note that (2.51) does not follow from the bound $\left\langle|\psi(\xi,\eta,\cdot)|^2\right\rangle \le \Lambda|e(\xi)|^2/|\eta|$ which we have already established. In view of (2.8) the inequality (2.51) is a consequence of the following lemma. ☐

Lemma 2.6. *Let $\varphi(\xi,\eta,\cdot)$ be the function defined by* (2.8). *Then there is the inequality,*

$$\int_0^\infty \left\langle|\varphi(\xi,\eta,\cdot)|^2\right\rangle d[\mathrm{Im}(\eta)] \le 2\pi\Lambda|\rho|^2.$$

Proof. Let $\varphi(t,\xi,\cdot),\ \ t>0$, be the solution to the initial value problem,

$$\begin{aligned}&\frac{\partial\varphi(t,\xi,\cdot)}{\partial t}+\mathcal{L}_\xi\varphi(t,\xi,\cdot)=0,\quad t>0,\\ &\varphi(0,\xi,\cdot)+\sum_{k=1}^d\rho_k\sum_{j=1}^d\left[\partial_j^*+e_j(-\xi)\right]e^{i\mathbf{e}_j\cdot\xi}\left[a_{k,j}(\cdot)-\langle a_{k,j}(\cdot)\rangle\right]=0.\end{aligned} \tag{2.52}$$

It is clear that

$$\varphi(\xi,\eta,\cdot)=\int_0^\infty e^{-\eta t}\varphi(t,\xi,\cdot)dt,\quad \mathrm{Re}(\eta)>0.$$

Hence the Plancherel Theorem yields

$$\int_0^\infty \left\langle|\varphi(\xi,\eta,\cdot)|^2\right\rangle d[\mathrm{Im}(\eta)] \le 2\pi\int_0^\infty \left\langle\varphi(t,\xi,\cdot)|^2\right\rangle dt.$$

We can estimate the RHS of this last equation by defining a function $\Phi(t,\xi,\cdot),\ t>0$, which satisfies the equation,

$$\mathcal{L}_\xi\Phi(t,\xi,\cdot)=\varphi(t,\xi,\cdot),\quad t>0.$$

Hence (2.52) yields

$$\left\langle\overline{\Phi(t,\xi,\cdot)}\frac{\partial\varphi(t,\xi,\cdot)}{\partial t}\right\rangle+\left\langle|\varphi(t,\xi,\cdot)|^2\right\rangle=0,\quad t>0. \tag{2.53}$$

Observe next that

$$\left\langle \overline{\Phi(t,\xi,\cdot)}\, \frac{\partial \varphi(t,\xi,\cdot)}{\partial t} \right\rangle = \frac{1}{2}\frac{\partial}{\partial t}\left\langle \overline{\Phi(t,\xi,\cdot)}\mathcal{L}_\xi \Phi(t,\xi,\cdot)\right\rangle .$$

Integrating (2.53) w.r. to t and using the positivity of $\mathcal{L}_\xi$ we conclude that for any $\delta > 0$, there is the inequality,

$$\int_\delta^\infty \langle \varphi(t,\xi,\cdot)|^2\rangle\, dt \le \frac{1}{2}\left\langle \overline{\Phi(\delta,\xi,\cdot)}\mathcal{L}_\xi \Phi(\delta,\xi,\cdot)\right\rangle .$$

We have now from (2.52) that

$$\left\langle \overline{\Phi(0,\xi,\cdot)}\mathcal{L}_\xi \Phi(0,\xi,\cdot)\right\rangle \le \Lambda|\rho|^2.$$

The result follows now on letting $\delta \to 0$. □

3. Proof of Theorem 1.5

For η real and positive, let $q(\xi,\eta)$ be the $d\times d$ matrix defined in (2.3). We shall show that the function $G_{\mathbf{a}}(x)$ of Theorem 1.5 is given by,

$$G_{\mathbf{a}}(x) = \lim_{\eta\to 0}\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} d\xi e^{-ix\cdot\xi}/e(\xi)q(\xi,\eta)e(-\xi), \quad x \in \mathbb{Z}^d\ . \tag{3.1}$$

In view of the fact that $q(\xi,\eta) \ge \lambda I_d$, we see from the following lemma that the limit (3.1) exists if $d \ge 3$.

Lemma 3.1. *The limit* $\lim_{\eta\to 0} q(\xi,\eta)$ *exists for all* $\xi \in \mathbb{R}^d$.

Proof. For $\eta > 0, x \in \mathbb{Z}^d$, let $G_\eta(x)$ be the Green's function satisfying the equation

$$\sum_{i=1}^d \nabla_i^*\nabla_i G_\eta(x) + \eta G_\eta(x) = \delta(x), \quad x \in \mathbb{Z}^d,$$

where $\delta(x)$ is the Kronecker δ function. Now for $\varphi \in L^2(\Omega)$ there is a unique solution $\psi \in L^2(\Omega)$ to the equation,

$$\sum_{i=1}^d [\partial_i^* + e_i(-\xi)]\,[\partial_i + e_i(\xi)]\,\psi(\omega) + \eta\psi(\omega) = \varphi(\omega), \quad \omega\in\Omega\ ,$$

which can be written as

$$\psi(\omega) = \sum_{x\in\mathbb{Z}^d} G_\eta(x)e^{-ix\cdot\xi}\varphi(\tau_x\omega), \quad \omega\in\Omega. \tag{3.2}$$

Observe the RHS of (3.2) is square integrable since $G_\eta(x)$ decreases exponentially as $|x| \to \infty$. Now for $1 \le k, k' \le d$ we define operators $T_{k,k',\eta,\xi}$ by $T_{k,k',\eta,\xi}(\varphi) = e^{-i\mathbf{e}_k\cdot\xi}[\partial_k + e_k(\xi)]\psi$, where ψ is the solution to the equation,

$$\begin{aligned}\sum_{i=1}^d [\partial_i^* + e_i(-\xi)]\,[\partial_i + e_i(\xi)]\,\psi(\omega) + \eta\psi(\omega) \\ = e^{i\mathbf{e}_{k'}\cdot\xi}\,[\partial_{k'}^* + e_{k'}(-\xi)]\,\varphi(\omega), \quad \omega\in\Omega.\end{aligned} \tag{3.3}$$

From (3.2) we see that

$$T_{k,k',\eta,\xi}\ \varphi(\omega) = \sum_{x\in\mathbb{Z}^d} \nabla_k^*\nabla_{k'}G_\eta(x)e^{-ix\cdot\xi}\varphi(\tau_x\omega), \quad \omega\in\Omega. \tag{3.4}$$

Since $\nabla_k^*\nabla_{k'}G_\eta(x)$ is exponentially decreasing as $|x|\to\infty$ it follows that $T_{k,k',\eta,\xi}$ is a bounded operator on $L^2(\Omega)$. Observe that the projection operator P on $L^2(\Omega)$ orthogonal to the constant function commutes with $T_{k,k',\eta,\xi}$. It follows from (3.3) that $\|T_{k,k',\eta,\xi}\|\le 1$, independent of η as $\eta\to 0$. We wish to show that there is an operator $T_{k,k',0,\xi}$ on $L^2(\Omega)$ with $\|T_{k,k',0,\xi}\|\le 1$ such that

$$\lim_{\eta\to 0}\|T_{k,k',\eta,\xi}\varphi - T_{k,k',0,\xi}\varphi\| = 0, \quad \varphi\in L^2(\Omega). \tag{3.5}$$

We follow the argument used to prove the von Neumann Ergodic Theorem [9]. Thus if $\varphi\in L^2(\Omega)$ satisfies $[\partial_{k'}^* + e_{k'}(-\xi)]\varphi = 0$, then $T_{k,k',\eta,\xi}\varphi = 0$. Thus we set $T_{k,k',0,\xi}\varphi = 0$ for φ in the null space of $[\partial_{k'}^* + e_{k'}(-\xi)]$. Now the range of $[\partial_{k'} + e_{k'}(\xi)]$ is dense in the subspace of $L^2(\Omega)$ orthogonal to the null space of $[\partial_{k'}^* + e_{k'}(-\xi)]$. If $\varphi = e^{-i\mathbf{e}_{k'}\cdot\xi}[\partial_{k'} + e_{k'}(\xi)]\psi$ with $\psi\in L^2(\Omega)$ then

$$T_{k,k',\eta,\xi}\varphi(\omega) = \sum_{x\in\mathbb{Z}^d}\nabla_{k'}^*\nabla_k^*\nabla_{k'}G_\eta(x)e^{-ix\cdot\xi}\psi(\tau_x\omega), \quad \omega\in\Omega.$$

It is clear from this representation that if we take

$$T_{k,k',0,\xi}\varphi(\omega) = \sum_{x\in\mathbb{Z}^d}\nabla_{k'}^*\nabla_k^*\nabla_{k'}G_0(x)e^{-ix\cdot\xi}\psi(\tau_x\omega), \quad \omega\in\Omega\ ,$$

then $T_{k,k',0,\xi}(\varphi)\in L^2(\Omega)$ and (3.5) holds. Thus $T_{k,k',0,\xi}$ is defined on a dense subspace of $L^2(\Omega)$ and $\|T_{k,k',0,\xi}\|\le 1$. If follows easily that one can extend the definition of $T_{k,k',0,\xi}$ to all of $L^2(\Omega)$ and (3.5) holds.

Suppose now $\mathbf{b}:\Omega\to\mathbb{R}^{d(d+1)/2}$ is a bounded measurable function from Ω to the space of symmetric $d\times d$ matrices. We define $\|\mathbf{b}\|$ to be

$$\|\mathbf{b}\| = \sup\left\{ \Big|\sum_{i,j=1}^d b_{i,j}(\omega)\lambda_i\lambda_j\Big| : \sum_{i=1}^d \lambda_i^2 = 1, \quad \omega\in\Omega\right\}.$$

Next let $\mathcal{H}(\Omega) = \{\varphi = (\varphi_1,\dots,\varphi_d) : \varphi_i\in L^2(\Omega), 1\le i\le d\}$ be the Hilbert space with norm $\|\varphi\|^2 = \|\varphi_1\|^2 + \cdots + \|\varphi_d\|^2, \varphi = (\varphi_1,\dots,\varphi_d)$. We define an operator $T_{\mathbf{b},\eta,\xi}$ on $\mathcal{H}(\Omega)$ by

$$\left[T_{\mathbf{b},\eta,\xi}\varphi(\cdot)\right]_k = \sum_{i,j=1}^d T_{k,i,\eta,\xi}\left[b_{i,j}(\cdot)\varphi_j(\cdot)\right], \quad 1\le k\le d.$$

Evidently,

$$\left[T_{\mathbf{b},\eta,\xi}\varphi(\cdot)\right]_k = e^{-i\mathbf{e}_k\cdot\xi}[\partial_k + e_k(\xi)]\psi(\cdot), \quad 1\le k\le d,$$

where $\psi(\cdot)$ satisfies the equation,

$$\sum_{r=1}^d\left[\partial_r^* + e_r(-\xi)\right]\left[\partial_r + e_r(\xi)\right]\psi(\omega) + \eta\psi(\omega) = \sum_{i,j=1}^d e^{i\mathbf{e}_i\cdot\xi}[\partial_i^* + e_i(-\xi)]\left[b_{i,j}(\cdot)\varphi_j(\cdot)\right].$$

If follows that $T_{\mathbf{b},\eta,\xi}$ is a bounded operator on $\mathcal{H}(\Omega)$ with norm $\|T_{\mathbf{b},\eta,\xi}\| \le \|\mathbf{b}\|$. In view of (3.5) there exists a bounded operator $T_{\mathbf{b},0,\xi}$ on $\mathcal{H}(\Omega)$ such that $\|T_{\mathbf{b},0,\xi}\| \le \|\mathbf{b}\|$ and

$$\lim_{\eta\to 0} \|T_{\mathbf{b},\eta,\xi}\varphi - T_{\mathbf{b},0,\xi}\varphi\| = 0, \quad \varphi \in \mathcal{H}(\Omega) \quad .$$

Let us take now $\mathbf{b}(\cdot) = \big[\Lambda I_d - \mathbf{a}(\cdot)\big]/\Lambda$, whence $\|\mathbf{b}\| < 1$. Let $\psi_k(\xi,\eta,\cdot)$ be the function satisfying (2.2). Define $\Psi_k(\xi,\eta,\cdot)$ to be

$$\Psi_k(\xi,\eta,\cdot) = \big(e^{-i\mathbf{e}_1\cdot\xi}[\partial_1 + e_1(\xi)]\psi_k(\xi,\eta,\cdot), \dots, e^{-i\mathbf{e}_d\cdot\xi}[\partial_d + e_d(\xi)]\psi_k(\xi,\eta,\cdot)\big).$$

Then $\Psi_k(\xi,\eta,\cdot) \in \mathcal{H}(\Omega)$ and satisfies the equation,

$$\Psi_k(\xi,\eta,\omega) - PT_{\mathbf{b},\eta/\Lambda,\xi}\Psi_k(\xi,\eta,\omega) + \frac{1}{\Lambda}\sum_{j=1}^d T_{j,\eta/\Lambda,\xi}\big[a_{k,j}(\omega) - \langle a_{k,j}(\cdot)\rangle\big] = 0, \tag{3.6}$$

where for $1 \le j \le d$, $T_{j,\eta,\xi}$ is a bounded operator from $L^2(\Omega)$ to $\mathcal{H}(\Omega)$ defined by $T_{j,\eta,\xi}(\varphi) = (T_{1,j,\eta,\xi}\varphi, \dots, T_{d,j,\eta,\xi}\varphi)$. Writing

$$\Psi_k(\xi,\eta,\cdot) = (\Psi_{k,1}(\xi,\eta,\cdot), \dots, \Psi_{k,d}(\xi,\eta,\cdot)),$$

we have from (2.3) that

$$q_{k,k'}(\xi,\eta) = \Big\langle a_{k,k'}(\cdot) + \sum_{j=1}^d a_{k,j}(\cdot)\Psi_{k',j}(\xi,\eta,\cdot)\Big\rangle \ . \tag{3.7}$$

It is clear now that $\lim_{\eta\to 0} q(\xi,\eta)$ exists. □

Our next goal is to assert some differentiability properties of $q(\xi,\eta)$ in ξ which are uniform as $\eta \to 0$.

Lemma 3.2. *Suppose $\eta > 0$, $1 \le k \le d$, $\varphi \in L^2(\Omega)$ and $\mathbf{b}(\cdot)$ a random symmetric matrix satisfying $\|\mathbf{b}\| < 1$. For $\xi \in \mathbb{R}^d$ let $\Psi(\xi,\eta,\cdot)$ be the solution to the equation,*

$$(I - PT_{\mathbf{b},\eta,\xi})\Psi(\xi,\eta,\cdot) = T_{k,\eta,\xi}\varphi(\cdot). \tag{3.8}$$

Then $\Psi(\xi,\eta,\cdot)$, regarded as a function of $\xi \in \mathbb{R}^d$ to $\mathcal{H}(\Omega)$ is differentiable. The derivative of $\Psi(\xi,\eta,\cdot)$ is given by

(3.9)
$$\frac{\partial\Psi}{\partial\xi_j}(\xi,\cdot) = (I - PT_{\mathbf{b},\eta,\xi})^{-1}\left(\frac{\partial}{\partial\xi_j}T_{k,\eta,\xi}\right)\varphi(\cdot)$$
$$+ (I - PT_{\mathbf{b},\eta,\xi})^{-1}\left(P\frac{\partial}{\partial\xi_j}T_{\mathbf{b},\eta,\xi}\right)(I - PT_{\mathbf{b},\eta,\xi})^{-1}T_{k,\eta,\xi}\varphi(\cdot), \ 1 \le j \le d.$$

Proof. Consider first the k' th component of $T_{k,\eta,\xi}\varphi(\cdot)$, which is

$$T_{k',k,\eta,\xi}\varphi(\cdot) = \sum_{x\in\mathbb{Z}^d} \nabla^*_{k'}\nabla_k G_\eta(x)e^{-ix\cdot\xi}\varphi(\tau_x\cdot).$$

Since $G_\eta(x)$ decreases exponentially as $|x| \to \infty$ it is clear that $T_{k',k,\eta,\xi}\varphi(\cdot)$, regarded as a mapping from $\mathbb{R}^d$ to $L^2(\Omega)$ is differentiable and

$$\frac{\partial}{\partial\xi_j}T_{k',k,\eta,\xi}\varphi(\cdot) = -\sum_{x\in\mathbb{Z}^d} ix_j\nabla^*_{k'}\nabla_k G_\eta(x)e^{ix\cdot\xi}\varphi(\tau_x\cdot). \tag{3.10}$$

We regard (3.10) as the definition of the operator $\partial/\partial\xi_j T_{k',k,\eta,\xi}$ which is clearly a bounded operator on $L^2(\Omega)$. Similarly we can define $\partial/\partial\xi_j T_{\mathbf{b},\eta,\xi}$ by

$$\left[\frac{\partial}{\partial\xi_j}T_{\mathbf{b},\eta,\xi}\psi(\cdot)\right]_r = \sum_{i,j'=1}^{d}\frac{\partial}{\partial\xi_j}T_{r,i,\eta,\xi}\left[b_{i,j'}(\cdot)\psi_{j'}(\cdot)\right], \tag{3.11}$$

$\psi = (\psi_1,\dots,\psi_d) \in \mathcal{H}(\Omega)$, $1 \le r \le d$. Again it is clear that $\partial/\partial\xi_j T_{\mathbf{b},\eta,\xi}$ is a bounded operator on $\mathcal{H}(\Omega)$. From (3.8) $\Psi(\xi,\eta,\cdot)$ is given by the Neumann series,

$$\Psi(\xi,\eta,\cdot) = \sum_{n=0}^{\infty}(PT_{\mathbf{b},\eta,\xi})^n T_{k,\eta,\xi}\varphi(\cdot),$$

which converges since $\|\mathbf{b}\| < 1$. Formally the derivative of $\Psi(\xi,\eta,\cdot)$ is given by

$$\begin{aligned}\frac{\partial}{\partial\xi_j}\Psi(\xi,\eta,\cdot) &= \sum_{n=0}^{\infty}(PT_{\mathbf{b},\eta,\xi})^n\frac{\partial}{\partial\xi_j}T_{k,\eta,\xi}\varphi(\cdot)\\ &\quad+\sum_{n,m=0}^{\infty}(PT_{\mathbf{b},\eta,\xi})^n(P\frac{\partial}{\partial\xi_j}T_{\mathbf{b},\eta,\xi})(PT_{\mathbf{b},\eta,\xi})^m T_{k,\eta,\xi}\varphi(\cdot).\end{aligned} \tag{3.12}$$

Since the RHS of (3.12) converges it is easy to see that $\Psi(\xi,\eta,\cdot)$, regarded as a mapping from $\mathbb{R}^d$ to $\mathcal{H}(\Omega)$ is differentiable and the derivative is given by (3.12). Finally observe that the RHS of (3.12) is the same as the RHS of (3.9). $\square$

For $2 \le p < \infty$ let $L^p(\Omega,\big[-\pi,\pi\big]^d)$ be the space of functions $\psi(\xi,\omega)$, $\xi \in \big[-\pi,\pi\big]^d$, $\omega \in \Omega$ such that $\|\psi\|_p < \infty$, where

$$\|\psi\|_p^p = \int_{[-\pi,\pi]^d} d\xi\,\left\langle|\psi(\xi,\cdot)|^2\right\rangle^{p/2}\ .$$

Suppose now $f : \big[-\pi,\pi\big]^d \to \mathbb{C}$ is a smooth periodic function. The Fourier transform $\hat{f}$ of f is given by

$$\hat{f}(x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} f(\xi)e^{-ix\cdot\xi}d\xi,\ x \in \mathbb{Z}^d.$$

Since $\hat{f}$ is rapidly decreasing we can define for $\varphi \in L^2(\Omega)$ an operator T_φ by

$$T_\varphi(f)(\xi,\omega) = \sum_{x\in\mathbb{Z}^d}\hat{f}(x)e^{-ix\cdot\xi}\varphi(\tau_x\omega),\quad \xi \in \big[-\pi,\pi\big]^d,\quad \omega\in\Omega\ . \tag{3.13}$$

Evidently $T_\varphi(f) \in L^\infty(\Omega,\big[-\pi,\pi\big]^d)$.

Lemma 3.3. *Suppose $2 \le p \le \infty$ and $\varphi \in L^2(\Omega)$. Then the operator T_φ extends to a bounded operator from $L^p(\big[-\pi,\pi\big]^d)$ to $L^p(\Omega,[-\pi,\pi]^d)$and the norm of T_φ satisfies $\|T_\varphi\| \le \|\varphi\|$.*

Proof. Now by Bochner's Theorem [9] there is a positive measure $d\mu_\varphi$ on $[-\pi,\pi]^d$ such that

$$\left\langle\varphi(\tau_x\cdot)\overline{\varphi(\tau_y\cdot)}\right\rangle = \int_{[-\pi,\pi]^d} e^{i(x-y)\cdot\zeta}d\mu_\varphi(\zeta),$$

whence

$$\left\langle |T_\varphi(f)(\xi,\cdot)|^2 \right\rangle = \int_{[-\pi,\pi]^d} |f(\zeta-\xi)|^2 d\mu_\varphi(\zeta). \tag{3.14}$$

Since $\int_{[-\pi,\pi]^d} d\mu_\varphi(\zeta) = \|\varphi\|^2$ the fact that $\|T_\varphi\| \le \|\varphi\|$ follows immediately from (3.14) for $p=2$, $p=\infty$. The fact that $\|T_\varphi\| \le \|\varphi\|$ when $2<p<\infty$ can be obtained by an application of Holder's inequality. $\square$

Next for $2 \le p < \infty$ let $\mathcal{H}^p(\Omega,[-\pi,\pi]^d)$ be the space of functions $\psi(\xi,\omega)$, $\xi \in [-\pi,\pi]^d$ with $\psi(\xi,\cdot) \in \mathcal{H}(\Omega)$ such that $\|\psi\|_p < \infty$, where

$$\|\psi\|_p^p = \int_{[-\pi,\pi]^d} d\xi \|\psi(\xi,\cdot)\|^p.$$

We define an operator for $\varphi \in L^2(\Omega)$, $T_{\varphi,\mathbf{b},\eta}$ by

$$T_{\varphi,\mathbf{b},\eta}(f)(\xi,\cdot) = \sum_{x\in\mathbb{Z}^d} \widehat{f}(x) e^{-ix\cdot\xi} \tau_x \Big[\mathbf{b}(\cdot)(I - PT_{\mathbf{b},\eta,\xi})^{-1} T_{k,\eta,\xi}\varphi(\cdot) \Big]. \tag{3.15}$$

It is clear that if f is smooth then $T_{\varphi,\mathbf{b},\eta}(f)$ is in $\mathcal{H}^\infty(\Omega,[-\pi,\pi]^d)$ provided $\|\mathbf{b}\| < 1$.

Lemma 3.4. *Suppose* $\|\mathbf{b}\| < 1$. *Then,*

(a) *If* $\varphi \in L^2(\Omega)$, $T_{\varphi,\mathbf{b},\eta}$ *extends to a bounded operator from* $L^\infty([-\pi,\pi]^d)$ *to* $\mathcal{H}^\infty(\Omega,[-\pi,\pi]^d)$. *The norm of* $T_{\varphi,\mathbf{b},\eta}$ *satisfies*

$$\|T_{\varphi,\mathbf{b},\eta}\| \le \frac{\|\mathbf{b}\|\|\varphi\|}{(1-\|\mathbf{b}\|)}.$$

(b) *If* $\varphi \in L^\infty(\Omega)$, $T_{\varphi,\mathbf{b},\eta}$ *extends to a bounded operator from* $L^2([-\pi,\pi]^d)$ *to* $\mathcal{H}^2(\Omega,[-\pi,\pi]^d)$. *The norm of* $T_{\varphi,\mathbf{b},\eta}$ *satisfies*

$$\|T_{\varphi,\mathbf{b},\eta}\| \le \frac{\sqrt{d}\|\mathbf{b}\|\|\varphi\|_\infty}{(1-\|\mathbf{b}\|)^2}.$$

Proof. To prove (a) observe that (3.14) implies

$$\begin{aligned} \|T_{\varphi,\mathbf{b},\eta}(f)(\xi)\|^2 &\le \|f\|_\infty^2 \|\mathbf{b}(\cdot)(I-PT_{\mathbf{b},\eta,\xi})^{-1}T_{k,\eta,\xi}\varphi(\cdot))\|^2 \\ &\le \frac{\|f\|_\infty^2\|\mathbf{b}\|^2\|\varphi\|^2}{(1-\|\mathbf{b}\|)^2}. \end{aligned}$$

To prove (b) we consider the integral

(3.16)

$$\begin{aligned} \int_{[-\pi,\pi]^d} d\xi \| \sum_{x\in\mathbb{Z}^d} \widehat{f}(x)e^{-ix\cdot\xi}\tau_x \left[\mathbf{b}(\cdot)(PT_{\mathbf{b},\eta,\xi})^m T_{k,\eta,\xi}\varphi\right] \|^2 &= (2\pi)^d \sum_{r\in\mathbb{Z}^d} \|\Theta_1\|^2 \\ &= (2\pi)^d \sum_{r\in\mathbb{Z}^d} \|\Theta_2\|^2, \end{aligned}$$

where Θ_1 is given by

$$\sum_{x_1+\cdots+x_{m+2}=r} \widehat{f}(x_1)\mathbf{b}(\tau_{x_1}\cdot)\left[\prod_{j=2}^{m+1} \nabla^*\nabla G_\eta(x_j) P\mathbf{b}(\tau_{x_1+\cdots+x_j}\cdot)\right] \nabla^*\nabla_k G_\eta(x_{m+2})\varphi(\tau_r\cdot)$$

and Θ_2 is given by

$$\sum_{y_1,\dots,y_{m+1}} \widehat{f}(y_1)\mathbf{b}(\tau_{y_1}\cdot)\Big[\prod_{j=2}^{m+1}\nabla^*\nabla G_\eta(y_j-y_{j-1})P\mathbf{b}(\tau_{y_j}\cdot)\Big]\nabla^*\nabla_k G_\eta(r-y_{m+1})\varphi(\tau_r\cdot).$$

Observe now that

$$\begin{aligned}
(3.17)\quad & \widehat{f}(y_1)\mathbf{b}(\tau_{y_1}\cdot)\Big[\prod_{j=2}^{m+1}\nabla^*\nabla G_\eta(y_j-y_{j-1})P\mathbf{b}(\tau_{y_j}\cdot)\Big]\nabla^*\nabla_k G_\eta(r-y_{m+1})\varphi(\tau_r\cdot)\\
&= \widehat{f}(y_1)\mathbf{b}(\tau_{y_1}\cdot)\Big[\prod_{j=2}^{m+1}\nabla^*\nabla G_\eta(y_j-y_{j-1})\mathbf{b}(\tau_{y_j}\cdot)\Big]\nabla^*\nabla_k G_\eta(r-y_{m+1})\varphi(\tau_r\cdot)\\
&\quad -\sum_{n=1}^{m}\widehat{f}(y_1)\mathbf{b}(\tau_{y_1}\cdot)\Big[\prod_{j=2}^{n}\nabla^*\nabla G_\eta(y_j-y_{j-1})\mathbf{b}(\tau_{y_j}\cdot)\Big]\\
&\quad \Big\langle \nabla^*\nabla G_\eta(y_{n+1}-y_n)\mathbf{b}(\tau_{y_n}\cdot)\Big[\prod_{j=n+2}^{m+1}\nabla^*\nabla G_\eta(y_j-y_{j-1})P\mathbf{b}(\tau_{y_j}\cdot)\Big]\\
&\qquad \nabla^*\nabla_k G_\eta(r-y_{m+1})\varphi(\tau_r\cdot)\Big\rangle.
\end{aligned}$$

Next let $\mathcal{M}$ be the space of complex $d\times d$ matrices and $L^2(\mathbb{Z}^d,\mathcal{M})$ the set of functions $A:\mathbb{Z}^d\to\mathcal{M}$. We can make $L^2(\mathbb{Z}^d,\mathcal{M})$ into a Hilbert space by defining the norm of A to be

$$\|A\|^2_{\mathcal{M}} = (2\pi)^d\sum_{x\in\mathbb{Z}^d} Tr(A^*(x)A(x)).$$

We can also define an operator T_η on $L^2(\mathbb{Z}^d,\mathcal{M})$by

$$T_\eta A(x) = \sum_y A(y)\nabla^*\nabla G_\eta(x-y),\quad x\in\mathbb{Z}^d.$$

It is easy to see that T_η is bounded on $L^2(\mathbb{Z}^d,\mathcal{M})$ and $\|T_\eta\|\le 1$. For $n=1,\dots,m+1$, $\omega\in\Omega$, $y_n\in\mathbb{Z}^d$, let us define $A_n(y_n,\omega)\in\mathcal{M}$ by

$$A_n(y_n,\omega) = \sum_{y_1,\cdots y_{n-1}}\widehat{f}(y_1)\mathbf{b}(\tau_{y_1}\omega)\Big[\prod_{j=2}^{n}\nabla^*\nabla G_\eta(y_j-y_{j-1})\mathbf{b}(\tau_{y_j}\omega)\Big].$$

It follows from the fact that $\|T_\eta\|\le 1$ that for any fixed $\omega\in\Omega$ the function $A_n(\cdot,\omega)\in L^2(\mathbb{Z}^d,\mathcal{M})$ and

$$(3.18)\qquad \|A_n(\cdot,\omega)\|_{\mathcal{M}} \le \|\mathbf{b}\|^n\|\widehat{f}I_d\|_{\mathcal{M}} = \sqrt{d}\|\mathbf{b}\|^n\|f\|_2\ .$$

Recall now that P is the projection operator, $P\psi(\cdot)=\psi(\cdot)-\langle\psi\rangle\,,\ \psi\in L^2(\Omega)$. If we introduce the notation P^* as $\psi(\cdot)P^*=P\psi(\cdot)$ then

$$\Big\langle \nabla^*\nabla G_\eta(y_{n+1}-y_n)\mathbf{b}(\tau_{y_n}\cdot)\Big[\prod_{j=n+2}^{m+1}\nabla^*\nabla G_\eta(y_j-y_{j-1})P\mathbf{b}(\tau_{y_j}\cdot)\Big]\nabla^*\nabla_k G_\eta(r-y_{m+1})\varphi(\tau_r\cdot)\Big\rangle$$

is equal to

$$\Big\langle \nabla^*\nabla G_\eta(y_{n+1}-y_n)\mathbf{b}(\tau_{y_n}\cdot)\Big[\prod_{j=n+2}^{m+1}\nabla^*\nabla G_\eta(y_j-y_{j-1})P^*\mathbf{b}(\tau_{y_j}\cdot)\Big]$$
$$\nabla^*\nabla_k G_\eta(r-y_{m+1})\varphi(\tau_r\cdot)\Big\rangle.$$

Let us denote now by $L^2(\mathbb{Z}^d\times\Omega,\mathcal{M})$ the set of functions $A:\mathbb{Z}^d\times\Omega\to\mathcal{M}$ with norm,

$$\|A\|^2_{\mathcal{M},ran}=(2\pi)^d\sum_{x\in\mathbb{Z}^d}\langle Tr(A^*(x,\cdot)A(x,\cdot))\rangle\,.$$

For $n=1,\dots,m$ define an operator T_n on this space by

$$T_nA(y_{m+1},\cdot)=\sum_{y_n,\dots,y_m}A(y_n,\cdot)\nabla^*\nabla G_\eta(y_{n+1}-y_n)$$
$$\mathbf{b}(\tau_{y_n}\cdot)\Big[\prod_{j=n+2}^{m+1}\nabla^*\nabla G_\eta(y_j-y_{j-1})P^*\mathbf{b}(\tau_{y_j}\cdot)\Big].$$

We can see as before that T_n is a bounded operator on $L^2(\mathbb{Z}^d\times\Omega,\mathcal{M})$ and

$$\|T_n\|\le\|\mathbf{b}\|^{m+1-n}. \tag{3.19}$$

Observe now from (3.17) that Θ_2 (the expression inside the norm in the last line of (3.16)) is given by

$$\sum_{y_1,\dots,y_{m+1}}\hat f(y_1)\mathbf{b}(\tau_{y_1}\cdot)\Big[\prod_{j=2}^{m+1}\nabla^*\nabla G_\eta(y_j-y_{j-1})P\mathbf{b}(\tau_{y_j}\cdot)\Big]\nabla^*\nabla_k G_\eta(r-y_{m+1})\varphi(\tau_r\cdot)$$
$$=[T_\eta A_{m+1}(r,\cdot)\mathbf{e}_k]\,\varphi(\tau_r\cdot)-\sum_{n=1}^m\big\langle\big[T_\eta T_nA_n^{ran}(r,\cdot)\mathbf{e}_k\big]\varphi(\tau_r\cdot)\big\rangle\,,$$

where A_n^{ran} denotes that $A_n(x,\omega),\ x\in\mathbb{Z}^d,\ \omega\in\Omega$ is to be regarded as a function of x only, with parameter ω, on which T_n acts. We have now

$$(2\pi)^d\sum_{r\in\mathbb{Z}^d}\|\,[T_\eta A_{m+1}(r,\cdot)\mathbf{e}_k]\,\varphi(\tau_r\cdot)\|^2\le\|\varphi\|^2_\infty\left\langle\|A_{m+1}(\cdot,\omega)\|^2_{\mathcal{M}}\right\rangle$$
$$\le\|\varphi\|^2_\infty d\|\mathbf{b}\|^{2(m+1)}\|f\|^2_2,$$

where we have used (3.18). Similarly we have

$$(2\pi)^d\sum_{r\in\mathbb{Z}^d}\|\big\langle\big[T_\eta T_nA_n^{ran}(r,\cdot)\mathbf{e}_k\big]\quad\varphi(\tau_r\cdot)\big\rangle\|^2\le\|\varphi\|^2_2\left\langle\|T_\eta T_nA_n^{ran}(\omega)\|^2_{\mathcal{M},ran}\right\rangle,$$

where ω denotes the random parameter for A_n^{ran}. The expectation is then to be taken with respect to this parameter. If we use now (3.18), (3.19) we have

$$(2\pi)^d\sum_{r\in\mathbb{Z}^d}\|\big\langle\big[T_\eta T_nA_n^{ran}(r,\cdot)\mathbf{e}_k\big]\varphi(\tau_r\cdot)\big\rangle\|^2\le\|\varphi\|^2_2 d\|\mathbf{b}\|^{2(m+1)}\|f\|^2_2\,.$$

We conclude therefore from (3.16) that

$$\int_{[-\pi,\pi]^d} d\xi \| \sum_{x\in\mathbb{Z}^d} \hat{f}(x) e^{ix\cdot\xi} \tau_x \left[\mathbf{b}(\cdot)(PT_{\mathbf{b},\eta,\xi})^m \; T_{k,\eta,\xi}\varphi \; \right\|^2$$
$$\le d(m+1) \; \|\varphi\|_\infty^2 \; \|\mathbf{b}\|^{2(m+1)} \|f\|_2^2, \; m \ge 0.$$

It follows that

$$\left[\int_{[-\pi,\pi]^d} d\xi \|T_{\varphi,\mathbf{b},\eta}(f)(\xi)\|^2\right]^{1/2} \le \sqrt{d}\|\varphi\|_\infty \|f\|_2 \sum_{m=0}^\infty \sqrt{m+1} \;\; \|\mathbf{b}\|^{m+1}$$
$$\le \sqrt{d}\|\varphi\|_\infty \|f\|_2 \|\mathbf{b}\|/(1-\|\mathbf{b}\|)^2 \; .$$

□

It follows from Lemma 3.3 and the Riesz-Thorin Interpolation Theorem [10] that if $\varphi \in L^\infty(\Omega)$ and $\|\mathbf{b}\| < 1$ then $T_{\varphi,\mathbf{b},\eta}$ is a bounded operator from $L^p([-\pi,\pi]^d)$ to $\mathcal{H}^p \; (\Omega, [-\pi,\pi]^d), \; 2 \le p \le \infty$ and the norm of $T_{\varphi,\mathbf{b},\eta}$ satisfies the inequality,

$$\|T_{\varphi,\mathbf{b},\eta}\| \le \frac{C_d \|\mathbf{b}\| \|\varphi\|_\infty}{(1-\|\mathbf{b}\|)^2},$$

where C_d is a constant depending only on d. Consider next the weak spaces $L_w^p([-\pi,\pi]^d)$ and $\mathcal{H}_w^p(\Omega,[-\pi,\pi]^d), \; 2 < p < \infty$. Thus $f \in L_w^p([-\pi,\pi]^d)$ if for all $\alpha > 0$ there is the inequality,

$$\text{meas}\{\xi \in [-\pi,\pi]^d : |f(\xi)| > \alpha\} \le C^p/\alpha^p. \tag{3.20}$$

The weak L^p norm of f, $\|f\|_{p,w}$ is then the minimum constant C such that (3.20) holds for all $\alpha > 0$. Similarly $\psi(\xi,\omega) \in \mathcal{H}_w^p(\Omega,[-\pi,\pi]^d)$ if for all $\alpha > 0$ there is the inequality,

$$\text{meas}\{\xi \in [-\pi,\pi]^d : \|\psi(\xi,\cdot)\| > \alpha\} \le C^p/\alpha^p. \tag{3.21}$$

The weak L^p norm of ψ, $\|\psi\|_{p,w}$ is again the minimum constant C such that (3.21) holds. Lemmas 3.3, 3.4 and Hunt's Interpolation Theorem [10] then imply the following:

Lemma 3.5. *Suppose* $2 < p < \infty$. *Then*

(a) *There is a constant* C_p *depending only on* p *such that* T_φ *is a bounded operator from* $L_w^p([-\pi,\pi]^d)$ *to* $L_w^p(\Omega,[-\pi,\pi]^d)$ *and* $\|T_\varphi\| \le C_p\|\varphi\|$.

(b) *There is a constant* $C_{p,d}$ *depending only on* p *and* d *such that* $T_{\varphi,\mathbf{b},\eta}$ *is a bounded operator from* $L_w^p([-\pi,\pi]^d)$ *to* $\mathcal{H}_w^p(\Omega,[-\pi,\pi]^d)$ *and*

$$\|T_{\varphi,\mathbf{b},\eta}\| \le \frac{C_{p,d}\|\mathbf{b}\|\|\varphi\|_\infty}{(1-\|\mathbf{b}\|)^2} \; .$$

We can use Lemma 3.5 to obtain bounds on the first two derivatives with respect to ξ of the function $q(\xi,\eta)$ defined by (3.7).

Lemma 3.6. *Let* $d = 3, \; \eta > 0, \; 1 \le k,k' \le d$. *Then* $q_{k,k'}(\xi,\eta)$ *is a* C^∞ *function of* $\xi \in \mathbb{R}^d$ *and for any* $i,j, \; 1 \le i,j \le d$ *the function* $\partial q_{k,k'}/\partial\xi_i \in L_w^3([-\pi,\pi]^d)$ *and* $\partial^2 q_{k,k'}/\partial\xi_i\partial\xi_j \in L_w^{3/2}([-\pi,\pi]^d)$. *Further, there is a constant* $C_{\lambda,\Lambda}$, *depending only on* λ,Λ *such that*

$$\|\partial q_{k,k'}/\partial\xi_i\|_{3,w} \le C_{\lambda,\Lambda}, \; \|\partial^2 q_{k,k'}/\partial\xi_i\partial\xi_j\|_{3/2,w} \le C_{\lambda,\Lambda}.$$

Proof. Observe that the function $\Psi_{k'}$ occurring in (3.7) satisfies (3.6) and hence corresponds to the function Ψ of Lemma 3.2 with $\varphi \in L^\infty(\Omega)$. Observe next from (3.10) that

$$\frac{\partial}{\partial \xi_i} T_{k',k,\eta,\xi}\varphi = T_\varphi(f),$$

where T_φ is defined by (3.13) and the function f is

$$f(\xi) = \frac{\partial}{\partial \xi_i}[e^{-i\mathbf{e}_k\cdot\xi} - 1)(e^{i\mathbf{e}_{k'}\cdot\xi} - 1)\widehat{G}_\eta(\xi)]. \tag{3.22}$$

Clearly $f \in L^3_w([-\pi,\pi]^d)$ and $\|f\|_{3,w} \le C$, where C is universal. Consider now the formula (3.9) for the derivative of Ψ. For the first term on the RHS of (3.9) we have

$$\|(I - PT_{\mathbf{b},\eta,\xi})^{-1}(\frac{\partial}{\partial \xi_i}T_{k,\eta,\xi})\ \varphi(\cdot)\| \ \le \frac{1}{1-\|\mathbf{b}\|}\|\frac{\partial}{\partial \xi_i}T_{k,\eta,\xi}\ \varphi(\cdot)\|.$$

It follows from Lemma 3.5 (a) that

$$\frac{\partial}{\partial \xi_i}\ T_{k,\eta,\xi}\varphi(\cdot) \in L^3_w(\Omega, [-\pi,\pi]^3)\ ,$$

$$\|\frac{\partial}{\partial \xi_i}T_{k,\eta,\xi}\varphi(\cdot)\|_{3,w} \le C\Lambda\ ,$$

for some universal constant C, since $\|\varphi\|$ is bounded by a constant times Λ. Hence the first term on the RHS of (3.9) is in $L^3_w(\Omega,[-\pi,\pi]^3)$ with norm bounded by a constant depending only on λ, Λ. Similarly the second term on the RHS of (3.9) is bounded by

$$\frac{1}{1-\|\mathbf{b}\|}\|\left(\frac{\partial}{\partial \xi_i}T_{\mathbf{b},\eta,\xi}\right)(I - PT_{\mathbf{b},\eta,\xi})^{-1}T_{k,\eta,\xi}\varphi\|\ .$$

It follows from (3.11),(3.15) that

$$\left(\frac{\partial}{\partial \xi_i}T_{\mathbf{b},\eta,\xi}\right)(I - PT_{\mathbf{b},\eta,\xi})^{-1}T_{k,\eta,\xi}\varphi = T_{\varphi,\mathbf{b},\eta}(\mathbf{f}),$$

where $T_{\varphi,\mathbf{b},\eta}$ is like the operator (3.15) but acts on matrix valued functions $\mathbf{f}(\xi) = [f_{i,j}(\xi)],\ \xi \in [-\pi,\pi]^d$. The functions $f_{i,j}(\xi)$ are similar to (3.22) and hence are in $L^3_w([-\pi,\pi]^3)$. It follows by the argument of Lemma 3.4 and Lemma 3.5 (b) that the second term on the RHS of (3.9) is in $L^3_w(\Omega,[-\pi,\pi]^3)$ with norm bounded by a constant depending only on λ,Λ. We conclude that $\partial q_{k,k'}/\partial\xi_i \in L^3_w([-\pi,\pi]^d)$ with norm bounded by a constant depending only on λ,Λ.

Next we turn to the second derivative, $\partial^2 q_{k,k'}/\partial\xi_i\partial\xi_j$. To estimate this we need a formula for the second derivative of the function Ψ of Lemma 3.2. One can see from (3.9) that

$$\frac{\partial^2\Psi}{\partial\xi_i\partial\xi_j}(\xi,\cdot) = \big(I - PT_{\mathbf{b},\eta,\xi}\big)^{-1}\big(P\frac{\partial}{\partial\xi_j}T_{\mathbf{b},\eta,\xi}\big)$$
$$\big(I - PT_{\mathbf{b},\eta,\xi}\big)^{-1}\big(\frac{\partial}{\partial\xi_i}T_{k,\eta,\xi}\big)\varphi(\cdot) + \big(I - PT_{\mathbf{b},\eta,\xi}\big)^{-1}\big(\frac{\partial^2}{\partial\xi_i\partial\xi_j}T_{k,\eta,\xi}\big)\varphi(\cdot)$$
$$+ \big(I - PT_{\mathbf{b},\eta,\xi}\big)^{-1}\big(P\frac{\partial}{\partial\xi_i}T_{\mathbf{b},\eta,\xi}\big)\big(I - PT_{\mathbf{b},\eta,\xi}\big)^{-1}\big(\frac{\partial}{\partial\xi_j}T_{k,\eta,\xi}\big)\varphi(\cdot)$$
$$+\big(I-PT_{\mathbf{b},\eta,\xi}\big)^{-1}\big(P\frac{\partial}{\partial\xi_j}T_{\mathbf{b},\eta,\xi}\big)\big(I-PT_{\mathbf{b},\eta,\xi}\big)^{-1}\big(P\frac{\partial}{\partial\xi_i}T_{\mathbf{b},\eta,\xi}\big)\big(I-PT_{\mathbf{b},\eta,\xi}\big)^{-1}T_{k,\eta,\xi}\varphi(\cdot)$$
$$+ \big(I - PT_{\mathbf{b},\eta,\xi}\big)^{-1}\big(P\frac{\partial^2}{\partial\xi_i\partial\xi_j}T_{\mathbf{b},\eta,\xi}\big)\big(I - PT_{\mathbf{b},\eta,\xi}\big)^{-1}T_{k,\eta,\xi}\varphi(\cdot).$$

Let $\varphi' \in \mathcal{H}^\infty(\Omega)$ and consider the expectation value $\left\langle\overline{\varphi'(\cdot)}\partial^2\Psi(\xi,\eta,\cdot)/\partial\xi_i\partial\xi_j\right\rangle$. From above this is a sum of five terms. The first term is given by

$$\left\langle\overline{\varphi'(\cdot)}(I - PT_{\mathbf{b},\eta,\xi})^{-1}(P\frac{\partial}{\partial\xi_j}T_{\mathbf{b},\eta,\xi})(I - PT_{\mathbf{b},\eta,\xi})^{-1}(\frac{\partial}{\partial\xi_i}T_{k,\eta,\xi})\varphi(\cdot)\right\rangle$$
$$= \left\langle\overline{\Big[(P\frac{\partial}{\partial\xi_j}T_{\mathbf{b},\eta,\xi})(I - PT_{\mathbf{b},\eta,\xi})^{-1}\varphi'(\cdot)\Big]}(I - PT_{\mathbf{b},\eta,\xi})^{-1}(\frac{\partial}{\partial\xi_i}T_{k,\eta,\xi})\varphi(\cdot)\right\rangle,$$

whence we have

$$\left|\left\langle\overline{\varphi'(\cdot)}(I - PT_{\mathbf{b},\eta,\xi})^{-1}(P\frac{\partial}{\partial\xi_j}T_{\mathbf{b},\eta,\xi})(I - PT_{\mathbf{b},\eta,\xi})^{-1}(\frac{\partial}{\partial\xi_i}T_{k,\eta,\xi})\varphi(\cdot)\right\rangle\right|$$
$$\le \|(P\frac{\partial}{\partial\xi_j}T_{\mathbf{b},\eta,\xi})(I - PT_{\mathbf{b},\eta,\xi})^{-1}\varphi'(\cdot)\| \;\; \|(\frac{\partial}{\partial\xi_i}T_{k,\eta,\xi})\varphi(\cdot)\|/(1 - \|\mathbf{b}\|).$$

We have already seen that both

$$\| \left(P\frac{\partial}{\partial\xi_j}T_{\mathbf{b},\eta,\xi}\right)(I - PT_{\mathbf{b},\eta,\xi})^{-1}\varphi'(\cdot)\|,\; \|(\frac{\partial}{\partial\xi_i}T_{k,\eta,\xi})\;\varphi(\cdot)\|$$

are in $L^3_w([-\pi,\pi]^3)$. We conclude that the first term in $\left\langle\overline{\varphi'(\cdot)}\partial^2\Psi(\xi,\eta,\cdot)/\partial\xi_i\partial\xi_j\right\rangle$ is in $L^{3/2}_w([-\pi,\pi]^3)$ with norm depending only on λ,Λ. Consider now the second term. To estimate this observe that if T_φ is given by (3.13) then

$$T_\varphi(fg)(\xi,\omega) = \sum_{x\in\mathbb{Z}^d}\widehat{f}(x)e^{ix\cdot\xi}T_\varphi(g)(\tau_x\omega)\ . \tag{3.23}$$

We have from (3.22) that

$$\frac{\partial^2}{\partial\xi_i\partial\xi_j}T_{k',k,\eta,\xi}\varphi = T_\varphi(fg),$$

where $fg \in L^{3/2}_w([-\pi,\pi]^3)$, whence we can choose f,g so that $f,g \in L^3_w([-\pi,\pi]^3)$. With this choice of f,g and using the formula (3.23) we can argue as for the first term of $\left\langle\overline{\varphi'(\cdot)}\partial^2\Psi(\xi,\eta,\cdot)/\partial\xi_i\;\partial\xi_j\right\rangle$ to conclude that the second term is also in $L^{3/2}_w([-\pi,\pi]^3)$ with norm depending only on λ,Λ. One can estimate the other three terms of $\left\langle\overline{\varphi'(\cdot)}\partial^2\Psi(\xi,\eta,\cdot)/\partial\xi_i\;\partial\xi_j\right\rangle$ similarly. □

Proposition 3.1. *Let $d = 3$. Then the function $G_{\mathbf{a}}(x)$ defined by* (3.1) *satisfies the inequalities*

$$0 \le G_{\mathbf{a}}(x) \le C_{\lambda,\Lambda}[1+|x|]^{2-d}, \quad x \in \mathbb{Z}^d, \tag{3.24}$$

$$|\nabla G_{\mathbf{a}}(x)| \le C_{\lambda,\Lambda} \log\,[2+|x|]/(1+|x|)^2, \quad x \in \mathbb{Z}^d. \tag{3.25}$$

The constant $C_{\lambda,\Lambda}$ depends only on λ, Λ.

Proof. For $\eta > 0$ let $G_{\mathbf{a},\eta}(x)$ be defined by

$$G_{\mathbf{a},\eta}(x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} d\xi e^{-ix\cdot\xi}/e(\xi)q(\xi,\eta)e(-\xi), \quad x \in \mathbb{Z}^d.$$

Then by Lemma 3.1 it follows that $\lim_{\eta\to 0} G_{\mathbf{a},\eta}(x) = G_{\mathbf{a}}(x)$. It will be sufficient therefore for us to obtain bounds on $G_{\mathbf{a},\eta}(x), \nabla G_{\mathbf{a},\eta}(x)$ which are uniform as $\eta \to 0$. Since $q(\xi,\eta) \ge \lambda I_d,\ \xi \in [-\pi,\pi]^d$, we clearly have $G_{\mathbf{a},\eta}(x) \le C_d$, where C_d is a constant depending only on $d \ge 3$. To obtain the decay in (3.24) we write

$$G_{\mathbf{a},\eta}(x) = \int_{|\xi|<\gamma/|x|} + \int_{|\xi|>\gamma/|x|}, \tag{3.26}$$

where γ is a parameter, $1 \le \gamma \le 2$. Evidently there is a constant C such that

$$\int_{|\xi|<\gamma/|x|} d\xi \le C/|x|.$$

To bound the second integral in (3.26) we integrate by parts. Thus

$$\begin{aligned}\int_{|\xi|>\gamma/|x|} &= \frac{1}{ix_1}\frac{1}{(2\pi)^d}\int_{|\xi|>\gamma/|x|}\frac{d\xi}{e(\xi)q(\xi,\eta)e(-\xi)}\left[-\frac{\partial}{\partial\xi_1}e^{-ix\cdot\xi}\right] \\ &= \frac{-1}{ix_1}\frac{1}{(2\pi)^d}\int_{|\xi|>\gamma/|x|} d\xi\frac{e^{-ix\cdot\xi}}{[e(\xi)q(\xi,\eta)e(-\xi)]^2}\frac{\partial}{\partial\xi_1}[e(\xi)q(\xi,\eta)e(-\xi)] \\ &\quad + \frac{1}{ix_1}\frac{1}{(2\pi)^d}\int_{|\xi|=\gamma/|x|} d\xi\frac{e^{-ix\cdot\xi}\xi_1}{[e(\xi)q(\xi,\eta)e(-\xi)]|\xi|},\end{aligned} \tag{3.27}$$

where we have assumed wlog that $|x_1| = \max[|x_1|,\dots,|x_d|]$. Evidently the surface integral on the RHS of the last expression is bounded by $C/|x|$. We estimate the volume integral by integrating by parts again. Thus,

(3.28)

$$\begin{aligned}&\int_{|\xi|>\gamma/|x|} d\xi\frac{e^{-ix\cdot\xi}}{[e(\xi)q(\xi,\eta)e(-\xi)]^2}\frac{\partial}{\partial\xi_1}[e(\xi)q(\xi,\eta)e(-\xi)] \\ &\quad = \frac{1}{ix_1}\int_{|\xi|>\gamma/|x|} d\xi\frac{1}{[e(\xi)q(\xi,\eta)e(-\xi)]^2}\frac{\partial}{\partial\xi_1}[e(\xi)q(\xi,\eta)e(-\xi)]\left[-\frac{\partial}{\partial\xi_1}e^{-ix\cdot\xi}\right] \\ &\quad = \frac{1}{ix_1}\int_{|\xi|>\gamma/|x|} d\xi e^{-ix\cdot\xi}\frac{\partial}{\partial\xi_1}\left\{\frac{1}{[e(\xi)q(\xi,\eta)e(-\xi)]^2}\frac{\partial}{\partial\xi_1}[e(\xi)q(\xi,\eta)e(-\xi)]\right\} \\ &\qquad + \frac{1}{ix_1}\int_{|\xi|=\gamma/|x|} d\xi e^{-ix\cdot\xi}\frac{\xi_1}{[e(\xi)q(\xi,\eta)e(-\xi)]^2|\xi|}\frac{\partial}{\partial\xi_1}[e(\xi)q(\xi,\eta)e(-\xi)].\end{aligned}$$

We divide the surface integral in the last expression into three parts corresponding to the three terms in the expression,

$$\frac{\partial}{\partial \xi_1}\left[e(\xi)q(\xi,\eta)e(-\xi)\right] = \frac{\partial e(\xi)}{\partial \xi_1}q(\xi,\eta)e(-\xi) + e(\xi)\frac{\partial q}{\partial \xi_1}(\xi,\eta)e(-\xi) + e(\xi)q(\xi,\eta)\frac{\partial e(-\xi)}{\partial \xi_1}.$$

The surface integral corresponding to the first and third terms in this expansion is bounded by

$$\frac{C}{|x|}\int_{|\xi|=\gamma/|x|}\frac{\Lambda d\xi}{\lambda^2|\xi|^3} \le C'\Lambda/\lambda^2,$$

where C, C' are universal constants. The surface integral corresponding to the middle term is bounded by

$$\frac{C}{|x|}\int_{|\xi|=\gamma/|x|}\frac{1}{\lambda^2|\xi|^2}\|\partial q(\xi,\eta)/\partial\xi_1\|d\xi, \tag{3.29}$$

for some universal constant C. To bound this we use the well known fact that if $f \in L^p_w([-\pi,\pi]^3)$, $1 < p < \infty$, then for any measurable set E one has

$$\int_E |f|d\xi \le C_p\|f\|_{p,w}m(E)^{1-1/p}, \tag{3.30}$$

where the constant C_p depends only on p. If we average the expression (3.29) over γ, $1<\gamma<2$, then we have from Lemma 3.6 that

$$\int_1^2 d\gamma\frac{C}{|x|}\int_{|\xi|=\gamma/|x|}\frac{1}{\lambda^2|\xi|^2}\|\partial q(\xi,\eta)/\partial\xi_1\|d\xi \le \frac{C'|x|^2}{\lambda^2}\int_{|x|^{-1}<|\xi|<2|x|^{-1}} d\xi\|\partial q(\xi,\eta)/\partial\xi_1\| \le C_{\lambda,\Lambda}\ ,$$

where we have used (3.30) with $p=3$.

Next we consider the volume integral on the RHS of (3.28). For any γ, $1\le\gamma\le 2$, this is bounded by

$$\frac{C(\lambda,\Lambda)}{|x|}\int_{|\xi|>1/|x|} d\xi\Big\{|\xi|^{-4} + |\xi|^{-3}\|\partial q(\xi,\eta)/\partial\xi_1\| + |\xi|^{-2}[\|\partial^2 q(\xi,\eta)/\partial\xi_1^2\| + \|\partial q(\xi,\eta)/\partial\xi_1\|^2]\Big\}\ , \tag{3.31}$$

for some constant $C(\lambda,\Lambda)$ depending only on λ,Λ. Evidently there is a constant $C'(\lambda,\Lambda)$ such that

$$\frac{C(\lambda,\Lambda)}{|x|}\int_{|\xi|>1/|x|}\frac{d\xi}{|\xi|^4} \le C'(\lambda,\Lambda).$$

Also we have

$$\begin{aligned}
&\frac{C(\lambda,\Lambda)}{|x|}\int_{|\xi|>1/|x|}\frac{d\xi}{|\xi|^3}\|\partial q(\xi,\eta)/\partial\xi_1\| \\
&\qquad\qquad = \frac{C(\lambda,\Lambda)}{|x|}\sum_{n=0}^{\infty}\int_{2^{n+1}/|x|>|\xi|>2^n/|x|} d\xi \\
&\qquad\qquad \le C(\lambda,\Lambda)|x|^2\sum_{n=0}^{\infty}2^{-3n}\int_{2^{n+1}/|x|>|\xi|>2^n/|x|}\|\partial q(\xi,\eta)/\partial\xi_1\|\ d\xi \\
&\qquad\qquad \le C'(\lambda,\Lambda)|x|^2\sum_{n=0}^{\infty}2^{-3n}\big[2^n/|x|\big]^2 \le C''(\lambda,\Lambda)\ ,
\end{aligned}$$

where we have used Lemma 3.6 and (3.30). We can similarly bound the third term in (3.31) using Lemma 3.6 and (3.30). We conclude that (3.31) is bounded by a constant depending only on λ,Λ. If we put this inequality together with the previous inequalities we obtain (3.24).

The proof of (3.25) is similar. For any unit vector $\mathbf{n}\in\mathbb{R}^d$ we have

$$\mathbf{n}.\nabla G_{\mathbf{a},\eta}(x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} d\xi e^{-ix\cdot\xi}[-\mathbf{n}\cdot e(-\xi)]/e(\xi)q(\xi,\eta)e(-\xi)\ . \tag{3.32}$$

We do a decomposition similar to (3.26) and it is clear that

$$\int_{|\xi|<\gamma/|x|} \le C/|x|^2.$$

For the $\{|\xi|>\gamma/|x|\}$ integral we do a decomposition analogous to (3.27). It is clear the surface integral which appears is bounded by $C/|x|^2$. For the other integral we do a separate integration by parts as in (3.28). The average of the corresponding surface integral over $\gamma,\ 1\le\gamma\le 2$, is bounded by $C(\lambda,\Lambda)/|x|$. The volume integral is bounded analogously to (3.31) by

$$\begin{aligned}
\frac{C(\lambda,\Lambda)}{|x|}\int_{|\xi|>1/|x|} d\xi\big\{&|\xi|^{-3}+|\xi|^{-2}\|\partial q(\xi,\eta)/\partial\xi_1\| \\
&+|\xi|^{-1}[\|\partial^2 q(\xi,\eta)/\partial\xi_1^2\|+\|\partial q(\xi,\eta)/\partial\xi_1\|^2]\big\}.
\end{aligned}$$

Arguing as before we see this is bounded by $C'(\lambda,\Lambda)\log[1+|x|]/|x|$. Putting this inequality together with the previous inequalities yields (3.25) $\square$

Proposition 3.1 gives an improvement of the estimate (1.14) when $d=3$. We wish now to obtain a corresponding improvement for all $d\ge 3$. To do this we need generalizations of Lemmas 3.4–3.6 appropriate for all $d\ge 3$. Let $A\in\mathcal{M}$, the space of complex $d\times d$ matrices. The norm of A is defined to be

$$\|A\|^2 = Tr(A^*A).$$

Similarly if $A:\Omega\to\mathcal{M}$ is a random function we define $\|A(\cdot)\|$ by

$$\|A(\cdot)\|^2 = \langle Tr(A^*(\cdot)A(\cdot))\rangle\ .$$

Consider now functions $A : [-\pi,\pi]^d \to \mathcal{M}$. For $2 \le p \le \infty$ we can define the space $L^p([-\pi,\pi]^d, \mathcal{M})$ with norm $\|A\|_p$ given by

$$\|A\|_p^p = \int_{[-\pi,\pi]^d} \|A(\xi)\|^p d\xi.$$

Similarly we can consider functions $A : [-\pi,\pi]^d \times \Omega \to \mathcal{M}$ and associate with them spaces $L^p([-\pi,\pi]^d \times \Omega, \mathcal{M})$ with norm $\|A\|_p$ given by

$$\|A\|_p^p = \int_{[-\pi,\pi]^d} \|A(\xi,\cdot)\|^p d\xi.$$

We define an operator which generalizes the operator given in (3.15). For $n = 1, 2, \dots$ the operator $T_{n,\varphi,\mathbf{b},\eta}$ acts on n functions $A_r : [-\pi,\pi]^d \to \mathcal{M}$, $1 \le r \le n$. The resulting quantity $T_{n,\varphi,\mathbf{b},\eta}(A_1, \dots A_n)$ is a function from $[-\pi,\pi]^d \times \Omega \to \mathcal{M}$. Specifically we define

$$(3.33) \quad T_{n,\varphi,\mathbf{b},\eta}(A_1,\dots,A_n)(\xi,\cdot) =$$
$$\sum_{x_1,\dots,x_n \in \mathbb{Z}^d} \prod_{r=1}^{n-1} \left[\widehat{A}_r(x_r) e^{-ix_r\cdot\xi} \tau_{x_r} \left\{ P\mathbf{b}(\cdot)(I - PT_{\mathbf{b},\eta,\xi})^{-1} \right\} \right] \widehat{A}_n(x_n) e^{-ix_n\cdot\xi} \tau_{x_n} \varphi(\cdot).$$

Evidently the operators (3.13),(3.15) correspond to the cases $n = 1$ and $n = 2$ in (3.33).

Lemma 3.7. *Suppose $\infty \ge p_1, \dots, p_n$, $p \ge 2$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{p}$. Then if $A_r \in L^{p_r}([-\pi,\pi]^d, \mathcal{M})$, $1 \le r \le n$, the function $T_{n,\varphi,\mathbf{b},\eta}(A_1,\dots,A_n) \in L^p([-\pi,\pi]^d \times \Omega, \mathcal{M})$ and*

$$(3.34) \qquad \|T_{n,\varphi,\mathbf{b},\eta}(A_1,\dots A_n)\|_p \le \frac{\|\mathbf{b}\|^{n-1}\|\varphi\|_\infty \prod_{r=1}^n \|A_r\|_{p_r}}{(1-\|\mathbf{b}\|)^{2n}}.$$

Proof. Consider first the case $n = 2$. If $A_1, A_2 \in L^\infty([-\pi,\pi]^d, \mathcal{M})$ then it is clear that $T_{2,\varphi,\mathbf{b},\eta}(A_1, A_2) \in L^\infty([-\pi,\pi]^d \times \Omega, \mathcal{M})$ and

$$\|T_{2,\varphi,\mathbf{b},\eta}(A_1,A_2)\|_\infty \le \frac{\|\mathbf{b}\|\|\varphi\|_2\|A_1\|_\infty\|A_2\|_\infty}{(1-\|\mathbf{b}\|)}\ .$$

If $A_2 \in L^\infty$, $A_1 \in L^2$ then we can see by the argument of Lemma 3.4 that $T_{2,\varphi,\mathbf{b},\eta}(A_1,A_2) \in L^2$ and

$$(3.35) \qquad \|T_{2,\varphi,\mathbf{b},\eta}(A_1,A_2)\|_2 \le \frac{\|\mathbf{b}\|\|\varphi\|_\infty\|A_2\|_\infty\|A_1\|_2}{(1-\|\mathbf{b}\|)^2}\ .$$

It follows therefore by interpolation theory that if $A_2 \in L^\infty$, $A_1 \in L^p$, $p \ge 2$, then $T_{2,\varphi,\mathbf{b},\eta}(A_1,A_2) \in L^p$ and

$$(3.36) \qquad \|T_{2,\varphi,\mathbf{b},\eta}(A_1,A_2)\|_p \le \frac{\|\mathbf{b}\|\|\varphi\|_\infty\|A_2\|_\infty\|A_1\|_p}{(1-\|\mathbf{b}\|)^2}\ .$$

Suppose next that $A_1 \in L^{p_1}$, $A_2 \in L^{p_2}$ and $1/p_1 + 1/p_2 = 1/2$. We have now that

$$\text{(3.37)} \quad \left[\int_{[-\pi,\pi]^d} d\xi \ \ \|T_{2,\varphi,\mathbf{b},\eta}(A_1,A_2)(\xi,\cdot)\|^2\right]^{1/2} \le \sum_{m=0}^{\infty}\left[\int_{[-\pi,\pi]^d} d\xi \| \sum_{x_1,x_2\in\mathbb{Z}^d} \widehat{A}_1(x_1)e^{-ix_1\cdot\xi}\tau_{x_1} P\mathbf{b}(\cdot)(PT_{\mathbf{b},\eta,\xi})^m \widehat{A}_2(x_2)e^{-ix_2\cdot\xi}\tau_{x_2}\varphi(\cdot)\|^2\right]^{1/2}.$$

Observe now that as in (3.16) one has

$$\int_{[-\pi,\pi]^d} d\xi \| \sum_{x_1,x_2,\in\mathbb{Z}^d} \widehat{A}_1(x_1)e^{-ix_1\cdot\xi}\tau_{x_1}\mathbf{b}(\cdot)(T_{\mathbf{b},\eta,\xi})^m \widehat{A}_2(x_2)e^{-ix_2\cdot\xi}\tau_{x_2}\varphi(\cdot)\|^2$$
$$= (2\pi)^d \sum_{r\in\mathbb{Z}^d} \| \sum_{y_1,\dots,y_{m+1}} \widehat{A}_1(y_1)\mathbf{b}(\tau_{y_1}\cdot)\left[\prod_{j=2}^{m+1} \nabla^*\nabla G_\eta(y_j-y_{j-1})\mathbf{b}(\tau_{y_j}\cdot)\right] \widehat{A}_2(r-y_{m+1})\varphi(\tau_r\cdot)\|^2$$
$$\le (2\pi)^d \|\varphi\|_\infty^2 \sum_{r\in\mathbb{Z}^d} || \sum_{y_1,\dots,y_{m+1}} \widehat{A}_1(y_1)\mathbf{b}(\tau_{y_1}\cdot)\left[\prod_{j=2}^{m+1} \nabla^*\nabla G_\eta(y_j-y_{j-1})\mathbf{b}(\tau_{y_j}\cdot)\right] \widehat{A}_2(r-y_{m+1})||^2$$
$$= \|\varphi\|_\infty^2 \int_{[-\pi,\pi]^d} d\xi \| \left[\sum_{x_1\in\mathbb{Z}^d} \widehat{A}_1(x_1)e^{-ix_1\cdot\xi}\tau_{x_1}\mathbf{b}(\cdot)(T_{\mathbf{b},\eta,\xi})^m I_d\right] A_2(\xi)\|^2$$
$$\le \|\varphi\|_\infty^2 \int_{[-\pi,\pi]^d} d\xi \| \sum_{x_1\in\mathbb{Z}^d} \widehat{A}_1(x_1)e^{-ix_1\cdot\xi}\tau_{x_1}\mathbf{b}(\cdot)(T_{\mathbf{b},\eta,\xi})^m I_d||^2 ||A_2(\xi)\|^2$$
$$\le \|\varphi\|_\infty^2 \left[\int_{[-\pi,\pi]^d} d\xi \| \sum_{x_1\in\mathbb{Z}^d} \widehat{A}_1(x_1)e^{-ix_1\cdot\xi}\tau_{x_1}\mathbf{b}(\cdot)(T_{\mathbf{b},\eta,\xi})^m I_d\|^{p_1}\right]^{2/p_1} \|A_2\|_{p_2}^2 \ .$$

We have already seen from interpolation theory that

$$\left[\int_{[-\pi,\pi]^d} d\xi \| \sum_{x_1\in\mathbb{Z}^d} \widehat{A}_1(x_1)e^{-ix_1\cdot\xi}\tau_{x_1}\mathbf{b}(\cdot)(T_{\mathbf{b},\eta,\xi})^m I_d\|^{p_1}\right]^{1/p_1} \le \|\mathbf{b}\|^{m+1}\|A_1\|_{p_1} \ .$$

We conclude therefore that

$$\int_{[-\pi,\pi]^d} d\xi \| \sum_{x_1\in\mathbb{Z}^d} \widehat{A}_1(x_1)e^{-ix_1\cdot\xi}\tau_{x_1}\mathbf{b}(\cdot)(T_{\mathbf{b},\eta,\xi})^m \widehat{A}_2(x_2)e^{-ix_2\cdot\xi}\tau_{x_2}\varphi(\cdot)\|^2 \le \|\mathbf{b}\|^{2m+2}\|\varphi\|_\infty^2\|A_1\|_{p_1}^2\|A_2\ \|_{p_2}^2 \ .$$

Arguing as in Lemma 3.4 we also see that

$$\int_{[-\pi,\pi]^d} d\xi \| \sum_{x_1,x_2\in\mathbb{Z}^d} \widehat{A}_1(x_1)e^{-ix_1\cdot\xi}\tau_{x_1}P\mathbf{b}(\cdot)(PT_{\mathbf{b},\eta,\xi})^m \widehat{A}_2(x_2)e^{-ix_2\cdot\xi}\tau_{x_2}\varphi(\cdot)\|^2 \le (m+1)^2\|\mathbf{b}\|^{2m+2}\|\varphi\|_\infty^2\|A_1\|_{p_1}^2\|A_2\|_{p_2}^2 \ .$$

It follows now from (3.37) that

$$\|T_{2,\varphi,\mathbf{b},\eta}(A_1,A_2)\|_2 \le \frac{\|\mathbf{b}\|\|\varphi\|_\infty\|A_1\|_{p_1}\|A_2\|_{p_2}}{(1-\|\mathbf{b}\|)^2}\ , \tag{3.38}$$

provided $1/p_1+1/p_2=1/2$. We can now use the inequalities (3.36),(3.38) to do a further interpolation. Thus we have if $p\ge 2$ and $1/p_1+1/p_2=1/p$ then

$$\|T_{2,\varphi,\mathbf{b},\eta}(A_1,A_2)\|_p \le \frac{\|\mathbf{b}\|\|\varphi\|_\infty\|A_1\|_{p_1}\|A_2\|_{p_2}}{(1-\|\mathbf{b}\|)^2}\ .$$

This proves the result for $n=2$.

To deal with $n=3$ we subdivide into 7 cases:

(a) $1/p_1=0,\ 1/p_2=0,\ 1/p_3=0,$
(b) $1/p_1=1/2,\ 1/p_2=0,\ 1/p_3=0,$
(c) $1/p_1<1/2,\ 1/p_2=0,\ 1/p_3=0,$
(d) $1/p_1+1/p_2=1/2,\ 1/p_3=0,$
(e) $1/p_1+1/p_2<1/2,\ 1/p_3=0,$
(f) $1/p_1+1/p_2+1/p_3=1/2,$
(g) $1/p_1+1/p_2+1/p_3<1/2.$

For (a) it is easy to see that

$$\|T_{3,\varphi,\mathbf{b},\eta}(A_1,A_2,A_3)\|_\infty \le \frac{\|\mathbf{b}\|^2\|\varphi\|_2\|A_1\|_\infty\|A_2\|_\infty\|A_3\|_\infty}{(1-\|\mathbf{b}\|)^2}\ .$$

For (b) we use the argument of Lemma 3.4 to conclude

$$\|T_{3,\varphi,\mathbf{b},\eta}(A_1,A_2,A_3)\|_2 \le \frac{\|\mathbf{b}\|^2\|\varphi\|_\infty\|A_1\|_2\|A_2\|_\infty\|A_3\|_\infty}{(1-\|\mathbf{b}\|)^4}\ .$$

The Riesz-Thorin Theorem applied to (a), (b) yield for (c) the inequality,

$$\|T_{3,\varphi,\mathbf{b},\eta}(A_1,A_2,A_3)\|_p \le \frac{\|\mathbf{b}\|^2\|\varphi\|_\infty\|A_1\|_{p_1}\|A_2\|_\infty\|A_3\|_\infty}{(1-\|\mathbf{b}\|)^4}\ ,$$

where $p=p_1$. For (d) we use the argument to obtain (3.38) to conclude that

$$\|T_{3,\varphi,\mathbf{b},\eta}(A_1,A_2,A_3)\|_2 \le \frac{\|\mathbf{b}\|^2\|\varphi\|_\infty\|A_1\|_{p_1}\|A_2\|_{p_2}\|A_3\|_\infty}{(1-\|\mathbf{b}\|)^4}\ .$$

Now the Riesz-Thorin Theorem applied to (c) and (d) yield for (e) the inequality

$$\|T_{3,\varphi,\mathbf{b},\eta}(A_1,A_2,A_3)\|_p \le \frac{\|\mathbf{b}\|^2\|\varphi\|_\infty\|A_1\|_{p_1}\|A_2\|_{p_2}\|A_3\|_\infty}{(1-\|\mathbf{b}\|)^4}\ ,$$

where $1/p=1/p_1+1/p_2$. To obtain an inequality for (f) we use the argument to obtain (3.38). This reduces us to the case dealt with in (e). Hence we can use the inequality for (e) to obtain the bound

$$\|T_{3,\varphi,\mathbf{b},\eta}(A_1,A_2,A_3)\|_2 \le \frac{\|\mathbf{b}\|^2\|\varphi\|_\infty\|A_1\|_{p_1}\|A_2\|_{p_2}\|A_3\|_{p_3}}{(1-\|\mathbf{b}\|)^4}\ .$$

Finally the Riesz-Thorin Theorem applied to (e) and (f) yields the inequality (3.34) with $n=3$ for (g). Since it is clear we can generalize the method for $n=3$ to all n, the result follows. ∎

Lemma 3.8. *Suppose* $\infty > p_1, \dots, p_n,\ p > 2$ *and* $\frac{1}{p_1} + \dots + \frac{1}{p_n} = \frac{1}{p}$. *Then if* $A_r \in L_w^{p_r}([-\pi,\pi]^d, \mathcal{M}),\ 1 \le r \le n$, *the function* $T_{n,\varphi,\mathbf{b},\eta}(A_1, \dots, A_n) \in L_w^p([-\pi,\pi]^d \times \Omega, \mathcal{M})$ *and*

$$\|T_{n,\varphi,\mathbf{b},\eta}(A_1, \dots, A_n)\|_{p,w} \le \frac{C\|\mathbf{b}\|^{n-1}\|\varphi\|_\infty \prod_{r=1}^n \|A_r\|_{p_r,w}}{(1-\|\mathbf{b}\|)^{2n}} \ , \tag{3.39}$$

where the constant C *depends only on* $p_1, \dots p_n$.

Proof. We use Lemma 3.7 and Hunt's Interpolation Theorem [10]. Suppose $n = 2$. Now from Lemma 3.7 we know that for a given $p_1,\ 2 \le p_1 \le \infty$, then

$$\|T_{2,\varphi,\mathbf{b},\eta}(A_1, A_2)\|_{p_1} \le \frac{\|\mathbf{b}\|\|\varphi\|_\infty\|A_1\|_{p_1}\|A_2\|_\infty}{(1-\|\mathbf{b}\|)^2} \ ,$$
$$\|T_{2,\varphi,\mathbf{b},\eta}(A_1, A_2)\|_2 \le \frac{\|\mathbf{b}\|\|\varphi\|_\infty\|A_1\|_{p_1}\|A_2\|_{q_1}}{(1-\|\mathbf{b}\|)^2} \ ,$$

where $1/p_1 + 1/q_1 = 1/2$. Hence the Hunt Theorem implies that

$$\|T_{2,\varphi,\mathbf{b},\eta}(A_1, A_2)\|_{p,w} \le \frac{C(p_1,p_2)\|\mathbf{b}\|\|\varphi\|_\infty\|A_1\|_{p_1}\|A_2\|_{p_2,w}}{(1-\|\mathbf{b}\|)^2} \ , \tag{3.40}$$

provided $1/p_1 + 1/p_2 = 1/p > 1/2,\ p_2 < \infty$, and $C(p_1, p_2)$ is a constant depending only on p_1, p_2. Suppose now that A_2 is fixed with $\|A_2\|_{p_2,w}$ finite for some $p_2,\ 2 < p_2 < \infty$. We consider the mapping

$$A_1 \to T_{2,\varphi,\mathbf{b},\eta}(A_1, A_2).$$

We see from (3.40) that this maps L^∞ to $L_w^{p_2}$. For $\varepsilon > 0$ let $p_1(\varepsilon)$ satisfy $1/p_1(\varepsilon) + 1/p_2 = 1/2 + \varepsilon = 1/p(\varepsilon)$. From (3.40) we also see that it maps $L^{p_1(\varepsilon)}$ to $L_w^{p(\varepsilon)}$. It follows again from interpolation theory that for any $p_1,\ p_1(\varepsilon) < p_1 < \infty$ it maps $L_w^{p_1}$ to L_w^p where $1/p_1 + 1/p_2 = 1/p$. The inequality (3.39) for $n = 2$ follows from this. It is clear this method can be generalised to all n. □

Lemma 3.9. *Let* $d \ge 3,\ \eta > 0,\ 1 \le k, k' \le d$. *Then* $q_{k,k'}(\xi,\eta)$ *is a* C^∞ *function of* $\xi \in [-\pi,\pi]^d$. *Further, let* $\alpha = (\alpha_1, \dots, \alpha_d)$, *where* $\alpha_i \ge 0,\ 1 \le i \le d$, *and* $|\alpha| = \alpha_1 + \dots + \alpha_d < d$. *Then the function* $\prod_{i=1}^d (\frac{\partial}{\partial \xi_i})^{\alpha_i} q_{k,k'}(\xi,\eta)$ *is in* $L_w^{d/|\alpha|}([-\pi,\pi]^d)$, *and* $\| \prod_{i=1}^d (\frac{\partial}{\partial \xi_i})^{\alpha_i} q_{k,k'}\|_{d/|\alpha|,w} \le C_{\lambda,\Lambda,d}$, *where the constant* $C_{\lambda,\Lambda,d}$ *depends only on* λ, Λ, d.

Proof. We argue as in Lemma 3.6. It is easy to see that the function Ψ of Lemma 3.2 has the property that

$$\prod_{i=1}^d (\frac{\partial}{\partial \xi_i})^{\alpha_i} \Psi(\xi, \eta, \cdot)$$

is a sum of terms $T_{n,\varphi,\mathbf{b},\eta}(A_1, \dots, A_n)(\xi, \cdot)\mathbf{e}_k$, where $1 \le n \le |\alpha|,\ A_r \in L_w^{p_r},\ 1 \le r \le n$, and $1/p_1 + \dots + 1/p_n = |\alpha|/d$. The result follows now from Lemma 3.8 if $|\alpha| < d/2$. To deal with the case $d/2 \le |\alpha| < d$, we argue exactly as for the $d = 3$ case with $|\alpha| = 2$. □

Proposition 3.2. *Let $d \ge 3$. Then the function $G_{\mathbf{a}}(x)$ defined by (3.1) satisfies the inequalities*

$$0 \le G_{\mathbf{a}}(x) \le C_{\lambda,\Lambda,d}[1+|x|]^{2-d}, \ x \in \mathbb{Z}^d,$$
$$|\nabla G_{\mathbf{a}}(x)| \le C_{\lambda,\Lambda,d} \log[2+|x|](1+|x|)^{1-d}, \quad x \in \mathbb{Z}^d,$$

where the constant $C_{\lambda,\Lambda,d}$ depends only on λ, Λ, d.

Proof. We argue as in Proposition 3.1, using Lemma 3.9. □

Proposition 3.2 gives an alternative derivation of the inequality (1.14). We can extend the argument in the proposition to obtain Theorem 1.5. To do this we need the following improvement of Lemma 3.9.

Lemma 3.10. *Let $q_{k,k'}(\xi,\eta)$ be the function of Lemma 3.9 and $|\alpha| = d-1$. Suppose $\rho \in \mathbb{R}^d$, $|\rho| \le 1$. Then for any ε, $0 < \varepsilon < 1$ the function*

$$\prod_{i=1}^{d} \left(\frac{\partial}{\partial \xi_i}\right)^{\alpha_i} \left[q_{k,k'}(\xi+\rho,\eta) - q_{k,k'}(\xi,\eta)\right] / |\rho|^{1-\varepsilon}$$

is in $L_w^{d/(d-\varepsilon)}([-\pi,\pi]^d)$, and

$$\left\| \prod_{i=1}^{d} \left(\frac{\partial}{\partial \xi_i}\right)^{\alpha_i} \left[q_{k,k'}(\xi+\rho,\eta) - q_{k,k'}(\xi,\eta)\right] / |\rho|^{1-\varepsilon} \right\|_{d/(d-\varepsilon),w} \le C_{\lambda,\Lambda,d,\epsilon} \ ,$$

where the constant $C_{\lambda,\Lambda,d,\epsilon}$ depends only on $\lambda, \Lambda, d, \epsilon$.

Before proving Lemma 3.10 we first show how Theorem 1.5 follows from it.

Proof of Theorem 1.5. We shall confine ourselves to the case $d = 3$ since the argument for $d > 3$ is similar. Consider the representation (3.32) for $\mathbf{n}.\nabla G_{\mathbf{a},\eta}(x)$. We have now,

$$\int_{|\xi|>\gamma/|x|} = \frac{1}{ix_1}\frac{1}{(2\pi)^d} \int_{|\xi|>\gamma/|x|} d\xi e^{-ix\cdot\xi} \frac{\partial}{\partial \xi_1}\left[\frac{-\mathbf{n}\cdot e(-\xi)}{e(\xi)q(\xi,\eta)e(-\xi)}\right]$$
$$- \frac{1}{ix_1}\frac{1}{(2\pi)^d} \int_{|\xi|=\gamma/|x|} d\xi e^{-ix\cdot\xi} \frac{\mathbf{n}\cdot e(-\xi)}{e(\xi)q(\xi,\eta)e(-\xi)} \frac{\xi_1}{|\xi|} \ .$$

Evidently one has a bound for the surface integral,

$$\left| \int_{|\xi|=\gamma/|x|} \right| \le C/|x|^2.$$

If we integrate by parts again in the volume integral we have

$$\frac{1}{ix_1}\frac{1}{(2\pi)^d} \int_{|\xi|>\gamma/|x|} d\xi e^{-ix\cdot\xi} \frac{\partial}{\partial \xi_1}\left[\frac{-\mathbf{n}\cdot e(-\xi)}{e(\xi)q(\xi,\eta)e(-\xi)}\right]$$
$$= \frac{1}{x_1^2}\frac{1}{(2\pi)^d} \int_{|\xi|>\gamma/|x|} d\xi e^{-ix\cdot\xi} \left(\frac{\partial}{\partial \xi_1}\right)^2 \left[\frac{\mathbf{n}\cdot e(-\xi)}{e(\xi)q(\xi,\eta)e(-\xi)}\right]$$
$$+ \frac{1}{x_1^2}\frac{1}{(2\pi)^d} \int_{|\xi|=\gamma/|x|} d\xi e^{-ix\cdot\xi} \frac{\partial}{\partial \xi_1}\left[\frac{\mathbf{n}\cdot e(-\xi)}{e(\xi)q(\xi,\eta)e(-\xi)}\right] \frac{\xi_1}{|\xi|} \ .$$

Since $q(\xi,\eta)$ is bounded and $\partial q(\xi,\eta)/\partial \xi_i$ is in L_w^3 it follows that the average of the surface integral over γ, $1 < \gamma < 2$, is bounded by $C/|x|^2$ for some constant C. To bound the volume integral let $\rho \in \mathbb{R}^3$ be such that $e^{-ix\cdot\rho} = -1$ and ρ has minimal magnitude. Then $|\rho| \le 10/|x|$ and

$$\begin{aligned}
&\frac{1}{x_1^2}\frac{1}{(2\pi)^d}\int_{|\xi|>\gamma/|x|} d\xi e^{-ix\cdot\xi}\left(\frac{\partial}{\partial\xi_1}\right)^2\left[\frac{\mathbf{n}\cdot e(-\xi)}{e(\xi)q(\xi,\eta)e(-\xi)}\right] \\
&\quad = \frac{1}{2x_1^2}\frac{1}{(2\pi)^d}\int_{|\xi|>100/|x|} d\xi e^{-ix\cdot\xi} \\
&\qquad\qquad \left(\frac{\partial}{\partial\xi_1}\right)^2\left[\frac{\mathbf{n}\cdot e(-\xi)}{e(\xi)q(\xi,\eta)e(-\xi)} - \frac{\mathbf{n}\cdot e(-\xi-\rho)}{e(\xi+\rho)q(\xi+\rho,\eta)e(-\xi-\rho)}\right] \\
&\qquad + \frac{1}{x_1^2}\int_{1/|x|<|\xi|<200/|x|} R(\xi)e^{-ix\cdot\xi}d\xi\ ,
\end{aligned}$$

where $R(\xi)$ is the remainder term. Using the fact that $\partial q/\partial\xi_1 \in L_w^3$, $\partial^2 q/\partial\xi_1^2 \in L_w^{3/2}$ we can easily see that

$$\frac{1}{x_1^2}\int_{1/|x|<\xi<200/|x|} |R(\xi)|d\xi \le C/|x|^2$$

for some constant C. We can bound the first term above using Lemma 3.9 and Lemma 3.10. Evidently we will get a term

$$\frac{C|\rho|^{1-\varepsilon}}{|x|^2}\int_{|\xi|>100/|x|}\left|\left(\frac{\partial}{\partial\xi_1}\right)^2[q(\xi+\rho,\eta)-q(\xi,\eta)]/|\rho|^{1-\varepsilon}\right|\frac{d\xi}{|\xi|}.$$

From Lemma 3.10 this is bounded by

$$\frac{C|\rho|^{1-\varepsilon}}{|x|^2}\sum_{n=0}^{\infty}\int_{2^{n+1}/|x|>|\xi|>2^n/|x|} \le \frac{C'|\rho|^{1-\varepsilon}}{|x|^2}|x|^{1-\varepsilon}\sum_{n=0}^{\infty}2^{-n(1-\varepsilon)} \le \frac{10C'}{|x|^2},$$

for some constant C'. Other terms are bounded using Lemma 3.9. We have proved the first inequality of Theorem 1.5. The second inequality of the theorem for $d=3$ is proved similarly. □

Proof of Lemma 3.10. The argument follows the same lines as in Lemma 3.9. The main point to observe is that if $f(\xi)$ is given by

$$f(\xi) = \prod_{i=1}^{d}\left(\frac{\partial}{\partial\xi_i}\right)^{\alpha_i}\left[e_k(-\xi)e_{k'}(\xi)\widehat{G}_\eta(\xi)\right],$$

with $|\alpha| < d$ then for any ε, $0<\varepsilon<1$, the function $[f(\xi+\rho)-f(\xi)]/|\rho|^{1-\varepsilon}$ is in $L_w^{d/(1+|\alpha|-\varepsilon)}$ and there is a constant $C_{d,\varepsilon}$ depending only on d,ε such that $\|[f(\xi+\rho)-f(\xi)]/|\rho|^{1-\varepsilon}\|_{d/(1+|\alpha|-\varepsilon),w} \le C_{d,\varepsilon}$, provided $|\rho|\le 1$. □

4. Proof of Theorem 1.4—Diagonal Case

Here we shall prove the inequalities of Theorem 1.4, but without the exponential falloff term. We shall call this the diagonal case. First we show that Theorem 1.6 already gives us the diagonal case of the inequality (1.10) in dimension $d=1$.

Corollary 4.1. *The function $G_{\mathbf{a}}(x,t)$ satisfies the inequality*

$$0 \le G_{\mathbf{a}}(x,t) \le C(\lambda,\Lambda)/[1+\sqrt{t}], \quad \text{if} \quad d=1, \tag{4.1}$$

where the constant $C(\lambda,\Lambda)$ depends only on λ,Λ. If $d>1$, it satisfies an inequality,

$$0 \le G_{\mathbf{a}}(x,t) \le C_\varepsilon(d,\lambda,\Lambda)/[1+t^{1-\varepsilon}], \tag{4.2}$$

where ε can be any number, $0<\varepsilon<1$, and $C_\varepsilon(d,\lambda,\Lambda)$ is a constant depending only on $\varepsilon, d, \lambda, \Lambda$.

Proof. We have

$$\begin{aligned} G_{\mathbf{a}}(x,t) &\le \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} |\hat{G}_{\mathbf{a}}(\xi,t)|d\xi \\ &\le \frac{1}{(2\pi)^d}\int_{|\xi|<1/\sqrt{t}} d\xi + \frac{1}{(2\pi)^d}\int_{|\xi|>1/\sqrt{t}} d\xi \ . \end{aligned}$$

Since $\hat{G}_{\mathbf{a}}(\xi,t)$ is bounded on $[-\pi,\pi]^d$ it follows that

$$\frac{1}{(2\pi)^d}\int_{|\xi|<1/\sqrt{t}} d\xi \le C(d,\lambda,\Lambda)/[1+t^{d/2}]. \tag{4.3}$$

The integral over $\{|\xi|>1/\sqrt{t}\}$ is nonzero only if $t>1/\pi^2 d$. In that case one has from Theorem 1.6,

$$\frac{1}{(2\pi)^d}\int_{|\xi|>1/\sqrt{t}} d\xi \le C(\delta,\lambda,\Lambda)\int_{|\xi|>1/\sqrt{t}} \frac{d\xi}{(\xi^2 t)^\delta} \ ,$$

for any δ satisfying $0\le\delta<1$. For $d=1$ we have on taking $\delta>1/2$ an inequality

$$\int_{|\xi|>1/\sqrt{t}} \frac{d\xi}{(\xi^2 t)^\delta} \ \le \ \frac{C}{\sqrt{t}} \ .$$

The inequality (4.1) follows from this and (4.3). For $d>1$ and any $p>d/2$ we have

$$\int_{|\xi|>1/\sqrt{t}} \frac{d\xi}{(\xi^2 t)^\delta} \le (2\pi)^{d(1-1/p)}\left[\int_{|\xi|>1/\sqrt{t}} \frac{d\xi}{(\xi^2 t)^{\delta p}}\right]^{1/p} \le C_p/t^{d/2p} \ ,$$

where the constant C_p depends only on p and we have chosen δ to satisfy $1>\delta>d/2p$. The inequality (4.2) follows now from this last inequality and (4.3) on choosing p to satisfy $d/2p = 1-\varepsilon$. □

We can similarly use Theorem 1.6 to obtain estimates on the t derivative of $G_{\mathbf{a}}(x,t)$.

Corollary 4.2. *The function $G_{\mathbf{a}}(x,t)$ is differentiable w.r. to t for $t>0$ and the derivative satisfies the inequality*

$$\left|\frac{\partial G_{\mathbf{a}}(x,t)}{\partial t}\right| \le C(\lambda,\Lambda)/[1+t^{3/2}] \ , \quad \text{if} \quad d=1 \ ,$$

where the constant $C(\lambda,\Lambda)$ depends only on λ,Λ. If $d>1$, it satisfies an inequality

$$\left|\frac{\partial G_{\mathbf{a}}(x,t)}{\partial t}\right| \le C_\varepsilon(d,\lambda,\Lambda)/[1+t^{2-\varepsilon}] \ ,$$

where ε can be any number, $0<\varepsilon<1$, and $C_\varepsilon(d,\lambda,\Lambda)$ is a constant depending only on $\varepsilon, d, \lambda, \Lambda$.

Observe now that Corollary 4.1 almost obtains the diagonal case of the inequality (1.10) for $d = 2$. In order to obtain this inequality for $d = 2$ we shall have to use the methods of Section 3.

Lemma 4.1. *For $d = 2$, there is a constant $C(\lambda, \Lambda)$ depending only on λ, Λ such that*

$$0 \le G_{\mathbf{a}}(x,t) \le C(\lambda,\Lambda)/[1+t] \ , \quad t > 0 \ .$$

Proof. We shall use the notation of Section 2, in particular the functions h, k defined by (2.18). It is easy to see that

(4.4)
$$\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] = \frac{1}{t^2}\int_0^\infty \frac{\partial^2 h(\xi,\eta)}{\partial[Im(\eta)]^2}\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)].$$

We have now from (2.44), (2.45), (2.47) that

$$\int_{|e(\xi)|^2}^\infty \left|\frac{\partial^2 h(\xi,\eta)}{\partial[Im(\eta)]^2}\right| d[\mathrm{Im}(\eta)] < \frac{2}{|e(\xi)|^4} \ .$$

Hence if $t > 1$, then

$$\begin{aligned}
&\int_{[-\pi,\pi]^d}\left|\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]\right| d\xi \\
&\quad = \int_{|e(\xi)|<1/\sqrt{t}} + \int_{|e(\xi)|>1/\sqrt{t}} \le \frac{C}{t} + \frac{2}{t^2}\int_{|e(\xi)|>1/\sqrt{t}} |e(\xi)|^{-4}\, d\xi \\
&\quad + \int_{|e(\xi)|>1/\sqrt{t}} d\xi \left|\frac{1}{t^2}\int_0^{|e(\xi)|^2} \frac{\partial^2 h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2}\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)]\right| ,
\end{aligned}$$

where we have used the fact that the LHS of (4.4) is bounded by a universal constant. Evidently the first two terms on the RHS of the last inequality are bounded by C/t, so we concentrate on the third term.

In view of (2.44) and the inequalities following it we have

$$\begin{aligned}
&\int_{|e(\xi)|>1/\sqrt{t}} d\xi \left|\frac{1}{t^2}\int_0^{|e(\xi)|^2} \frac{\partial^2 h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2}\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)]\right| \le \\
&\frac{C_{\lambda,\Lambda}}{t} + C_{\lambda,\Lambda}\int_{|e(\xi)|>1/\sqrt{t}} d\xi \frac{1}{t^2}\int_0^{|e(\xi)|^2} \frac{\langle|\psi(\xi,\eta,\cdot)|^2\rangle}{|e(\xi)|^4\mathrm{Im}(\eta)}\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)] \ ,
\end{aligned}$$

for some constant $C_{\lambda,\Lambda}$ depending only on λ, Λ. Now for $\mathrm{Re}(\eta) > 0$ and $2 < p < \infty$ let $h_{\eta,p}(\xi)$, $\xi \in [-\pi,\pi]^2$, be the function

$$h_{\eta,p}(\xi) = |\mathrm{Im}(\eta)|^{1/2\,-\,1/p}\left[\sum_{k=1}^2 \langle|\psi_k(\xi,\eta,\cdot)|^2\rangle\right]^{1/2} ,$$

where $\psi_k(\xi,\eta,\cdot)$ is given by (2.2). We shall show that $h_{\eta,p} \in L^p_w([-\pi,\pi]^2)$ and there is a constant $C_{p,\lambda,\Lambda}$ depending only on p, λ, Λ such that

(4.5)
$$\|h_{\eta,p}\|_{p,w} \le C_{p,\lambda,\Lambda}, \quad Re(\eta) > 0, \quad 2 < p < \infty.$$

Observe now from (2.8),(2.26) that

$$\frac{1}{t^2}\int_0^{|e(\xi)|^2}\frac{\langle|\psi(\xi,\eta,\cdot)|^2\rangle}{|e(\xi)|^4\mathrm{Im}(\eta)}\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)]$$
$$\le\frac{1}{t^{1+1/p}}\int_0^{|e(\xi)|^2}\frac{h_{\eta,p}(\xi)^2}{|e(\xi)|^2\mathrm{Im}(\eta)^{1-1/p}}\frac{\{1-\cos[\mathrm{Im}(\eta)t]\}}{[\mathrm{Im}(\eta)t]^{1-1/p}}d[\mathrm{Im}(\eta)]$$
$$\le\frac{C}{t^{1+1/p}}\int_0^{|e(\xi)|^2}\frac{h_{\eta,p}(\xi)^2}{|e(\xi)|^2\mathrm{Im}(\eta)^{1-1/p}}d[\mathrm{Im}(\eta)],$$

for some universal constant C. Next we have that

$$(4.6)\quad \int_{|e(\xi)|>1/\sqrt{t}}d\xi\frac{1}{t^{1+1/p}}\int_0^{|e(\xi)|^2}\frac{h_{\eta,p}(\xi)^2}{|e(\xi|^2\mathrm{Im}(\eta)^{1-1/p}}d[\mathrm{Im}(\eta)]\le$$
$$\frac{C}{t^{1/p}}\sum_{n=0}^{\infty}2^{-2n}\int_0^{2^{2n+2}/t}\frac{d[\mathrm{Im}(\eta)]}{\mathrm{Im}(\eta)^{1-1/p}}\int_{|e(\xi)|<2^{n+1}/\sqrt{t}}h_{\eta,p}(\xi)^2d\xi\ .$$

From (3.30) it follows that

$$\int_{|e(\xi)|<2^{n+1}/\sqrt{t}}h_{\eta,p}(\xi)^2d\xi\le C_p\|h_{\eta,p}\|_{p,w}^2 2^{2n(1-2/p)}/t^{1-2/p}.$$

If we use the inequality (4.5) we see from the last inequality that the RHS of (4.6) is bounded by $C_{p,\lambda,\Lambda}/t$ for some constant $C_{p,\lambda,\Lambda}$ depending only on p,λ,Λ. We conclude that if (4.5) holds then

$$\int_{[-\pi,\pi]^2}d\xi\left|\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]\right|\le C_{\lambda,\Lambda}/t,\quad t>0,$$

for some constant $C_{\lambda,\Lambda}$ depending only on λ,Λ.

To prove (4.5) note that $\psi_k(\xi,\eta,\cdot)$ is a sum of terms $PT_{\varphi,\mathbf{b},\eta}(f)$ where $T_{\varphi,\mathbf{b},\eta}$ is the operator (3.15), φ is an entry of the matrix $\mathbf{a}(\cdot)$ and f is the Fourier transform of $\nabla_j G_\eta(x),\ x\in\mathbb{Z}^d$, $1\le j\le d$. Hence

$$|f(\xi)|\le\frac{|e(\xi)|}{|e(\xi)|^2+|\mathrm{Im}(\eta)|}\le\frac{1}{\left[2\sqrt{|\mathrm{Im}(\eta)|}\right]^{1-2/p}}\ \frac{1}{|e(\xi)|^{2/p}},\quad \xi\in[-\pi,\pi]^2,$$

whence $f\in L_w^p([-\pi,\pi]^2)$ with norm bounded by a constant times $|\mathrm{Im}(\eta)|^{1/p-1/2}$. The inequality (4.5) follows now from Lemma 3.5.

Next, observe that for finite N,

$$\int_0^N k(\xi,\eta)\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]=\frac{-k(\xi,Re(\eta)+iN)\cos Nt}{t}$$
$$+\frac{1}{t}\int_0^N\frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)],$$

where we have used the fact that $k(\xi,\eta)=0$ if $\mathrm{Im}(\eta)=0$. Integrating again by parts and letting $N\to\infty$ we conclude that

$$\lim_{N\to\infty}\int_0^N k(\xi,\eta)\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]=\frac{-1}{t^2}\int_0^\infty\frac{\partial^2k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2}\ \sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)].$$

Since the estimates on $\partial^2 k(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^2$ are similar to those on $\partial^2 h(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^2$ we can argue as previously to conclude that

$$\int_{[-\pi,\pi]^2} d\xi \lim_{N\to\infty} \left| \int_0^N k(\xi,\eta)\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] \right| \le C_{\lambda,\Lambda}/t, \quad t>0,$$

for some constant $C_{\lambda,\Lambda}$ depending only on λ,Λ. □

We can similarly sharpen the result of Corollary 4.2 when $d=2$.

Lemma 4.2. *For $d=2$, there is a constant $C(\lambda,\Lambda)$ depending only on λ,Λ such that*

$$\left| \frac{\partial G_{\mathbf{a}}(x,t)}{\partial t} \right| \le C(\lambda,\Lambda)/[1+t^2] \ , \quad t>0 \ .$$

Proof. Integrating by parts in (2.42) we have that

$$\frac{\partial}{\partial t}\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] =$$
$$\lim_{N\to\infty} \frac{-1}{t^3}\int_0^N \left\{ 3\ \frac{\partial^2 h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2} + \mathrm{Im}(\eta)\ \frac{\partial^3 h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^3} \right\} \{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)] \ .$$

We can compute $\partial^3 h(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^3$ from (2.44). It is clear we can derive similar estimates on $\mathrm{Im}(\eta)\partial^3 h(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^3$ to the ones on $\partial^2 h(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^2$ which we used in the proof of Lemma 4.1. We conclude therefore that there is a constant $C_{\lambda,\Lambda}$ depending only on λ,Λ such that

$$\int_{[-\pi,\pi]^2} d\xi \left| \frac{\partial}{\partial t}\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] \right| \le C_{\lambda,\Lambda}/t^2 \ , \quad t>1 \ .$$

From (2.48) we have that

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{-1}{t}\int_0^\infty \frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)] \\
&= \frac{1}{t^2}\int_0^\infty \left[\frac{\partial}{\partial[\mathrm{Im}(\eta)]}\left\{ \mathrm{Im}(\eta)\frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]} \right\} + \frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]} \right] \{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)] \\
&= \lim_{N\to\infty}\frac{-1}{t^2}\int_0^N \left[\frac{\partial}{\partial[\mathrm{Im}(\eta)]}\left\{ \mathrm{Im}(\eta)\frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]} \right\} + \frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]} \right] \cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)],
\end{aligned}$$

where we have used the fact that $k(\xi,\eta)=0$ if $\eta>0$ is real. Integrating now by parts we conclude that

$$\frac{\partial}{\partial t}\frac{-1}{t}\int_0^\infty \frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)] =$$
$$\lim_{N\to\infty}\frac{1}{t^3}\int_0^N \left\{ 3\ \frac{\partial^2 k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2} + \mathrm{Im}(\eta)\ \frac{\partial^3 k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^3} \right\} \sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] \ .$$

We can then argue just as for the integral in h that

$$\int_{[-\pi,\pi]^2} d\xi \left| \frac{\partial}{\partial t}\frac{-1}{t}\int_0^\infty \frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)] \right| \le C_{\lambda,\Lambda}/t^2, \quad t>1 \ ,$$

for a constant $C_{\lambda,\Lambda}$ depending only on λ,Λ. □

So far we have not obtained diagonal estimates, even in dimension 1, on spatial derivatives of $G_{\mathbf{a}}(x,t)$, which correspond to the estimates of Theorem 1.4. Next we shall prove these estimates for the case $d=2$. The method readily extends to the case $d=1$.

Lemma 4.3. *For $d=2$ there is a constant $C(\lambda,\Lambda)$ depending only on λ,Λ such that*

$$|\nabla_i G_{\mathbf{a}}(x,t)| \le C(\lambda,\Lambda)/[1+t^{3/2}] \ .$$

Let δ satisfy $0 \le \delta < 1$. Then there is a constant $C(\delta,\lambda,\Lambda)$ depending only on δ,λ,Λ such that

$$|\nabla_i \nabla_j G_{\mathbf{a}}(x,t)| \le C(\delta,\lambda,\Lambda)/[1+t^{(3+\delta)/2}] \ .$$

Proof. If we use the fact that for $0 \le \delta < 1$, $|e_j(-\xi)| \le 2^{1-\delta}|e_j(-\xi)|^{\delta}$, then we see from the proof of Lemma 4.1 that it is sufficient to show that for $t>1$,

(4.7)

$$\int_{|e(\xi)|>1/\sqrt{t}} d\xi |e(-\xi)|^{1+\delta} \left| \frac{1}{t^2} \int_0^{|e(\xi)|^2} \frac{\partial^2 h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2} \{1-\cos[\mathrm{Im}(\eta)t]\} d[\mathrm{Im}(\eta)] \right| \le \frac{C}{t^{(3+\delta)/2}} .$$

This follows by the argument of Lemma 4.1 if we can choose $h_{\eta,p} \in L^p_w$ with $p < 2/(1+\delta)$. We cannot do this since $h_{\eta,p} \in L^p_w$ only for $p>2$. To get around this we can argue similarly to Lemma 3.6 in the proof that $\|\partial^2 q_{k,k'}/\partial\xi_i\partial\xi_j\| \in L^{3/2}_w$. For $\mathrm{Re}(\eta) > 0$ and $2 < p < \infty$, let $g_{\eta,p}(\xi)$, $\xi \in [-\pi,\pi]^2$ be the function

$$g_{\eta,p}(\xi) = |\mathrm{Im}(\eta)|^{1-1/p} \left[\sum_{k,k'=1}^{2} |\langle \psi_k(-\xi,\eta,\cdot)[\partial\psi_{k'}(\xi,\eta,\cdot)/\partial\eta]\rangle| \right]^{1/2} .$$

Then from Section 3 we see that $g_{\eta,p} \in L^p_w([-\pi,\pi]^2)$ and there is a constant $C_{p,\lambda,\Lambda}$ depending only on p,λ,Λ such that

$$\|g_{\eta,p}\|_{p,w} \le C_{p,\lambda,\Lambda}, \quad Re(\eta) > 0, \quad 2 < p < \infty \ .$$

Observe now that the contribution of the last term on the RHS of (2.44) to the integral on the LHS of (4.7) is bounded by a constant times

$$\int_{|e(\xi)|>1/\sqrt{t}} d\xi \frac{1}{t^2} \int_0^{|e(\xi)|^2} \frac{g_{\eta,p}(\xi)^2}{|e(\xi)|^{1-\delta}\mathrm{Im}(\eta)^{2-2/p}} d[\mathrm{Im}(\eta)] \ .$$

Arguing as in Lemma 4.1 we see that this is bounded by the RHS of (4.7) provided $\delta < 1$. Note that as $\delta \to 1$ the estimate diverges. Since we can make similar estimates for the other terms on the RHS of (2.44), the result follows. □

We wish to extend Lemmas 4.1, 4.2, 4.3 to $d \ge 3$. To do this we need the following lemma.

Lemma 4.4. *The functions $h(\xi,\eta), k(\xi,\eta)$, $Re(\eta) > 0$, have the property that*

$$\begin{aligned} \partial^m h(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^m &= 0, \quad m \text{ odd}, \quad \eta > 0 \text{ real}, \\ \partial^m k(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^m &= 0, \quad m \text{ even}, \quad \eta > 0 \text{ real}. \end{aligned}$$

Proof. Note that we can see $\partial h(\xi,\eta)/\partial[\text{Im}(\eta)] = 0$ for real $\eta > 0$ from (2.37) if we use (2.15) and the fact that $\psi(-\xi,\eta,\cdot) = \overline{\psi(\xi,\bar{\eta},\cdot)}$. More generally the result follows from the fact that the function $g(\eta) = h(\xi,\eta) + ik(\xi,\eta)$ is analytic for $\text{Re}(\eta) > 0$ and real when η is real. In fact from the Cauchy-Riemann equations we have that

$$\frac{\partial^m g(\eta)}{\partial\eta^m} = (-i)^m \left[\frac{\partial^m h(\xi,\eta)}{\partial[\text{Im}(\eta)]^m} + i\,\frac{\partial^m k(\xi,\eta)}{\partial[\text{Im}(\eta)]^m}\right], \quad m = 0,1,2,\ldots .$$

Evidently the LHS of this identity is real for all $\eta > 0$ real. □

Lemma 4.5. *For $d \geq 3$ there is a constant $C(\lambda,\Lambda,d)$ depending only on λ,Λ,d such that*

$$0 \leq G_{\mathbf{a}}(x,t) \leq C(\lambda,\Lambda,d)/[1+t^{d/2}], \quad t > 0.$$

Proof. From Lemma 4.4 we see that

$$\int_0^\infty h(\xi,\eta)\cos[\text{Im}(\eta)t]d[\text{Im}(\eta)] \;, \tag{4.8}$$

is, for d odd, one of the integrals,

$$\begin{aligned} &\pm \;\frac{1}{t^{(d+1)/2}}\int_0^\infty \frac{\partial^{(d+1)/2}h(\xi,\eta)}{\partial[\text{Im}(\eta)]^{(d+1)/2}}\sin[\text{Im}(\eta)t]d[\text{Im}(\eta)] \\ &\pm \;\frac{1}{t^{(d+1)/2}}\int_0^\infty \frac{\partial^{(d+1)/2}h(\xi,\eta)}{\partial[\text{Im}(\eta)]^{(d+1)/2}}\{1-\cos[\text{Im}(\eta)t]\}d[\text{Im}(\eta)]. \end{aligned} \tag{4.9}$$

For d even it is one of the integrals,

$$\begin{aligned} &\pm \;\frac{1}{t^{d/2+1}}\int_0^\infty \frac{\partial^{d/2+1}h(\xi,\eta)}{\partial[\text{Im}(\eta)]^{d/2+1}}\sin[\text{Im}(\eta)t]d[\text{Im}(\eta)] \\ &\pm \;\frac{1}{t^{d/2+1}}\int_0^\infty \frac{\partial^{d/2+1}h(\xi,\eta)}{\partial[\text{Im}(\eta)]^{d/2+1}}\{1-\cos[\text{Im}(\eta)t]\}d[\text{Im}(\eta)]. \end{aligned} \tag{4.10}$$

We can see from (2.44) that for $m = 1,2,\ldots$, the derivative $\partial^m h(\xi,\eta)/\partial[\text{Im}(\eta)]^m$ is bounded in absolute value by a sum of terms

$$\prod_{r=0}^{m-1} \frac{\left\langle \left|\frac{\partial^r\psi(\pm\xi,\eta,\cdot)}{\partial\eta^r}\right|^2\right\rangle^{\alpha_r/2}}{|\eta+e(\xi)q(\xi,\eta)e(-\xi)|^{m+1-\sum_{r=1}^{m-1}r\alpha_r}}, \tag{4.11}$$

where the α_r, $r = 0,\ldots,m-1$, are nonnegative integers satisfying the inequality

$$\sum_{r=0}^{m-1}(2r+1)\alpha_r \leq 2m. \tag{4.12}$$

By differentiating (2.46) sufficiently often and using Cauchy-Schwarz we obtain the inequality,

$$\left\langle\left|\frac{\partial^r\psi(\pm\xi,\eta,\cdot)}{\partial\eta^r}\right|^2\right\rangle \leq \frac{(r!)^2}{|\eta|^{2r}}\left\langle|\psi(\pm\xi,\eta,\cdot)|^2\right\rangle, \quad r = 0,1,2,\ldots \quad .$$

Observe next that (4.12) implies that

$$m + 1 - \sum_{r=1}^{m-1} r\alpha_r > \frac{1}{2}\sum_{r=0}^{m-1}\alpha_r.$$

It follows therefore from (2.15) that there is a constant C_m depending only on m such that (4.11) is bounded by $C_m/|\mathrm{Im}(\eta)|^{m+1}$. Hence,

$$\int_{|e(\xi)|^2}^{\infty} \left| \frac{\partial^m h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^m} \right| d[\mathrm{Im}(\eta)] \le \frac{C_m}{|e(\xi)|^{2m}} \ ,$$

for some constant C_m depending only on m.

Let us assume now that d is odd and that the first integral in (4.9) is the correct representation of (4.8). Then, following the argument of Lemma 4.1, we have that

$$\begin{aligned}
&\int_{[-\pi,\pi]^d} \left| \int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] \right| d\xi \\
&\quad = \int_{|e(\xi)|<1/\sqrt{t}} + \int_{|e(\xi)|>1/\sqrt{t}} \\
&\quad \le \frac{C_{d,\lambda,\Lambda}}{t^{d/2}} + \frac{C_d}{t^{(d+1)/2}} \int_{|e(\xi)|>1/\sqrt{t}} |e(\xi)|^{-d-1} d\xi \\
&\quad\quad + \int_{|e(\xi)|>1/\sqrt{t}} d\xi \left| \frac{1}{t^{(d+1)/2}} \int_0^{|e(\xi)|^2} \frac{\partial^{(d+1)/2} h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^{(d+1)/2}} \sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] \right| .
\end{aligned}$$

The first two terms on the RHS of the last inequality are bounded by $C_{d,\lambda,\Lambda}/t^{d/2}$ for some constant $C_{d,\lambda,\Lambda}$ depending only on d, λ, Λ. We are left therefore to deal with the final term.

Consider the simplest case of (4.11) when $\alpha_r = 0$, $r > 0$ and $\alpha_0 \le 2m$. Then (4.11) is bounded by

$$\left\langle |\psi(\pm\xi,\eta,\cdot)|^2 \right\rangle^{\alpha_0/2} / \lambda^{2(m+1)} |e(\xi)|^{2(m+1)}.$$

We shall show that

(4.13)

$$\int_{|e(\xi)|>1/\sqrt{t}} d\xi \left| \frac{1}{t^{(d+1)/2}} \int_0^{|e(\xi)|^2} \frac{\left\langle |\psi(\xi,\eta,\cdot)|^2 \right\rangle^{\alpha_0/2}}{|e(\xi)|^{d+3}} \sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] \right| \le C_{d,\lambda,\Lambda}/t^{d/2},$$

provided $\alpha_0 \le d+1$. To do this we define for $d \le p < \infty$ the function $h_{\eta,p}(\xi)$, $\xi \in [-\pi,\pi]^d$, by

$$h_{\eta,p}(\xi) = |[\mathrm{Im}(\eta)]|^{1/2\,-\,d/2p} \left[\sum_{k=1}^d \left\langle |\psi_k(\xi,\eta,\cdot)|^2 \right\rangle \right]^{1/2} , \tag{4.14}$$

where $\psi_k(\xi,\eta,\cdot)$ is given by (2.2). We can see similarly to the argument of Lemma 4.1 that $h_{\eta,p} \in L^p_w([-\pi,\pi]^d)$ and that there is a constant $C_{p,\lambda,\Lambda,d}$ depending on p, λ, Λ, d such that

$$\|h_{\eta,p}\|_{p,w} \le C_{p,\lambda,\Lambda,d}, \quad Re(\eta) > 0, \ d \le p < \infty \ .$$

We have now that

$$\begin{aligned}
&\int_{|e(\xi)|>1/\sqrt{t}} d\xi \left| \frac{1}{t^{(d+1)/2}} \int_0^{|e(\xi)|^2} \frac{\left\langle |\psi(\xi,\eta,\cdot)|^2 \right\rangle^{\alpha_0/2}}{|e(\xi)|^{d+3}} \sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)] \right| \\
&\quad\quad \le \int_{|e(\xi)|>1/\sqrt{t}} d\xi \ \frac{1}{t^{(d+1)/2}} \int_0^{|e(\xi)|^2} \frac{h_{\eta,p}(\xi)^{\alpha_0}}{|e(\xi)|^{d+3-\alpha_0}} \ \frac{d[\mathrm{Im}(\eta)]}{[\mathrm{Im}(\eta)]^{\alpha_0(1/2\,-\,d/2p)}} \ .
\end{aligned}$$

Since $\alpha_0 \le d+1$ we can find $p > d+1$ such that $\alpha_0(1/2 - d/2p) < 1$. The proof of the inequality (4.13) follows now exactly the same lines as in (4.6).

Next we consider the general case of (4.11). To do this we define for $r = 0,1,2,\dots$, $\mathrm{Re}(\eta) > 0$ and p satisfying $d/(2r+1) \le p < \infty$, functions $h_{\eta,p,r}(\xi)$ by

$$h_{\eta,p,r}(\xi) = |\mathrm{Im}(\eta)|^{r+1/2\ -\ d/2p}\left[\sum_{k=1}^{d}\left\langle\left|\frac{\partial^r\psi_k(\xi,\eta,\cdot)}{\partial\eta^r}\right|^2\right\rangle\right]^{1/2}.$$

Observe that $h_{\eta,p,0}(\xi)$ is the function (4.14). We shall show that for $p > \max[2, \frac{d}{2r+1}]$, the function $h_{\eta,p,r} \in L^p_w([-\pi,\pi]^d)$ and there is a constant $C_{p,r,\lambda,\Lambda,d}$ depending only on $p, r, \lambda, \Lambda, d$ such that

$$\|h_{\eta,p,r}\|_{p,w} \le C_{p,r,\lambda,\Lambda,d}, \quad \mathrm{Re}(\eta) > 0,\ \max[2, d/(2r+1)] < p < \infty. \tag{4.15}$$

To prove this note that $\partial^r\psi_k(\xi,\eta,\cdot)/\partial\eta^r$ is a sum of terms $PT_{n,\varphi,\mathbf{b},\eta}\,(A_1,\dots,A_n)(\xi,\cdot)$, where $T_{n,\varphi,\mathbf{b},\eta}$ is the operator (3.33) and φ is an entry of the matrix $\mathbf{a}(\cdot)$. Furthermore, there are positive integers $r_1,\dots,r_n$ such that $r_1+\cdots+r_n = r+1$ and

$$\begin{aligned}\|A_1(\xi)\| &\le |e(\xi)|\Big/[|e(\xi)|^2 + |\mathrm{Im}(\eta)|]^{r_1},\\ \|A_j(\xi)\| &\le |e(\xi)|^2\Big/[|e(\xi)|^2 + |\mathrm{Im}(\eta)|]^{r_j+1},\ 2 \le j \le n.\end{aligned}$$

Suppose now $p_1,\dots,p_n$ are positive numbers satisfying

$$p_1 \ge \frac{d}{2r_1 - 1}, \quad p_j \ge \frac{d}{2r_j}, \quad 2 \le j \le n. \tag{4.16}$$

If also $p_j > 1,\ j = 1,\dots,n$, then we see that $A_j \in L^{p_j}_w([-\pi,\pi]^d)$, $1 \le j \le n$, and

$$\begin{aligned}\|A_1\|_{p_1,w} &\le C_{d,r}/|\mathrm{Im}(\eta)|^{r_1-1/2-d/2p_1},\\ \|A_j\|_{p_j,w} &\le C_{d,r}/|\mathrm{Im}(\eta)|^{r_j-d/2p_j},\ 2 \le j \le n,\end{aligned}$$

where $C_{d,r}$ is a constant depending only on d,r. Note now that since

$$\frac{2r_1-1}{d} + \sum_{j=2}^{n}\frac{2r_j}{d} = \frac{2r+1}{d},$$

if p satisfies $p > \max[2,\ d/(2r+1)]$, it is possible to choose $p_1,\dots,p_n$ satisfying $p_j > 2$, $1 \le j \le n$, the inequalities (4.16) and the identity

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{p}.$$

The inequality (4.15) follows now from Lemma 3.7.

For the general case of (4.11) we need to show that

$$\int_{|e(\xi)|>1/\sqrt{t}} d\xi \frac{1}{t^{(d+1)/2}}\int_0^{|e(\xi)|^2} \frac{\prod_{r=0}^{(d-1)/2} h_{\eta,q_r,r}(\xi)^{\alpha_r}}{|e(\xi)|^{d+3-\sum_{r=0}^{(d-1)/2}(2r+1)\alpha_r}} \frac{d[\mathrm{Im}(\eta)]}{[\mathrm{Im}(\eta)]^{\sum_{r=0}^{(d-1)/2}(r+1/2-d/2q_r)\alpha_r}} \le \frac{C_{d,\lambda,\Lambda}}{t^{d/2}}, \tag{4.17}$$

where the $q_r,\ r = 0,\dots,(d-1)/2$ are restricted to satisfy

$$q_r > \max[2,\ d/(2r+1)],\ r = 0,\dots,(d-1)/2. \tag{4.18}$$

Define now q by

$$\frac{1}{q} = \sum_{r=0}^{(d-1)/2} \frac{\alpha_r}{q_r}. \tag{4.19}$$

It follows then from (4.12) that

$$\sum_{r=0}^{(d-1)/2} \left(r + \frac{1}{2} - \frac{d}{2q_r}\right) \alpha_r = \frac{1}{2}\left[d' + 1 - \frac{d}{q}\right] ,$$

where d' is an integer satisfying $-1 \le d' \le d$. Observe that the RHS of the last inequality is strictly less than 1 if $d' \le 1$ or $d' > 1,\ q < d/(d'-1)$. Suppose now $q_r,\ r = 0, \dots, (d-1)/2$ and q satisfy (4.18), (4.19), with $q > 1$. Then it follows from (4.15) that

$$\prod_{r=0}^{(d-1)/2} h_{\eta, q_r, r}(\xi)^{\alpha_r} \in L^q_w([-\pi,\pi]^d) ,$$

with norm bounded by a constant depending only on $\lambda, \Lambda, d\, q, q_r, r = 0, \dots (d-1)/2$. If we can also arrange that for $d' > 1$, q satisfies $q < d/(d'-1)$ then we can see that (4.17) holds by arguing just as we did in Lemma 4.1. Evidently it is possible to choose the $q_r,\ r = 0, \dots, (d-1)/2$, such that $q_r > d/(2r+1)$ and $1 < q < d/(d'-1)$. It is not possible, however, to satisfy the condition $q_r > 2$, $r = 0, \dots, (d-1)/2$ in general.

To deal with this problem we need to use a sharper estimate on the derivative $\partial^m h(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^m$ than sums of terms of the form (4.11). For r, j satisfying $0 \le j \le r,\ \ 0 \le r \le m-1$, let $\alpha_{r,j}$ be nonnegative integers satisfying the inequality

$$\sum_{r=0}^{m-1} \sum_{j=0}^{r} (1 + r + j)\alpha_{r,j} \le m. \tag{4.20}$$

If we define α_r for $0 \le r \le m-1$, by

$$\alpha_r = \sum_{j=0}^{r} \alpha_{r,j} + \sum_{j=r}^{m-1} \alpha_{j,r} , \tag{4.21}$$

then we see from (4.20) that α_r defined by (4.21) satisfies (4.12). It is also easy to see that $\partial^m h(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^m$ is bounded in absolute value by a sum of terms,

$$\frac{\prod_{r=0}^{m-1} \prod_{j=0}^{r} \left| \left\langle \frac{\partial^r \psi(\pm\xi,\eta,\cdot)}{\partial \eta^r} \ \frac{\partial^j \psi(\pm\xi,\eta,\cdot)}{\partial \eta^j} \right\rangle \right|^{\alpha_{r,j}}}{|\eta + e(\xi)\ q(\xi,\eta) e(-\xi)|^{m+1-\sum_{r=1}^{m-1} r\alpha_r}} , \tag{4.22}$$

where the $\alpha_{r,j}$ satisfy (4.20) and the α_r are defined by (4.21). Evidently the Schwarz inequality implies that (4.22) is bounded by (4.11). We define for $r, j = 0, 1, 2, \dots$, $\mathrm{Re}(\eta) > 0$ and p satisfying $d/(r+j+1) \le p < \infty$, functions $h_{\eta,p,r,j}(\xi)$ by

$$h_{\eta,p,r,j}(\xi) = |\mathrm{Im}(\eta)|^{(r+j+1-d/p)/2} \left[\sum_{k,k'=1}^{d} \left| \left\langle \frac{\partial^r \psi_k(\xi,\eta,\cdot)}{\partial \eta^r} \frac{\partial^j \psi_{k'}(\xi,\eta,\cdot)}{\partial \eta^j} \right\rangle \right| \right]^{1/2} .$$

We shall show that for $p > \max[2, d/(r+j+1)]$, the function $h_{\eta,p,r,j} \in L^p_w([-\pi,\pi]^d)$ and there is a constant $C_{p,r,j,\lambda,\Lambda,d}$ depending only on $p, r, j, \lambda, \Lambda, d$ such that

$$(4.23) \qquad \|h_{\eta,p,r,j}\|_{p,w} \le C_{p,r,j,\lambda,\Lambda,d}, \quad \mathrm{Re}(\eta) > 0, \ \max[2, d/(r+j+1)] < p < \infty.$$

In fact it is easy to see that (4.23) follows by arguing exactly as we did in Lemma 3.9 for the case $d/2 \le |\alpha| < d$.

On replacing (4.11) by (4.22) the inequality (4.17) gets replaced by

$$(4.24) \qquad \int_{|e(\xi)|>1/\sqrt{t}} d\xi \frac{1}{t^{(d+1)/2}} \int_0^{|e(\xi)|^2} \frac{\prod_{r=0}^{(d-1)/2}\prod_{j=0}^{r} h_{\eta,q_{r,j},r,j}(\xi)^{2\alpha_{r,j}}}{|e(\xi)|^{d+3-\sum_{r=0}^{(d-1)/2}(2r+1)\alpha_r}} \frac{d[\mathrm{Im}(\eta)]}{[\mathrm{Im}(\eta)]^{\sum_{r=0}^{(d-1)/2}\sum_{j=0}^{r}(r+j+1-d/q_{r,j})\alpha_{r,j}}} \le \frac{C_{d,\lambda,\Lambda}}{t^{d/2}} ,$$

where the $\alpha_{r,j}, \alpha_r$ satisfy (4.20), (4.21). Let q be defined

$$(4.25) \qquad \frac{1}{q} = \sum_{r=0}^{(d-1)/2} \sum_{j=0}^{r} \frac{2\alpha_{r,j}}{q_{r,j}}.$$

Then from (4.25) it follows that if $q_{r,j} > \max[2, \ d/(r+j+1)], \ 0 \le j \le r, \ 0 \le r \le (d-1)/2$, and $q > 1$ then the function

$$\prod_{r=0}^{(d-1)/2} \prod_{j=0}^{r} h_{\eta,q_{r,j},r,j}(\xi)^{2\alpha_{r,j}} \in L^q_w([-\pi,\pi]^d) \ ,$$

with norm bounded by a constant depending only on λ, Λ, d and the $q_{r,j}$. We have now from (4.20), (4.25) that

$$\sum_{r=0}^{(d-1)/2} \sum_{j=0}^{r} (r+j+1-d/q_{r,j})\alpha_{r,j} = \frac{(d'+1)}{2} - \frac{d}{2q} \ ,$$

where d' is an integer satisfying $-1 \le d' \le d$. As before, the RHS of the last inequality is strictly less than 1 if $d' \le 1$ or $d' > 1, \ q < d/(d'-1)$. Hence if $q < d/(d'-1)$ the power of $\mathrm{Im}(\eta)$ on the LHS of (4.24) is strictly less than 1 and hence integrable. Finally, observe that in (4.22) one has $\alpha_{r,j} = 0$ unless $r + j \le m - 1 = (d-1)/2$. Note that if $r + j < (d-1)/2$ then $d/(r+j+1) > 2$. On the other hand if $\alpha_{r,j} \ne 0$ for some (r,j) with $r + j = m - 1$ then $\alpha_{r,j} = 1$ and $\alpha_{r',j'} = 0$ for $(r',j') \ne (r,j)$. In that case $d' = d$ and (4.25) becomes $1/q = 2/q_{r,j}$, whence the condition $q < d/(d-1)$ becomes $q_{r,j} < 2d/(d-1)$, so we may still choose $q_{r,j} > 2$. Thus (4.24) holds on appropriate choice of the $q_{r,j}$.

The proof of the lemma for d odd is complete if we make the observation that from Lemma 4.4 one has

$$\frac{1}{t}\int_0^\infty \frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)] =$$
$$\pm \frac{1}{t^{(d+1)/2}}\int_0^\infty \frac{\partial^{(d+1)/2}k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^{(d+1)/2}} \sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]$$
$$\pm \frac{1}{t^{(d+1)/2}}\int_0^\infty \frac{\partial^{(d+1)/2}k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^{(d+1)/2}} \{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)],$$

depending on the value of d. Since the estimates on $\partial^m k(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^m$ are the same as those on $\partial^m h(\xi,\eta)/\partial[\mathrm{Im}(\eta)]^m$ the argument proceeds as before. The case for d even is similar. $\Box$

Lemma 4.6. *For $d \ge 3$ there is a constant $C(\lambda,\Lambda,d)$ depending only on λ,Λ,d such that*

$$\left|\frac{\partial G_{\mathbf{a}}(x,t)}{\partial t}\right| \le C(\lambda,\Lambda,d)/[1+t^{1+d/2}], \quad t>0.$$

Proof. Suppose d is odd and the first integral in (4.9) is the appropriate representation for this value of d. Then we have that

$$\frac{\partial}{\partial t}\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]$$

is a sum of the terms,

$$\pm \quad \frac{(d+1)}{2t^{(d+3)/2}}\int_0^\infty \frac{\partial^{(d+1)/2}h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^{(d+1)/2}}\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)],$$
$$\pm \quad \frac{1}{t^{(d+1)/2}}\int_0^\infty \frac{\partial^{(d+1)/2}h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^{(d+1)/2}}\ \mathrm{Im}(\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)].$$

Evidently the first term is bounded by $C(\lambda,\Lambda,d)/[1+t^{1+d/2}]$ by Lemma 4.5. On integration by parts we can write the second term as a sum of two integrals,

$$\pm \quad \frac{1}{t^{(d+3)/2}}\int_0^\infty \frac{\partial^{(d+1)/2}h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^{(d+1)/2}}\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)],$$
$$\pm \quad \frac{1}{t^{(d+3)/2}}\int_0^\infty \frac{\partial^{(d+3)/2}h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^{(d+3)/2}}\ \mathrm{Im}(\eta)\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)].$$

Again Lemma 4.5 implies that the first integral is bounded by $C(\lambda,\Lambda,d)/[1+t^{1+d/2}]$, so we are left to deal with the second integral. Using the method of Lemma 4.1 we see that we are left to deal with

$$\int_{|e(\xi)|>1/\sqrt{t}} d\xi\ \left|\frac{1}{t^{(d+3)/2}}\int_0^{|e(\xi)|^2} \frac{\partial^{(d+3)/2}h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^{(d+3)/2}}\ \mathrm{Im}(\eta)\sin[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]\right|.$$

Arguing exactly as in Lemma 4.5 we see this integral is bounded by $C(\lambda,\Lambda,d)/[1+t^{1+d/2}]$. We can similarly bound the corresponding integral in $k(\xi,\eta)$ and hence the result follows. $\Box$

Lemma 4.7. *For $d \ge 3$ there is a constant $C(\lambda,\Lambda,d)$ depending only on λ,Λ,d such that*

$$|\nabla_i G_{\mathbf{a}}(x,t)| \le C(\lambda,\Lambda,d)/[1+t^{(d+1)/2}]\ .$$

Let δ satisfy $0\le\delta<1$. Then there is a constant $C(\delta,\lambda,\Lambda,d)$ depending only on δ,λ,Λ,d such that

$$|\nabla_i\nabla_j G_{\mathbf{a}}(x,t)| \le C(\delta,\lambda,\Lambda,d)/[1+t^{(d+1+\delta)/2}]\ .$$

Proof. Same as for Lemma 4.6. $\Box$

5. Proof of Theorem 1.4—Off Diagonal case

Here we shall complete the proof of Theorem 1.4. Thus we need to establish the exponential falloff in (1.10) and in the inequalities of Theorem 1.4. Evidently (1.10) implies that the periodic function $\hat{G}_a(\xi,t),\ \xi\in\mathbb{R}^d$, can be analytically continued to $\mathbb{C}^d$. Our goal will be to establish analyticity properties of $\hat{G}_a(\xi,t)$ and derive from them the inequality (1.10) and Theorem 1.4.

Lemma 5.1. *Suppose ε satisfies $0<\varepsilon<1$. Then the periodic function $\hat{G}_a(\xi,t),\ \xi\in\mathbb{R}^d$, can be analytically continued to the strip $\{\xi\in\mathbb{C}^d : |\mathrm{Im}(\xi)|<\varepsilon\}$. There are constants $C_1(\lambda,\Lambda,d),\ C_2(\lambda,\Lambda,d)$ depending only on λ,Λ,d such that*

$$|\hat{G}_a(\xi,t)|\le C_1(\lambda,\Lambda,d)\exp[C_2(\lambda,\Lambda,d)\varepsilon^2 t],\ |\mathrm{Im}(\xi)|<\varepsilon,\ t>0\ .$$

Proof. We have already seen in Section 2 that the matrix $q(\xi,\eta)$ of (2.3) is defined for all $\xi\in\mathbb{R}^d,\ \mathrm{Re}(\eta)>0$, is continuous in (ξ,η) and analytic in η for fixed ξ. We shall show now that for any $\delta>0$ there exists a constant $C(\lambda,\Lambda,d,\delta)>0$ depending only on λ,Λ,d,δ such that if $\mathrm{Re}(\eta)>C(\lambda,\Lambda,d,\delta)\varepsilon^2$ then $q(\xi,\eta),\ \xi\in\mathbb{R}^d$, can be analytically continued to the strip $\{\xi\in\mathbb{C}^d : |\mathrm{Im}(\xi)|<\varepsilon\}$ and

$$\|q(\xi,\eta)-q(\mathrm{Re}(\xi),\eta)\|<\delta,\ |\mathrm{Im}(\xi)|<\varepsilon,\ \mathrm{Re}(\eta)>C(\lambda,\Lambda,d,\delta)\varepsilon^2\ . \tag{5.1}$$

In view of (3.6), (3.7) this will follow if we can show that for any $\delta>0$ there exists a constant $C(d,\delta)>0$ depending only on d,δ, such that if $\mathrm{Re}(\eta)>C(d,\delta)\varepsilon^2$, then the operator $T_{k,k',\eta,\xi}$ of (3.4), which is bounded on $L^2(\Omega)$ for $\xi\in\mathbb{R}^d$, extends analytically to a bounded operator on $L^2(\Omega)$ for $|\mathrm{Im}(\xi)|<\varepsilon$ and

$$\|T_{k,k',\eta,\xi}-T_{k,k',\eta,\mathrm{Re}(\xi)}\|<\delta,\quad |\mathrm{Im}(\xi)|<\varepsilon,\quad \mathrm{Re}(\eta)>C(d,\delta)\varepsilon^2. \tag{5.2}$$

To prove (5.2) observe that the Green's function $G_\eta(x)$ satisfies an inequality

$$|\nabla_k^*\nabla_{k'}G_\eta(x)|\le\frac{C_d\exp\big[-g(\mathrm{Re}(\eta))|x|\big]}{[1+|x|]^d}\ ,\qquad x\in\mathbb{Z}^d,$$

where $g(z),\ z>0$, is the function

$$g(z)=\begin{cases} c_d\sqrt{z}, & 0<z<1,\\[2ex] c_d\log(1+z), & z\ge 1.\end{cases}$$

Here C_d,c_d are positive constants depending only on d. It follows in particular that there is a constant $C_1(d)$, depending only on d, such that if $0<\varepsilon<1$ and $\mathrm{Re}(\eta)>C_1(d)\varepsilon^2$, then the function $\nabla_k^*\nabla_{k'}G_\eta(x)e^{ix\cdot\xi}$ decreases exponentially in x as $|x|\to\infty$, provided $|\mathrm{Im}(\xi)|<\varepsilon$. It follows from (3.4) that if $\mathrm{Re}(\eta)>C_1(d)\varepsilon^2$ then the bounded operators $T_{k,k',\eta,\xi}$ on $L^2(\Omega),\ \xi\in\mathbb{R}^d$, extend analytically to bounded operators on $L^2(\Omega)$ provided $\xi\in\mathbb{C}^d$ satisfies $|\mathrm{Im}(\xi)|<\varepsilon$.

To prove (5.2) we use Bochner's Theorem [9]. Thus for any $\varphi\in L^2(\Omega)$ there is a positive finite measure $d\mu_\varphi$ on $[-\pi,\pi]^d$ such that

$$\Big\langle\varphi(\tau_x\cdot)\overline{\varphi(\tau_y\cdot)}\Big\rangle=\int_{[-\pi,\pi]^d}e^{i(x-y)\cdot\zeta}d\mu_\varphi(\zeta).$$

Hence,

$$\|T_{k,k',\eta,\xi}\varphi - T_{k,k',\eta,\mathrm{Re}(\xi)}\varphi\|^2 = \int_{[-\pi,\pi]^d} d\mu_\varphi(\zeta)$$

$$\Big|\sum_{x\in\mathbb{Z}^d} \nabla_k^*\nabla_{k'} G_\eta(x)\Big[\exp\{ix\cdot(\zeta+\xi)\} - \exp\{ix\cdot(\zeta+Re(\xi))\}\Big]\Big|^2$$

$$= \int_{[-\pi,\pi]^d} d\mu_\varphi(\zeta)\Big|e_k(\zeta+\xi)e_{k'}(-\zeta-\xi)\hat{G}_\eta(\zeta+\xi)$$

$$- e_k(\zeta+\mathrm{Re}(\xi))e_{k'}(-\zeta-\mathrm{Re}(\xi))\hat{G}_\eta(\zeta+\mathrm{Re}(\xi))\Big|^2 .$$

We have now that

$$e_k(\zeta+\xi)e_{k'}(-\zeta-\xi)\hat{G}_\eta(\zeta+\xi) = \frac{e_k(\zeta+\xi)e_{k'}(-\zeta-\xi)}{\sum_{j=1}^d e_j(\zeta+\xi)e_j(-\zeta-\xi)+\eta} .$$

Observe that

$$|e_k(\zeta+\xi) - e_k(\zeta+\mathrm{Re}(\xi))| \le e^\varepsilon - 1 < 2\varepsilon,\ \zeta\in[-\pi,\pi]^d,\ |\mathrm{Im}\xi| < \varepsilon < 1.$$

Hence if $C_1(d) > 24d$ and $\mathrm{Re}(\eta) > C_1(d)\varepsilon^2$ then

$$\text{(5.3)}\qquad \Big|\sum_{j=1}^d e_j(\zeta+\xi)e_j(-\zeta-\xi)+\eta\Big| \ge \frac{1}{2}\Big|\sum_{j=1}^d e_j(\zeta+\mathrm{Re}(\xi))e_j(-\zeta-\mathrm{Re}(\xi))+\eta\Big|$$

$$\ge \frac{1}{2}C_1(d)\varepsilon^2,\ \zeta\in[-\pi,\pi]^d,\quad |\mathrm{Im}\xi|<\varepsilon.$$

Similarly we see that

$$\Big|e_k(\zeta+\xi)e_{k'}(-\zeta-\xi) - e_k(\zeta+\mathrm{Re}(\xi))e_{k'}(-\zeta-\mathrm{Re}(\xi))\Big|$$

$$\le 1000\varepsilon^2 + 100\varepsilon\Big[\sum_{j=1}^d e_j(\zeta+\mathrm{Re}(\xi))e_j(-\zeta-\mathrm{Re}(\xi))\Big]^{1/2},\ \zeta\in[-\pi,\pi]^d,\ |\mathrm{Im}\xi|<\varepsilon.$$

We conclude therefore that the integrand in the $d\mu_\varphi$ integral is bounded as

$$\Big|e_k(\zeta+\xi)e_{k'}(-\zeta-\xi)\hat{G}_\eta(\zeta+\xi) - e_k(\zeta+\mathrm{Re}(\xi))e_{k'}(-\zeta-\mathrm{Re}(\xi))\hat{G}_\eta(\zeta+\mathrm{Re}(\xi))\Big|$$

$$\le \frac{1000\varepsilon^2 + 100\varepsilon\left[\sum_{j=1}^d e_j(\zeta+\mathrm{Re}(\xi))e_j(-\zeta-\mathrm{Re}(\xi))\right]^{1/2}}{|\sum_{j=1}^d e_j(\zeta+\xi)e_j(-\zeta-\xi)+\eta|}$$

$$+ \frac{|e_k(\zeta+\mathrm{Re}(\xi))||e_{k'}(-\zeta-\mathrm{Re}(\xi))|\left[4d\varepsilon^2 + 2\varepsilon\sqrt{d}\left\{\sum_{j=1}^d |e_j(\zeta+\mathrm{Re}(\xi))|^2\right\}^{1/2}\right]}{|\sum_{j=1}^d e_j(\zeta+\xi)e_j(-\zeta-\xi)+\eta||\sum_{j=1}^d e_j(\zeta+\mathrm{Re}(\xi))e_j(-\zeta-\mathrm{Re}(\xi))+\eta|}.$$

We can see from (5.3) that the expression on the RHS of the last inequality is bounded by

$$\frac{2000}{C_1(d)} + \frac{200}{\sqrt{C_1(d)}} + \frac{4d}{C_1(d)} + \frac{2\sqrt{d}}{\sqrt{C_1(d)}} .$$

Evidently this last expression can be made smaller than δ by choosing $C_1(d)$ sufficiently large, whence (5.2) follows.

We have shown that (5.1) holds. Next consider the function $h(\xi,\eta)$ defined by (2.18) for $\xi \in \mathbb{R}^d$, $\mathrm{Re}(\eta) > 0$. Furthermore, (2.19) holds. We show now that there is a constant $C_{\lambda,\Lambda,d}$ depending only on λ, Λ, d such that if $\mathrm{Re}(\eta) > C_{\lambda,\Lambda,d}\varepsilon^2$ then $h(\xi,\eta)$ may be analytically continued to the strip $\{\xi \in \mathbb{C}^d : |\mathrm{Im}(\xi)| < \varepsilon\}$ and

$$\int_0^\infty |h(\xi,\eta)| d[\mathrm{Im}(\eta)] \le C_{\lambda,\Lambda,d} \ , \quad |\mathrm{Im}(\xi)| < \varepsilon \ . \tag{5.4}$$

To do this we need to rewrite the identities (2.14), (2.15) in such a way that they extend analytically in ξ from $\xi \in \mathbb{R}^d$ to the strip $|\mathrm{Im}(\xi)| < \varepsilon$. First consider (2.14). We define a function $A_j(\xi,\eta,\omega)$ for $\xi \in \mathbb{R}^d, \ \mathrm{Re}(\eta) > 0, \ \omega \in \Omega$, by

$$A_j(\xi,\eta,\cdot) = e_j(-\xi) + e^{-i\mathbf{e}_j\cdot\xi}[\partial_j + e_j(\xi)]\frac{1}{2}\{\psi(\xi,\eta,\cdot) + \psi(\xi,\bar{\eta},\cdot)\} \ ,$$

where $\psi(\xi,\eta,\cdot)$ is defined just before (2.26). It is easy to see that the complex conjugate $\overline{A_j(\xi,\eta,\cdot)} = A_j(-\xi,\eta,\cdot)$. We conclude from this and (2.14), (2.26) that

$$\begin{aligned} \mathrm{Re}[e(\xi)q(\xi,\eta)e(-\xi)] &= \left\langle \sum_{i,j=1}^d a_{i,j}(\cdot)A_i(-\xi,\eta,\cdot)A_j(\xi,\eta,\cdot)\right\rangle \\ &+ \ \frac{\mathrm{Re}(\eta)}{4}\left\langle[\psi(\xi,\eta,\cdot)+\psi(\xi,\bar{\eta},\cdot)]\,[\psi(-\xi,\eta,\cdot)+\psi(-\xi,\bar{\eta},\cdot)]\right\rangle \\ &+ \ \frac{1}{4}\Big\langle [\psi(-\xi,\bar{\eta}.\cdot)-\psi(-\xi,\eta,\cdot)]\,[\mathcal{L}_\xi + \ \mathrm{Re}(\eta)]\,[\psi(\xi,\eta,\cdot)-\psi(\xi,\bar{\eta},\cdot)]\Big\rangle. \end{aligned} \tag{5.5}$$

Similarly we have from (2.15) that

$$\begin{aligned} \mathrm{Im}[e(\xi)q(\xi,\eta)e(-\xi)] &= \frac{1}{2}\ \mathrm{Im}(\eta)\left\langle\psi(\xi,\eta,\cdot)\psi(-\xi,\bar{\eta},\cdot)\right\rangle \\ &+ \frac{1}{2}\ \mathrm{Im}(\eta)\left\langle\psi(-\xi,\eta,\cdot)\psi(\xi,\bar{\eta},\cdot)\right\rangle . \end{aligned} \tag{5.6}$$

We write now $h(\xi,\eta)$ as

$$h(\xi,\eta) = \frac{\mathrm{Re}(\eta) + \ \mathrm{Re}[e(\xi)q(\xi,\eta)e(-\xi)]}{[\mathrm{Re}(\eta) + \ \mathrm{Re}[e(\xi)q(\xi,\eta)e(-\xi)]]^2 + [\mathrm{Im}(\eta) + \ \mathrm{Im}[e(\xi)q(\xi,\eta)e(-\xi)]]^2} \ , \tag{5.7}$$

and use the expressions (5.5), (5.6) to analytically continue $h(\xi,\eta), \ \xi \in \mathbb{R}^d$, to complex $\xi \in \mathbb{C}^d$. Note now that it follows from our proof of (5.1) that for any $\delta > 0$ there exists a constant $C(\lambda,\Lambda,d,\delta) > 0$ depending only on λ,Λ,d,δ such that if $\mathrm{Re}(\eta) > C(\lambda,\Lambda,d,\delta)\varepsilon^2$ then the function $\psi_k(\xi,\eta,\cdot)$ of (2.2) from $\mathbb{R}^d$ to $L^2(\Omega)$ can be analytically continued to the strip $\{\xi \in \mathbb{R}^d : |\mathrm{Im}(\xi)| < \varepsilon\}$ and

$$\begin{gathered} \left\langle\left|e^{-i\mathbf{e}_j\cdot\xi}[\partial_j + e_j(\xi)]\psi_k(\xi,\eta,\cdot) - e^{-i\mathbf{e}_j\cdot\mathrm{Re}(\xi)}[\partial_j + e_j(\mathrm{Re}(\xi))]\psi_k(\mathrm{Re}(\xi),\eta,\cdot)\right|^2\right\rangle \le \delta, \\ |\eta|\left\langle|\psi_k(\xi,\eta,\cdot) - \psi_k(\mathrm{Re}(\xi),\eta,\cdot)|^2\right\rangle \le \delta, \\ 1 \le j,k \le d, \ |\mathrm{Im}(\xi)| < \varepsilon, \ \mathrm{Re}(\eta) > C(\lambda,\Lambda,d,\delta)\varepsilon^2. \end{gathered} \tag{5.8}$$

It is easy to see from this and (5.5), (5.6), (5.7) that there is a constant $C(\lambda,\Lambda,d)$ depending only on λ,Λ,d such that

$$(5.9)\qquad |h(\xi,\eta)| \le 2\min\left[\frac{1}{\mathrm{Re}(\eta)+\lambda|e(\mathrm{Re}(\xi))|^2}\ ,\ \frac{\mathrm{Re}(\eta)+4\Lambda|e(\mathrm{Re}(\xi))|^2}{[\mathrm{Im}(\eta)]^2}\right],$$
$$|\mathrm{Im}(\xi)|<\varepsilon,\ \mathrm{Re}(\eta)>C(\lambda,\Lambda,d,\delta)\varepsilon^2.$$

It is clear now from this last inequality that $h(\xi,\eta)$ as defined by (5.7), (5.6), (5.5) can be analytically continued to $|\mathrm{Im}(\xi)|<\varepsilon$ and that (5.4) holds.

Next consider the function $k(\xi,\eta)$ defined by (2.18). We shall show that there is a constant $C_{\lambda,\Lambda,d}$ depending only on λ,Λ,d such that the function $\partial k(\xi,\eta)/\partial[\mathrm{Im}(\xi)]$ can be analytically continued to the strip $|\mathrm{Im}(\xi)|<\varepsilon$, provided $\mathrm{Re}(\eta)>C_{\lambda,\Lambda,d}\varepsilon^2$. Furthermore there is the inequality

$$(5.10)\qquad |\partial k(\xi,\eta)/\partial[\mathrm{Im}(\eta)]| \le \frac{2}{|\mathrm{Im}(\eta)|^2},\ |\mathrm{Im}(\xi)|<\varepsilon,\ \mathrm{Re}(\eta)>C_{\lambda,\Lambda,d}\varepsilon^2.$$

To see this we use the identity,

$$\frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]} = \mathrm{Re}\left[\frac{1+<\psi(-\xi,\eta,\cdot)\psi(\xi,\eta,\cdot)>}{[\eta+e(\xi)q(\xi,\eta)e(-\xi)]^2}\right]\ ,\quad \xi\in\mathbb{R}^d\ .$$

The inequality (5.10) follows now from the proof of (2.27) and the inequalities (5.8).

The proof of the lemma is completed by using the representation (2.17) with $\mathrm{Re}(\eta)=C_{\lambda,\Lambda,d}\varepsilon^2$ and the inequalities (5.4), (5.10). □

Corollary 5.1. *There are constants $C_1(d,\lambda,\Lambda)$ and $C_2(d,\lambda,\Lambda)>0$ depending only on d,λ,Λ such that*

$$0\le G_{\mathbf{a}}(x,t)\le C_1(d,\lambda,\Lambda)\exp\left[-C_2(d,\lambda,\Lambda)\min\{|x|,\ |x|^2/t\}\right]\ ,\quad x\in\mathbb{Z}^d,\quad t>0.$$

Proof. Follows from Lemma 5.1 on writing

$$G_{\mathbf{a}}(x,t)=\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\hat{G}_a(\xi,t)e^{-i\xi\cdot x}d[Re(\xi)]\ ,$$

and deforming the contour of integration to $\xi\in\mathbb{C}^d$ with $|\mathrm{Im}(\xi)|=\min[1,|x|/2C_2t]$, where $C_2=C_2(\lambda,\Lambda,d)$ is the constant in the statement of Lemma 5.1. □

Next we extend Lemma 2.3 to $\xi\in\mathbb{C}^d$.

Lemma 5.2. *Let ε satisfy $0\le\varepsilon<1$ and $\hat{G}_a(\xi,t)$, $\xi\in\mathbb{C}^d$, $|\mathrm{Im}(\xi)|<\varepsilon$ the function of Lemma 5.1. Then for any δ, $0<\delta<1$, there is a constant $C_1(\lambda,\Lambda,d,\delta)$ depending only λ,Λ,d,δ and a constant $C_2(\lambda,\Lambda,d)$ depending only on λ,Λ,d such that*

$$|\hat{G}_a(\xi,t)|\le\frac{C_1(\lambda,\Lambda,d,\delta)}{[1+|e(\mathrm{Re}(\xi))|^2t]^\delta},\ \exp[C_2(\lambda,\Lambda,d)\varepsilon^2t],\ |\mathrm{Im}(\xi)|<\varepsilon,\quad t>0.$$

Proof. Observe that one may choose $C_{\lambda,\Lambda,d}$ sufficiently large, depending only on λ,Λ,d such that both (5.10) holds and the inequality

$$(5.11)\quad |\partial k(\xi,\eta)/\partial[\mathrm{Im}(\eta)]|\le\frac{2(1+\Lambda)}{\lambda^2|e(\mathrm{Re}(\xi))|^2|\eta|}\ ,\quad |\mathrm{Im}(\xi)|<\varepsilon,\ \mathrm{Re}(\eta)>C_{\lambda,\Lambda,d}\varepsilon^2\ .$$

The inequality (5.11) is analogous to (2.30) and is proved using (5.8), We conclude then from (5.10), (5.8) just as we did in Lemma 2.3 that the LHS of (2.28) is bounded by $C_1(\lambda,\Lambda,d,\delta)/[1+|e(\mathrm{Re}(\xi))|^2t]^\delta$ if $|\mathrm{Im}(\xi)|<\varepsilon$ provided $\mathrm{Re}(\eta)>C_{\lambda,\Lambda,d}\varepsilon^2$. Since we can do similar estimates on $\partial h(\xi,\eta)/\partial[\mathrm{Im}(\eta)]$ the result follows just as in Lemma 2.3. □

Corollary 5.2. *The function $G_{\mathbf{a}}(x,t)$ satisfies the inequality*

$$0\le G_{\mathbf{a}}(x,t)\le\frac{C_1(\lambda,\Lambda)}{(1+\sqrt{t})}\exp\left[-C_2(\lambda,\Lambda)\min\{|x|,\ |x|^2/t\}\right]\ ,\quad\text{if}\quad d=1,$$

where the constants $C_1(\lambda,\Lambda)$ and $C_2(\lambda,\Lambda)$ depend only on λ,Λ. For $d>1$ it satisfies an inequality,

$$0\le G_{\mathbf{a}}(x,t)\le\frac{C_1(\lambda,\Lambda,d,\delta)}{(1+t^\delta)}\exp\left[-C_2(\lambda,\Lambda,d)\min\{|x|,\ |x|^2/t\}\right]\ ,$$

for any δ, $0\le\delta<1$. The constant $C_1(\lambda,\Lambda,d,\delta)$ depends only on λ,Λ,d,δ and $C_2(\lambda,\Lambda,d)$ only on λ,Λ,d.

Proof. Same as for Corollary 4.1 on using the method of proof of Corollary 5.1 and Lemma 5.2. □

Next we generalize Lemma 2.4.

Lemma 5.3. *Let ε satisfy $0<\varepsilon<1$ and $\hat{G}_a(\xi,t)$, $\xi\in\mathbb{C}^d$, $|\mathrm{Im}(\xi)|<\varepsilon$, be the function of Lemma 5.1. Then $\hat{G}_a(\xi,t)$ is differentiable for $t>0$. For any δ, $0\le\delta<1$, there is a constant $C_1(\lambda,\Lambda,d,\delta)$ depending only λ,Λ,d,δ and a constant $C_2(\lambda,\Lambda,d)$ depending only on λ,Λ such that*

$$\left|\frac{\partial\hat{G}_a(\xi,t)}{\partial t}\right|\le\frac{C_1(\lambda,\Lambda,d,\delta)}{t[1+|e(\mathrm{Re}(\xi))|^2t]^\delta}\exp[C_2(\lambda,\Lambda,d)\varepsilon^2t]\ ,\quad|\mathrm{Im}(\xi)|<\varepsilon\ ,\ t>0\ .$$

Proof. In analogy to the proof of the inequality (2.41) we see that there are constants $C_1(\lambda,\Lambda,d)>0$ and $C_2(\lambda,\Lambda,d)>0$ such that

(5.12)
$$|\partial h(\xi,\eta)/\partial[\mathrm{Im}(\eta)]|\le\frac{C_1(\lambda,\Lambda,d)}{|\mathrm{Im}(\eta)|}\min\left[\frac{1}{\mathrm{Re}(\eta)+|e(\mathrm{Re}(\xi))|^2}\ ,\frac{\mathrm{Re}(\eta)+|e(\mathrm{Re}(\xi))|^2}{|\mathrm{Im}(\eta)|^2}\right],$$
$$|\mathrm{Im}(\xi)|<\varepsilon,\quad\mathrm{Re}(\eta)>C_2(\lambda,\Lambda,d)\varepsilon^2\ .$$

It follows then just as in Lemma 2.4 that

$$\left|\frac{\partial}{\partial t}\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]\right|\le C_{\lambda,\Lambda,d}/t,\ |\mathrm{Im}(\xi)|<\varepsilon\ . \tag{5.13}$$

We can also see that there are constants $C_1(\lambda,\Lambda,d)>0$ and $C_2(\lambda,\Lambda,d)>0$ such that

$$\left|\frac{\partial^2h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2}\right|\le\frac{C_1(\lambda,\Lambda,d)}{|\mathrm{Im}(\eta)|^2}\min\left[\frac{1}{\mathrm{Re}(\eta)+|e(\mathrm{Re}(\xi))|^2}\ ,\ \frac{1}{|\mathrm{Im}(\eta)|}\right], \tag{5.14}$$
$$|\mathrm{Im}(\eta)|<\varepsilon\ ,\quad\mathrm{Re}(\eta)>C_2(\lambda,\Lambda,d)\varepsilon^2\ .$$

We conclude from this last inequality and (5.13) that for any $\delta>0$ there is a constant $C(\lambda,\Lambda,d,\delta)$ such that

$$\left|\frac{\partial}{\partial t}\int_0^\infty h(\xi,\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]\right|\le\frac{C(\lambda,\Lambda,d,\delta)}{t[1+|e(\mathrm{Re}(\xi))|^2t]^\delta},\quad|\mathrm{Im}(\xi)|<\varepsilon,$$

provided $\mathrm{Re}(\eta) > C_2(\lambda,\Lambda,d)\varepsilon^2$. Arguing similarly we see also that

$$\left|\frac{\partial}{\partial t}\frac{1}{t}\int_0^\infty \frac{\partial k(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)]\right| \le \frac{C(\lambda,\Lambda,d,\delta)}{t[1+|e(\mathrm{Re}(\xi))|^2t]^\delta},$$
$$|\mathrm{Im}(\xi)|<\varepsilon,\ \mathrm{Re}(\eta) > C(\lambda,\Lambda,d)\varepsilon^2.$$

The result of the lemma follows from these last two inequalities and

$$\left|\frac{\partial}{\partial t}\exp[\mathrm{Re}(\eta)t]\right| \le \frac{1}{t}\exp[2\mathrm{Re}(\eta)t].$$

□

Corollary 5.3. *The function $G_{\mathbf{a}}(x,t)$ is differentiable with respect to t for $t>0$ and satisfies the inequality*

$$\left|\frac{\partial G_{\mathbf{a}}(x,t)}{\partial t}\right| \le \frac{C_1(\lambda,\Lambda)}{t(1+\sqrt{t})}\exp\left[-C_2(\lambda,\Lambda)\min\{|x|,\ |x|^2/t\}\right],\quad \text{if}\quad d=1,$$

where the constants $C_1(\lambda,\Lambda)$ and $C_2(\lambda,\Lambda)$ depend only on λ,Λ. For $d>1$ it satisfies an inequality,

$$\left|\frac{\partial G_{\mathbf{a}}(x,t)}{\partial t}\right| \le \frac{C_1(\lambda,\Lambda,d,\delta)}{t[1+t^\delta]}\exp\left[-C_2(\lambda,\Lambda,d)\min\{|x|,\ |x|^2/t\}\right],$$

for any δ, $0\le\delta<1$. The constant $C_1(\lambda,\Lambda,d,\delta)$ depends only λ,Λ,d,δ and $C_2(\lambda,\Lambda,d)$ only on λ,Λ,d.

Proof. Same as for Corollary 5.2 on using Lemma 5.3. □

Lemma 5.4. *There are constants $C_1(\lambda,\Lambda,d)$ and $C_2(\lambda,\Lambda,d)$ depending only on λ,Λ,d such that*

$$\left|\frac{\partial \hat{G}_{\mathbf{a}}(\xi,t)}{\partial t}\right| \le C_1(\lambda,\Lambda,d)[|e(\mathrm{Re}(\xi))|^2+\varepsilon^2]\exp[C_2(\lambda,\Lambda,d)\varepsilon^2 t],\quad |\mathrm{Im}(\xi)|<\varepsilon,\quad t>0.$$

Proof. In analogy to the inequalities (2.49) we have from (5.9), (5.12) the inequalities

(5.15)
$$\frac{1}{t}\int_0^\infty |h(\xi,\eta)|\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)] \le C_1(\lambda,\Lambda,d)\ [|e(\mathrm{Re}(\xi))|^2+\mathrm{Re}(\eta)],$$
$$\frac{1}{t}\int_0^\infty \left|\frac{\partial h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]}\right||\mathrm{Im}(\eta)|\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)] \le C_1(\lambda,\Lambda,d)[|e(\mathrm{Re}(\xi))|^2+\mathrm{Re}(\eta)],$$

provided $\xi\in\mathbb{C}^d$ and $\eta\in\mathbb{C}$ satisfy

(5.16) $$|\mathrm{Im}(\xi)|<\varepsilon,\quad \mathrm{Re}(\eta) > C_2(\lambda,\Lambda,d)\varepsilon^2,$$

where $C_1(\lambda,\Lambda,d)$ and $C_2(\lambda,\Lambda,d)$ depend only on λ,Λ,d.

Next we need to deal with the integral (2.50) in $k(\xi,\eta)$. Observe first from (5.8) that there are constants $C_1(\lambda,\Lambda,d)$ and $C_2(\lambda,\Lambda,d)$ depending only on λ,Λ,d such that if (5.16) holds then

$$|\mathrm{Im}(\eta)||k(\xi,\eta)| \le C_1(\lambda,\Lambda,d),\ |\mathrm{Im}(\eta)| \le\ \mathrm{Re}(\eta)+|e(\mathrm{Re}(\xi))|^2.$$

We can write $k(\xi,\eta)$ as in Lemma 2.5 to be

$$k(\xi,\eta) = \frac{b(\xi,\eta)}{a(\xi,\eta)^2 + b(\xi,\eta)^2} \ ,$$

where $a(\xi,\eta)$ is the sum of $\mathrm{Re}(\eta)$ and the RHS of (5.5) whence $b(\xi,\eta)$ is the sum of $\mathrm{Im}(\eta)$ and the RHS of (5.6). We have now from (5.8) that

$$|b(\xi,\eta) - b(\mathrm{Re}(\xi),\eta)| \le \frac{1}{10}|\mathrm{Im}(\eta)|, \quad |\mathrm{Im}(\eta)| > \mathrm{Re}(\eta) + |e(\mathrm{Re}(\xi))|^2 \ ,$$

provided (5.16) holds and the constant $C_2(\lambda,\Lambda,d)$ is sufficiently large. We also have that

$$|a(\xi,\eta) - a(\mathrm{Re}(\xi),\eta)| \le \frac{1}{10} \ a(\mathrm{Re}(\xi),\eta),$$

provided (5.16) holds and $C_2(\lambda,\Lambda,d)$ is sufficiently large. It follows from these last two inequalities that

$$\int_{\mathrm{Re}(\eta)+|e(\mathrm{Re}(\xi))|^2}^{\infty} \frac{|\mathrm{Im}(\eta)|}{|b(\xi,\eta)|} \frac{|a(\xi,\eta)|^2}{|a(\xi,\eta)^2 + b(\xi,\eta)^2|} d[\mathrm{Im}(\eta)] \le C(\lambda,\Lambda,d)[\mathrm{Re}(\eta)+|e(\mathrm{Re}(\xi))|^2]$$

for some constant $C(\lambda,\Lambda,d)$, provided (5.16) holds with sufficiently large constant $C_2(\lambda,\Lambda,d)$. Now the following lemma implies that

$$\int_0^\infty \langle|\psi(\xi,\eta,\cdot)|^2\rangle d[\mathrm{Im}(\eta)] \le C(\lambda,\Lambda,d)[|e(\mathrm{Re}(\xi))|^2 + \varepsilon^2] \ ,$$

for some constant $C(\lambda,\Lambda,d)$ depending only on λ,Λ,d, provided (5.16) holds with sufficiently large $C_2(\lambda,\Lambda,d)$. We conclude then from these last inequalities that

(5.17)
$$\lim_{m\to\infty}\left|\int_0^{\pi m/t} k(\xi,\eta)\mathrm{Im}(\eta)\cos[\mathrm{Im}(\eta)t]d[\mathrm{Im}(\eta)]\right| \le C(\lambda,\Lambda,d)[\mathrm{Re}(\eta) + |e(\mathrm{Re}(\xi))|^2] \ ,$$

for some constant $C(\lambda,\Lambda,d)$ provided (5.16) holds. The Lemma follows now from (5.15), (5.17). □

Lemma 5.5. *Let $\psi_k(\xi,\eta,\cdot)$ be the function defined by (2.2), and $0<\varepsilon<1$. There is a constant $C_1(\lambda,\Lambda,d)$ depending on λ,Λ,d such that if $\mathrm{Re}(\eta) > C_1(\lambda,\Lambda,d)\varepsilon^2$ then $\psi_k(\xi,\eta,\cdot)$, regarded as a mapping from $\mathbb{R}^d$ to $L^2(\Omega)$, can be analytically continued to $\{\xi\in\mathbb{C}^d : |\mathrm{Im}(\xi)|<\varepsilon\}$. Furthermore, there is a constant $C(\lambda,\Lambda,d)$ depending only on λ,Λ,d such that*

$$\int_0^\infty \left\langle|\psi_k(\xi,\eta,\cdot)|^2\right\rangle d[\mathrm{Im}(\eta)] \le C_2(\lambda,\Lambda,d), \ \mathrm{Re}(\eta) > C_1(\lambda,\Lambda,d)\varepsilon^2, \quad |\mathrm{Im}(\eta)| < \varepsilon.$$

Proof. We proceed as in Lemma 2.6. Let $\psi_k(t,\xi,\cdot), t>0$, be the solution to the initial value problem,

$$\frac{\partial\psi_k(t,\xi,\cdot)}{\partial t} + [\mathcal{L}_\xi + \ \mathrm{Re}(\eta)]\psi_k(t,\xi,\cdot) = 0, \quad t>0, \tag{5.18}$$

$$\psi_k(0,\xi,\cdot) + \sum_{j=1}^d [\partial_j^* + e_j(-\xi)]e^{i\mathbf{e}_j\cdot\xi}[a_{k,j}(\cdot) - \langle a_{k,j}(\cdot)\rangle] = 0.$$

It is clear that (5.18) is soluble for $\xi \in \mathbb{R}^d$, $\operatorname{Re}(\eta) > 0$, and

$$\psi_k(\xi,\eta,\cdot) = \int_0^\infty e^{-i\,\operatorname{Im}(\eta)t}\psi_k(t,\xi,\cdot)dt. \tag{5.19}$$

Taking $\operatorname{Re}(\eta) > C_1(\lambda,\Lambda,d)\varepsilon^2$ for sufficiently large $C_1(\lambda,\Lambda,d)$ it is clear that one can solve (5.18) for $\xi \in \mathbb{C}^d$ with $|\operatorname{Im}(\xi)| < \varepsilon$ and the resulting function is analytic in ξ. Evidently the corresponding function $\psi_k(\xi,\eta,\cdot)$ defined by (5.19) is analytic in ξ for $|\operatorname{Im}(\xi)| < \varepsilon$. Furthermore, from the Plancherel Theorem we have that

$$\int_0^\infty \left\langle |\psi_k(\xi,\eta,\cdot)|^2 \right\rangle d[\operatorname{Im}(\eta)] \le 2\pi \int_0^\infty \left\langle |\psi_k(t,\xi,\cdot)|^2 \right\rangle dt.$$

Now for $\xi \in \mathbb{C}^d$ let $\mathcal{L}_\xi^*$ be the adjoint of $\mathcal{L}_\xi$ acting on $L^2(\Omega)$. Thus $\mathcal{L}_\xi^* = \mathcal{L}_{\bar\xi}$ where $\bar\xi$ is the complex conjugate of ξ. Let $\varphi_k(t,\xi,\cdot)$ be the solution of the equation

$$[\mathcal{L}_\xi^* + \operatorname{Re}(\eta)]\varphi_k(t,\xi,\cdot) = \psi_k(t,\xi,\cdot), \quad t > 0,\ |\operatorname{Im}(\xi)| < \varepsilon.$$

It follows from (5.18) that

$$\left\langle \overline{\varphi_k(t,\xi,\cdot)} \frac{\partial \psi_k(t,\xi,\cdot)}{\partial t} \right\rangle + \left\langle |\psi_k(t,\xi,\cdot)|^2 \right\rangle = 0, \quad t > 0. \tag{5.20}$$

We also have that

(5.21)

$$\begin{aligned}\left\langle \overline{\varphi_k(t,\xi,\cdot)} \frac{\partial \psi_k(t,\xi,\cdot)}{\partial t} \right\rangle &= \operatorname{Re}\left\langle \overline{[\mathcal{L}_\xi + \operatorname{Re}(\eta)]\varphi_k} \frac{\partial \varphi_k}{\partial t} \right\rangle = \operatorname{Re}\left\langle \frac{\partial \bar\varphi_k}{\partial t}[\mathcal{L}_\xi + \operatorname{Re}(\eta)]\varphi_k \right\rangle \\ &= \operatorname{Re}\left\langle \overline{([\mathcal{L}_\xi + \operatorname{Re}(\eta)]\frac{\partial \varphi_k}{\partial t}} \varphi_k \right\rangle + \operatorname{Re}\left\langle \frac{\partial \bar\varphi_k}{\partial t}[\mathcal{L}_\xi - \mathcal{L}_\xi^*]\varphi_k \right\rangle.\end{aligned}$$

We conclude that

$$\left\langle \bar\varphi_k \frac{\partial \psi_k}{\partial t} \right\rangle = \frac{1}{2}\frac{\partial}{\partial t}\operatorname{Re}\left\langle \overline{[\mathcal{L}_\xi + \operatorname{Re}(\eta)]\varphi_k}\varphi_k \right\rangle + \frac{1}{2}\operatorname{Re}\left\langle \frac{\partial \bar\varphi_k}{\partial t}[\mathcal{L}_\xi - \mathcal{L}_\xi^*]\varphi_k \right\rangle.$$

Observe now that

$$\begin{aligned}&\left\langle \frac{\partial \bar\varphi_k}{\partial t}[\mathcal{L}_\xi - \mathcal{L}_\xi^*]\varphi_k \right\rangle = \\ &\qquad - \left\langle \bar\psi_k[\mathcal{L}_\xi^* + \operatorname{Re}(\eta)][\mathcal{L}_\xi + \operatorname{Re}(\eta)]^{-1}[\mathcal{L}_\xi - \mathcal{L}_\xi^*][\mathcal{L}_\xi^* + \operatorname{Re}(\eta)]^{-1}\psi_k \right\rangle.\end{aligned}$$

Hence if $C_1(\lambda,\Lambda,d)$ is sufficiently large one has

$$\left| \left\langle \frac{\partial \bar\varphi_k}{\partial t}[\mathcal{L}_\xi - \mathcal{L}_\xi^*]\varphi_k \right\rangle \right| \le \left\langle |\psi_k(t,\xi,\cdot)|^2 \right\rangle,\ t > 0,\ |\operatorname{Im}(\xi)| < \varepsilon.$$

Putting this inequality together with (5.20), (5.21) we have that

$$\frac{1}{2}\frac{\partial}{\partial t}\operatorname{Re}\left\langle \overline{[\mathcal{L}_\xi + \operatorname{Re}(\eta)]\varphi_k}\varphi_k \right\rangle + \frac{1}{2}\left\langle |\psi_k(t,\xi,\cdot)|^2 \right\rangle \le 0, \quad |\operatorname{Im}(\xi)| < \varepsilon,\ t > 0.$$

Integrating this inequality with respect to t we conclude

$$\int_0^\infty \left\langle |\psi_k(t,\xi,\cdot)|^2 \right\rangle dt \le \operatorname{Re} \left\langle \overline{\varphi_k(0,\xi,\cdot)}\psi_k(0,\xi,\cdot) \right\rangle.$$

Arguing as in Lemma 2.6 we see that the RHS of this last inequality is bounded by a constant $C(\lambda,\Lambda,d)$ depending only on λ,Λ,d. $\square$

Corollary 5.4. *There are constants $C_1(\lambda,\Lambda,d)$ and $C_2(\lambda,\Lambda,d)>0$ depending only on d,λ,Λ such that*

$$\left|\frac{\partial G_{\mathbf{a}}(x,t)}{\partial t}\right| \le C_1(d,\lambda,\Lambda)\exp\left[-C_2(d,\lambda,\Lambda)\min\{|x|,\ |x|^2/t\}\right], \quad x\in\mathbb{Z}^d, \quad t>0.$$

Proof. Same as for Corollary 5.1 on using Lemma 5.4. □

Corollary 5.2 proves (1.10) for $d=1$. Following the argument of Lemma 4.1 we shall use our methods to prove (1.10) for $d=2$.

Lemma 5.6. *For $d=2$ there are positive constants $C_1(\lambda,\Lambda)$, $C_2(\lambda,\Lambda)$ depending only on λ,Λ such that*

$$0\le G_{\mathbf{a}}(x,t)\le \frac{C_1(\lambda,\Lambda)}{1+t}\exp\left[-C_2(\lambda,\Lambda)\min\{|x|,\ |x|^2/t\}\right], \quad x\in\mathbb{Z}^2,\ t>0.$$

Proof. It will be sufficient to show that there are constants $C_1(\lambda,\Lambda)$, $C_2(\lambda,\Lambda)$ such that

$$(5.22)\quad \int_{[-\pi,\pi]^2}|\hat{G}_{\mathbf{a}}(\xi,t)|d[\mathrm{Re}(\xi)]\le\frac{C_1(\lambda,\Lambda)}{1+t}\exp[C_2(\lambda,\Lambda)\varepsilon^2 t],\ |\mathrm{Im}(\xi)|<\varepsilon,\ t>0.$$

In view of Lemma 5.1 it will be sufficient to prove (5.22) for $t\ge 1$. It is also evident from Lemma 5.1 that

$$\int_{|e(\mathrm{Re}(\xi))|<1/\sqrt{t}}|\hat{G}_{\mathbf{a}}(\xi,t)|d[\mathrm{Re}(\xi)]\le\frac{C_1(\lambda,\Lambda)}{t}\exp[C_2(\lambda,\Lambda)\varepsilon^2 t],\ |\mathrm{Im}(\xi)|<\varepsilon,\ t>1,$$

whence we are left to show that

$$(5.23)\quad \int_{|e(\mathrm{Re}(\xi))|>1/\sqrt{t}}|\hat{G}_{\mathbf{a}}(\xi,t)|d[\mathrm{Re}(\xi)]\le\frac{C_1(\lambda,\Lambda)}{t}\exp[C_2(\lambda,\Lambda)\varepsilon^2 t],\ |\mathrm{Im}(\xi)|<\varepsilon,\ t>1.$$

Now the integral on the LHS of (5.23) is a sum of an integral in $h(\xi,\eta)$ and $k(\xi,\eta)$. We first consider the integral in $h(\xi,\eta)$. Following the argument of Lemma 4.1 and using (5.14) we see that it is sufficient to show that

$$\int_{|e(\mathrm{Re}(\xi))|>1/\sqrt{t}}d[\mathrm{Re}(\xi)]\left|\frac{1}{t^2}\int_0^{|e(\mathrm{Re}(\xi))|^2}\frac{\partial^2 h(\xi,\eta)}{\partial[\mathrm{Im}(\eta)]^2}\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)]\right| \le\frac{C_1(\lambda,\Lambda)}{t},\ |\mathrm{Im}(\xi)|<\varepsilon,\ \mathrm{Re}(\eta)>C_2(\lambda,\Lambda)\varepsilon^2,$$

for sufficiently large constant $C_2(\lambda,\Lambda)$ depending only on λ,Λ. Again, arguing as in Lemma 4.1, we see it is sufficient to show that

$$(5.24)\quad \int_{|e(\mathrm{Re}(\xi))|>1/\sqrt{t}}d[\mathrm{Re}(\xi)]\frac{1}{t^2}\int_0^{|e(\mathrm{Re}(\xi))|^2}\frac{<|\psi(\xi,\eta,\cdot)|^2>}{|e(\mathrm{Re}(\xi))|^4\mathrm{Im}(\eta)}\{1-\cos[\mathrm{Im}(\eta)t]\}d[\mathrm{Im}(\eta)] \le\frac{C_1(\lambda,\Lambda)}{t},\ |\mathrm{Im}(\xi)|<\varepsilon,\ \mathrm{Re}(\eta)>C_2(\lambda,\Lambda)\varepsilon^2.$$

Define for $2 < p < \infty$ a function $h_{\eta,p}(\xi)$ by

$$h_{\eta,p}(\xi) = |\mathrm{Im}(\eta)|^{1/2\,-\,1/p}\left[\sum_{k=1}^{2}\left\langle|\psi_k(\xi,\eta,\cdot)|^2\right\rangle\right]^2, \quad \xi\in\mathbb{C}^d,\ |\mathrm{Im}(\xi)|<\varepsilon.$$

Suppose now $\mathrm{Im}(\xi)\in\mathbb{R}^2$ is fixed and regard $h_{2,p}(\xi)$ as a function of $\mathrm{Re}(\xi)\in[-\pi,\pi]^2$. Then one can see just as in Lemma 4.1 that if $|\mathrm{Im}(\xi)|<\varepsilon$ and $\mathrm{Re}(\eta)>C_2(\lambda,\Lambda)\varepsilon^2$ for sufficiently large constant $C_2(\lambda,\Lambda)$ depending only on λ,Λ, then $h_{\eta,p}\in L^p_w([-\pi,\pi]^2)$ and there is a constant $C_{p,\lambda,\Lambda}$ such that

$$\|h_{\eta,p}\|_{p,w}\le C_{p,\lambda,\Lambda},\ |\mathrm{Im}(\xi)|<\varepsilon,\ \mathrm{Re}(\eta)>C_2(\lambda,\Lambda)\varepsilon^2.$$

The inequality (5.24) follows from this last inequality just as in Lemma 4.1. Hence we have found an appropriate bound for the contribution to the LHS of (5.23) from the integral in $h(\xi,\eta)$. The contribution from the integral in $k(\xi,\eta)$ can be estimated similarly. $\square$

We can similarly generalize Lemma 4.2 to obtain the following.

Lemma 5.7. *For $d=2$ there are positive constants $C_1(\lambda,\Lambda)$, $C_2(\lambda,\Lambda)$ depending only on λ,Λ such that*

$$\left|\frac{\partial G_{\mathbf{a}}(x,t)}{\partial t}\right|\le\frac{C_1(\lambda,\Lambda)}{(1+t^2)}\exp\left[-C_2(\lambda,\Lambda)\min\{|x|,\ |x|^2/t\}\right],\quad x\in\mathbb{Z}^2,\ t>0.$$

Next we wish to consider derivatives of $G_{\mathbf{a}}(x,t)$ with respect to $x\in\mathbb{Z}^d$. If $d=1$ then it is clear from Corollary 5.2 that

$$|\nabla_i G_{\mathbf{a}}(x,t)|\ \le\frac{C_1(\lambda,\Lambda)}{(1+\sqrt{t})}\exp\left[-C_2(\lambda,\Lambda)\min\{|x|,\ |x|^2/t\}\right],\quad x\in\mathbb{Z}^2,\ t>0.$$

We can use Lemma 5.2 to obtain an improvement on this inequality.

Lemma 5.8. *Suppose $d=1$ and $0\le\delta<1$. Then there exist constants $C_1(\lambda,\Lambda,\delta)$ depending only on λ,Λ,δ and $C_2(\lambda,\Lambda)$ depending only on λ,Λ such that*

$$|\nabla_i G_{\mathbf{a}}(x,t)|\ \le\frac{C_1(\lambda,\Lambda,\delta)}{(1+t^{\delta})}\exp\left[-C_2(\lambda,\Lambda)\min\{|x|,\ |x|^2/t\}\right],\quad x\in\mathbb{Z}^2,\ t>0.$$

Proof. The result follows from Lemma 5.2 and the observation that

$$\int_{-\pi}^{\pi}\frac{|e(\mathrm{Re}(\xi))|}{[1+|e(\mathrm{Re}(\xi))|^2t]^{\delta'}}\ d[\mathrm{Re}(\xi)]\le\frac{C(\delta',\delta)}{1+t^{\delta}},$$

for any δ',δ satisfying $1/2<\delta<\delta'<1$, where $C(\delta',\delta)$ is a constant depending only on δ',δ. $\square$

We can improve Lemma 5.8 by using the techniques developed in Section 3.

Lemma 5.9. *Suppose $d=1$ and $0\le\delta<1$. Then there are constants $C_1(\lambda,\Lambda)$, $C_2(\lambda,\Lambda)$ depending only on λ,Λ and a constant $C_3(\lambda,\Lambda,\delta)$ depending on λ,Λ,δ such that*

$$|\nabla_i G_{\mathbf{a}}(x,t)|\ \le\frac{C_1(\lambda,\Lambda)}{(1+t)}\exp\left[-C_2(\lambda,\Lambda)\min\{|x|,\ |x|^2/t\}\right]\ , \tag{5.25}$$

(5.26)

$$|\nabla_i\nabla_j G_{\mathbf{a}}(x,t)|\le\frac{C_3(\lambda,\Lambda,\delta)}{[1+t^{1+\delta/2}]}\exp\left[-C_2(\lambda,\Lambda)\min\{|x|,\ |x|^2/t\}\right],\quad x\in\mathbb{Z}^2,\ t>0.$$

Proof. It will be sufficient to show that for any δ, $0 \le \delta < 1$, there are constants $C_3(\lambda, \Lambda, \delta)$ and $C_2(\lambda, \Lambda)$ such that

$$(5.27) \qquad \int_{-\pi}^{\pi} |e(\mathrm{Re}(\xi))|^{1+\delta} |\hat{G}_{\mathbf{a}}(\xi, t)| d[\mathrm{Re}(\xi)] \le \frac{C_3(\lambda, \Lambda, \delta)}{1 + t^{1+\delta/2}} \exp[C_2(\lambda, \Lambda)\varepsilon^2 t],$$

for $|\mathrm{Im}(\xi)| < \varepsilon$, $t > 1$. Now from Lemma 5.1 we have

$$\int_{|e(\mathrm{Re}(\xi))| < 1/\sqrt{t}} |e(\mathrm{Re}(\xi))|^{1+\delta} |\hat{G}_{\mathbf{a}}(\xi, t)| d[\mathrm{Re}(\xi)] \le \frac{C_3(\lambda, \Lambda)}{1 + t^{1+\delta/2}} \exp[C_2(\lambda, \Lambda)\varepsilon^2 t],$$

for $|\mathrm{Im}(\xi)| < \varepsilon$, so we are left to prove

(5.28)
$$\int_{|e(\mathrm{Re}(\xi))| > 1/\sqrt{t}} |e(\mathrm{Re}(\xi))|^{1+\delta} |\hat{G}_{\mathbf{a}}(\xi, t)| d[\mathrm{Re}(\xi)] \le \frac{C_3(\lambda, \Lambda, \delta)}{t^{1+\delta/2}} \exp[C_2(\lambda, \Lambda)\varepsilon^2 t],$$

for $|\mathrm{Im}(\xi)| < \varepsilon$, $t > 1$. Proceeding now as in Lemma 5.6 we write the integral on the LHS of (5.28) as an integral in $h(\xi, \eta)$ and an integral in $k(\xi, \eta)$. We first consider the integral in $h(\xi, \eta)$. If we use (5.14) we see that it is sufficient to show that

$$(5.29) \qquad \int_{|e(\mathrm{Re}(\xi))| > 1/\sqrt{t}} d[\mathrm{Re}(\xi)] |e(\mathrm{Re}(\xi))|^{1+\delta}$$
$$\left| \frac{1}{t^2} \int_0^{|e(\mathrm{Re}(\xi))|^2} \frac{\partial^2 h(\xi, \eta)}{\partial [\mathrm{Im}(\eta)]^2} \{1 - \cos[\mathrm{Im}(\eta) t]\} d[\mathrm{Im}(\eta)] \right|$$
$$\le \frac{C_3(\lambda, \Lambda, \delta)}{t^{1+\delta/2}}, \quad |\mathrm{Im}(\xi)| < \varepsilon, \ \mathrm{Re}(\eta) > C_2(\lambda, \Lambda)\varepsilon^2, \ 0 \le \delta < 1,$$

for sufficiently large $C_2(\lambda, \Lambda)$ depending only on λ, Λ. Arguing as in Lemma 5.6 we see that to prove (5.29) it is sufficient to show that

$$(5.30) \qquad \int_{|e(\mathrm{Re}(\xi))| > 1/\sqrt{t}} d[\mathrm{Re}(\xi)] |e(\mathrm{Re}(\xi))|^{1+\delta}$$
$$\frac{1}{t^2} \int_0^{|e(\mathrm{Re}(\xi))|^2} \frac{< |\psi(\xi, \eta, \cdot)|^2 >}{|e(\mathrm{Re}(\xi))|^4 \mathrm{Im}(\eta)} \{1 - \cos[\mathrm{Im}(\eta) t]\} d[\mathrm{Im}(\eta)]$$
$$\le \frac{C_3(\lambda, \Lambda, \delta)}{t^{1+\delta/2}}, \quad |\mathrm{Im}(\xi)| < \varepsilon, \ \mathrm{Re}(\eta) > C_2(\lambda, \Lambda)\varepsilon^2, \ 0 \le \delta < 1.$$

Define for $2 < p < \infty$ a function $h_{\eta,p}(\xi)$ by

$$h_{\eta,p}(\xi) = |\mathrm{Im}(\eta)|^{1/2 - 1/2p} \left\langle |\psi_1(\xi, \eta, \cdot)|^2 \right\rangle^{1/2}, \ \xi \in \mathbb{C}^d, \quad |\mathrm{Im}(\xi)| < \varepsilon.$$

We fix now $\mathrm{Im}(\xi) \in \mathbb{R}$ and regard $h_{\eta,p}(\xi)$ as a function of $\mathrm{Re}(\xi) \in [-\pi, \pi]$. Then one can see just as in Lemma 5.6 that if $|\mathrm{Im}(\xi)| < \varepsilon$ and $\mathrm{Re}(\eta) > C_2(\lambda, \Lambda)\varepsilon^2$ for sufficiently large constant $C_2(\lambda, \Lambda)$ depending only on λ, Λ, then $h_{\eta,p} \in L^p_w([-\pi, \pi])$ and there is a constant $C_{p,\lambda,\Lambda}$ such that

$$\|h_{\eta,p}\|_{p,w} \le C_{p,\lambda,\Lambda}, \quad |\mathrm{Im}(\xi)| < \varepsilon, \quad \mathrm{Re}(\eta) > C_2(\lambda, \Lambda)\varepsilon^2.$$

Arguing as in Lemma 4.1 we see that (5.30) holds if

$$\int_{|e(\mathrm{Re}(\xi))|>1/\sqrt{t}} d[\mathrm{Re}(\xi)] \frac{1}{|e(\mathrm{Re}(\xi))|^{1-\delta}\ t^{1/2+2/p}} \int_0^{|e(\mathrm{Re}(\xi))|^2} \frac{h_{\eta,p}(\xi)^2}{[\mathrm{Im}(\eta)]^{1/2+1/p}} d[\mathrm{Im}(\eta)] \leq \frac{C_3(\lambda,\Lambda,\delta)}{t^{1+\delta/2}}, \quad t>1.$$

The LHS of this last expression is bounded by

$$\frac{C}{t^{1/2+2/p}} \sum_{n=0}^{\infty} \left(\frac{\sqrt{t}}{2^n}\right)^{1-\delta} \int_0^{2^{2n+2}/t} \frac{d[\mathrm{Im}(\eta)]}{[\mathrm{Im}(\eta)]^{1/2+1/p}} \int_{|e(\xi))|<2^{n+1}/\sqrt{t}} h_{\eta,p}(\xi)^2 d[\mathrm{Re}(\xi)]$$

$$\leq \frac{C_{p,\lambda,\Lambda}}{t^{1/2+2/p}} \sum_{n=0}^{\infty} \left(\frac{\sqrt{t}}{2^n}\right)^{1-\delta} \left(\frac{2^{2n+2}}{t}\right)^{1/2\,-\,1/p} \left(\frac{2^{n+1}}{\sqrt{t}}\right)^{1\,-\,2/p} \leq \frac{C(\lambda,\Lambda,\delta)}{t^{1+\delta/2}},$$

provided we choose p to satisfy $2<p<4/(1+\delta)$. Hence the contribution to the LHS of (5.28) from $h(\xi,\eta)$ is bounded appropriately. Since we can similarly estimate the contribution from $k(\xi,\eta)$ we have proved (5.28) and hence (5.27). Now (5.25) follows by taking $\delta=0$ in (5.27), and (5.26) by taking δ close to 1. $\square$

We have proven Theorem 1.4 for $d=1$. It is clear by now that we can use the methods developed in Section 4 to extend the results of Lemmas 5.6, 5.7, 5.9 to all dimensions $d\geq 1$.

6. Proof of Theorem 1.2

For $\eta>0,\ x\in\mathbb{R}^d$, let $G_\eta(x)$ be the Green's function which satisfies the equation,

$$-\sum_{i=1}^{d} \frac{\partial^2 G_\eta(x)}{\partial x_i^2} + \eta G_\eta(x) = \delta(x), \quad x\in\mathbb{R}^d,$$

where $\delta(x)$ is the Dirac δ function. Analogously to (3.4) we define an operator $T_{k,k',\eta,\xi}$ on $L^2(\Omega)$ by

$$T_{k,k',\eta,\xi}\varphi(\omega) = -\int_{\mathbb{R}^d} dx \frac{\partial^2 G_\eta(x)}{\partial x_k \partial x_{k'}} e^{-ix\cdot\xi}\varphi(\tau_x\omega), \quad \omega\in\Omega. \tag{6.1}$$

Evidently $T_{k,k',\eta,\xi}$ is a bounded operator on $L^2(\Omega)$. Just as in Section 3 we can define operators $T_{\mathbf{b},\eta,\xi}$ and $T_{j,\eta,\xi},\ j=1,\dots,d$ associated with the operators (6.1). We then have the following.

Lemma 6.1. *Let $T_{\mathbf{b},\eta,\xi}$ and $T_{j,\eta,\xi}, j=1,\dots,d$, be the operators associated with (6.1). Then if $\|\mathbf{b}\|<\mathbf{1}$ the equation (3.6) has a unique solution $\Psi_k(\xi,\eta,\cdot)\in\mathcal{H}(\Omega),\ \xi\in\mathbb{R}^d,\ \eta>0$, which satisfies an inequality,*

$$\|\Psi_k(\xi,\eta,\cdot)\| \leq C(\lambda,\Lambda,d)/[1-\|\mathbf{b}\|], \qquad k=1,\dots,d,$$

where the constant $C(\lambda,\Lambda,d)$ depends only on λ,Λ,d. The function $(\xi,\eta)\to\Psi_k(\xi,\eta,\cdot)$ from $\mathbb{R}^d\times\mathbb{R}^+$ to $\mathcal{H}(\Omega)$ is continuous.

Next we put $\mathbf{b}(\cdot)=[\Lambda I_d-\mathbf{a}(\cdot)]/\Lambda$ and define a $d\times d$ matrix $q(\xi,\eta)$ by (3.7), where $\Psi_k(\xi,\eta,\cdot),\ k=1,2,\dots,d$ are the functions of Lemma 6.1.

Lemma 6.2. *For $\xi \in \mathbb{R}^d$, $\eta > 0$, the matrix $q(\xi,\eta)$ is Hermitian and is a continuous function of (ξ,η). Furthermore, there is the inequality*

$$\lambda I_d \le q(\xi,\eta) \le \Lambda I_d. \tag{6.2}$$

Proof. We use the operators $T_{k,k',\eta,\xi}$ of (6.1) to define an operator $T_{\eta,\xi}$ on $\mathcal{H}(\Omega)$. Thus if $\varphi = (\varphi_1, \dots, \varphi_d) \in \mathcal{H}(\Omega)$ then

$$(T_{\eta,\xi}\varphi)_k = \sum_{k'=1}^{d} T_{k,k',\eta,\xi}\varphi_{k'} \ .$$

It is easy to see that $T_{\eta,\xi}$ is a bounded self-adjoint operator on $\mathcal{H}(\Omega)$ which is nonnegative definite and has norm $\|T_{\eta,\xi}\| \le 1$. We can see this by using Bochner's Theorem as in Lemma 3.3. Now for $\rho \in \mathbb{C}^d$, we put Ψ_ρ to be

$$\Psi_\rho = \sum_{k=1}^{d} \rho_k \Psi_k \ ,$$

where Ψ_k is the solution of (3.6). Then from (3.7) we have that

$$\bar{\rho} q(\xi,\eta)\rho = \langle \bar{\rho}\mathbf{a}(\cdot)\rho + \bar{\rho}\mathbf{a}(\cdot)\Psi_\rho(\xi,\eta,\cdot)\rangle \, .$$

It is also clear from (3.6) that Ψ_ρ satisfies

$$\Psi_\rho(\xi,\eta,\omega) - PT_{\mathbf{b},\eta/\Lambda,\,\xi}\Psi_\rho(\xi,\eta,\omega) + \frac{1}{\Lambda}T_{\eta/\Lambda,\,\xi}[\mathbf{a}(\omega)\rho - < \mathbf{a}(\cdot)\rho >] = 0. \tag{6.3}$$

To obtain the upper bound in (6.2) we observe that

$$\Big\langle \bar{\rho}\mathbf{a}(\cdot)\Psi_\rho(\xi,\eta,\cdot)\Big\rangle \le 0. \tag{6.4}$$

To prove (6.4) we generate Ψ_ρ from (6.3) by a Neumann series. The first order approximation to the solution is then

$$\Psi_\rho(\xi,\eta,\cdot) \simeq -\frac{1}{\Lambda}T_{\eta/\Lambda,\,\xi}[\mathbf{a}(\omega)\rho - \langle \mathbf{a}(\cdot)\rho\rangle] \ .$$

Putting this approximation into (6.4) yields

$$\begin{aligned}
&\left\langle \rho\mathbf{a}(\cdot)\left[-\frac{1}{\Lambda}T_{\eta/\Lambda,\,\xi}[\mathbf{a}(\omega)\rho - \langle \mathbf{a}(\cdot)\rho\rangle\right]\right\rangle \\
&\qquad\qquad = -\frac{1}{\Lambda}\left\langle \rho[\mathbf{a}(\cdot) - \langle \mathbf{a}\rangle]T_{\eta/\Lambda,\,\xi}[\mathbf{a}(\cdot) - \langle \mathbf{a}\rangle]\rho \ \right\rangle \le 0,
\end{aligned}$$

since $T_{\eta/\Lambda,\xi}$ is nonnegative definite. One can similarly argue that each term in the Neumann series makes a negative contribution to (6.4).

To prove the lower bound in (6.2) we use the fact that

$$T_{\eta,\xi}^2 = T_{\eta,\xi}[1 - \eta A_{\eta,\xi}],$$

where $A_{\eta,\xi}$ is the bounded operator on $L^2(\Omega)$ defined by

$$A_{\eta,\xi} \ \phi(\omega) = \int_{\mathbb{R}^d} dx G_\eta(x) e^{-ix\cdot\xi}\phi(\tau_x\omega).$$

Note that $A_{\eta,\xi}$ is self-adjoint, nonnegative definite, and commutes with $T_{\eta,\xi}$. It follows now from (6.3) that

$$T_{\eta/\Lambda,\,\xi}\Psi_\rho = \left[1 - \frac{\eta}{\Lambda}A_{\eta/\Lambda,\,\xi}\right]\Psi_\rho \ .$$

Hence if we multiply (6.3) by $\bar{\Psi}_\rho$ and take the expectation value we have that

$$\|\Psi_\rho(\xi,\eta,\cdot)\|^2 - \left\langle \overline{\Psi_\rho(\xi,\eta,\cdot)}\mathbf{b}(\cdot)\Psi_\rho(\xi,\eta,\cdot)\right\rangle + \frac{\eta}{\Lambda}\left\langle \overline{\Psi_\rho(\xi,\eta,\cdot)}A_{\eta/\Lambda,\,\xi}\mathbf{b}(\cdot)\Psi_\rho(\xi,\eta,\cdot)\right\rangle$$
$$+\frac{1}{\Lambda}\left\langle \overline{\Psi_\rho(\xi,\eta,\cdot)}\left[1-\frac{\eta}{\Lambda}A_{\eta/\Lambda,\,\xi}\right][\mathbf{a}(\cdot)\rho - \langle\mathbf{a}\rangle\rho]\right\rangle = 0.$$

If we define the operator K by

$$K = 1 - \frac{\eta}{\Lambda}A_{\eta/\Lambda,\,\xi},$$

then the previous equation can be written as

$$\|\Psi_\rho(\xi,\eta,\cdot)\|^2 = \left\langle \overline{K\Psi_\rho(\xi,\eta,\cdot)}\mathbf{b}(\cdot)\Psi_\rho(\xi,\eta,\cdot)\right\rangle - \frac{1}{\Lambda}\left\langle \overline{K\Psi_\rho(\xi,\eta,\cdot)}[\mathbf{a}(\cdot)-\lambda I_d]\rho\right\rangle.$$

Applying the Schwarz inequality to this last equation, we obtain

$$\|\Psi_\rho(\xi,\eta,\cdot)\|^2 \le \frac{1}{2}\left\langle \overline{K\Psi_\rho(\xi,\eta,\cdot)}\mathbf{b}(\cdot)K\Psi_\rho(\xi,\eta,\cdot)\right\rangle + \frac{1}{2}\left\langle \overline{\Psi_\rho(\xi,\eta,\cdot)}\mathbf{b}(\cdot)\Psi_\rho(\xi,\eta,\cdot)\right\rangle$$
$$+\frac{1}{2\Lambda}\left\langle \overline{K\Psi_\rho(\xi,\eta,\cdot)}[\mathbf{a}(\cdot)-\lambda I_d]K\Psi_\rho(\xi,\eta,\cdot)\right\rangle + \frac{1}{2\Lambda}\langle\bar{\rho}[\mathbf{a}(\cdot)-\lambda I_d]\rho\rangle.$$

Observing now that K is also nonnegative definite and bounded above by the identity, we see from this last inequality that

$$\left\langle \overline{\Psi_\rho(\xi,\eta,\cdot)}\mathbf{a}(\cdot)\Psi_\rho(\xi,\eta,\cdot)\right\rangle \le \langle\bar{\rho}[\mathbf{a}(\cdot)-\lambda I_d]\rho\rangle.$$

The lower bound in (6.2) follows now from the Schwarz inequality on writing

$$\langle\bar{\rho}\mathbf{a}(\cdot)\Psi_\rho(\xi,\eta,\cdot)\rangle = \langle\bar{\rho}[\mathbf{a}(\cdot)-\lambda I_d]\Psi_\rho(\xi,\eta,\cdot)\rangle.$$

□

We have defined the functions $\Psi_k(\xi,\eta,\cdot),\ k=1,\dots,d$ corresponding to the solutions of (3.6). Next we wish to define functions $\psi_k(\xi,\eta,\cdot)$ corresponding to the solutions of (2.2). To do this we consider an equation adjoint to (3.6). Since $T_{\mathbf{b},\eta,\xi} = T_{\eta,\xi}\mathbf{b}(\cdot)$, the adjoint $T^*_{\mathbf{b},\eta,\xi}$ of $T_{\mathbf{b},\eta,\xi}$ is $T^*_{\mathbf{b},\eta,\xi} = \mathbf{b}(\cdot)T_{\eta,\xi}$. For $k=1,\dots,d$ let $\Psi^*_k(\xi,\eta,\omega)\in\mathcal{H}(\Omega)$ be the solution to the equation,

$$\Psi^*_k(\xi,\eta,\omega) - PT^*_{\mathbf{b},\eta/\Lambda,\,\xi}\Psi^*_k(\xi,\eta,\omega) + \frac{1}{\Lambda}[a_k(\omega)-\langle a_k(\cdot)\rangle] = 0, \tag{6.5}$$

where $a_k(\omega)$ is the k th column vector of the matrix $\mathbf{a}(\omega)$. Just as in Lemma 6.1 we see that Ψ^*_k regarded as a mapping from $\mathbb{R}^d\times\mathbb{R}^+$ to $\mathcal{H}(\Omega)$ is continuous. We also define an operator $S_{\eta,\xi}:\mathcal{H}(\Omega)\to L^2(\Omega)$ by

$$S_{\eta,\xi}\varphi(\omega) = \sum_{k'=1}^{d}\int_{\mathbb{R}^d}dx\,\frac{\partial G_\eta(x)}{\partial x_{k'}}e^{-ix\cdot\xi}\varphi_{k'}(\tau_x\omega), \tag{6.6}$$

where $\varphi=(\varphi_1,\dots,\varphi_d)\in\mathcal{H}(\Omega)$. Evidently $S_{\eta,\xi}$ is a bounded operator. We define the functions $\psi_k(\xi,\eta,\omega),\ k=1,\dots,d$, then by

$$\psi_k(\xi,\eta,\omega) = S_{\eta/\Lambda,\,\xi}\Psi^*_k(\xi,\eta,\omega),\quad \omega\in\Omega, \tag{6.7}$$

where $\Psi^*_k(\xi,\eta,\omega)$ is the solution to (6.5). It is easy to see that there is a constant $C(\lambda,\Lambda,d)$ depending only on λ,Λ,d, such that

$$\|\psi_k(\xi,\eta,\cdot)\| \le C(\lambda,\Lambda,d)/\sqrt{\eta}\ . \tag{6.8}$$

Let us define $\hat{G}_{\mathbf{a}}(\xi,\eta)$ similarly to (2.4) by

$$\hat{G}_{\mathbf{a}}(\xi,\eta) = \frac{1}{\eta + \xi q(\xi,\eta)\xi}, \quad \xi \in \mathbb{R}^d, \quad \eta > 0, \tag{6.9}$$

where the matrix $q(\xi,\eta)$ is as in Lemma 6.2. Suppose now $f : \mathbb{R}^d \to \mathbb{R}$ is a C^∞ function with compact support and Fourier transform $\hat{f}(\xi)$. We define $\hat{v}(\xi,\eta,\omega)$ to be

$$\hat{v}(\xi,\eta,\omega) = \left[1 - i\sum_{k=1}^d \xi_k \psi_k(\xi,\eta,\omega)\right]\hat{G}_{\mathbf{a}}(\xi,\eta)\hat{f}(\xi).$$

Let $\eta > 0$ be fixed. Then $\hat{v}(\xi,\eta,\omega)$, regarded as a function of $\xi \in \mathbb{R}^d$ to $L^2(\Omega)$, is continuous and rapidly decreasing. Hence the Fourier inverse

$$v(x,\eta,\omega) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d\xi e^{-ix\cdot\xi}\hat{v}(\xi,\eta,\omega), \quad x \in \mathbb{R}^d,$$

regarded as a mapping of $x \in \mathbb{R}^d$ to $L^2(\Omega)$ is C^∞. In particular it follows that $v(x,\eta,\omega)$, regarded as a function of $(x,\omega) \in \mathbb{R}^d \times \Omega$ to $\mathbb{C}$, is measurable and in $L^2(\mathbb{R}^d \times \Omega)$. Define now the function $u(x,\eta,\omega) = v(x,\eta,\tau_x\omega)$. It is clear that $u(x,\eta,\omega)$, regarded as a function of $(x,\omega) \in \mathbb{R}^d \times \Omega$ to $\mathbb{C}$ is measurable and in $L^2(\mathbb{R}^d \times \Omega)$, with the same norm as v.

Lemma 6.3. *With probability one the function $u(x,\eta,\cdot)$, $x \in \mathbb{R}^d$, is in $L^2(\mathbb{R}^d)$ and its distributional gradient $\nabla u(x,\eta,\cdot)$ is also in $L^2(\mathbb{R}^d)$. Furthermore $u(x,\eta,\cdot)$ is a weak solution of the equation*

$$-\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left[a_{i,j}(\tau_x\cdot)\frac{\partial u}{\partial x_j}(x,\eta,\cdot)\right] + \eta u(x,\eta,\cdot) = f(x), \quad x \in \mathbb{R}^d. \tag{6.10}$$

Proof. Since $u(x,\eta,\omega) \in L^2(\mathbb{R}^d \times \Omega)$, it follows that with probability one $u(x,\eta,\cdot)$, $x \in \mathbb{R}^d$ is in $L^2(\mathbb{R}^d)$. To see that the distributional gradient of $u(x,\eta,\cdot)$ is also in $L^2(\mathbb{R}^d)$ with probability one, we shall establish a formula for $\nabla u(x,\eta,\cdot)$. To do this we define for any C^∞ function of compact support $g : \mathbb{R}^d \to \mathbb{C}$, an operator $A_{g,\xi}$ on $L^2(\Omega)$ by

$$A_{g,\xi}\varphi(\omega) = \int_{\mathbb{R}^d} dx g(x) e^{-ix\cdot\xi}\varphi(\tau_x\omega), \quad \varphi \in L^2(\Omega).$$

Evidently $A_{g,\xi}$ is a bounded operator on $L^2(\Omega)$. Suppose now $\Psi_k(\xi,\eta,\cdot) \in \mathcal{H}(\Omega)$ is the function of Lemma 6.1 with components $\Psi_k = (\Psi_{k,1},\dots,\Psi_{k,d})$ and $\psi_k(\xi,\eta,\cdot)$ is given by (6.7). Then

$$A_{\nabla_j g,\xi}\psi_k(\xi,\eta,\cdot) = -A_{g,\xi}\Psi_{k,j}(\xi,\eta,\cdot), \tag{6.11}$$

where $\nabla_j g$ is the j th partial of g. To see that (6.11) holds, observe that if $S_{\eta,\xi}$ is the operator of (6.6) then for $\varphi \in \mathcal{H}(\Omega)$,

$$A_{\nabla_j g,\xi}S_{\eta,\xi}\varphi = -A_{g,\xi}\sum_{k'=1}^d T_{j,k',\eta,\xi}\varphi_{k'}, \tag{6.12}$$

where $\varphi = (\varphi_1, \ldots, \varphi_d)$ and $T_{j,k',\eta,\xi}$ are the operators (6.1). The identity (6.11) follows now from (6.12) by observing that $\Psi_k(\xi,\eta,\cdot) = T_{\eta/\Lambda,\,\xi}\Psi_k^*(\xi,\eta,\cdot)$. We have now that

$$(6.13)\qquad \begin{aligned} -\int_{\mathbb{R}^d} dx \nabla_j g(x) u(x,\eta,\cdot) &= -\int_{\mathbb{R}^d} dx \nabla_j g(x) v(x,\eta,\tau_x\cdot) \\ &= -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d\xi A_{\nabla_j g,\xi}\hat{v}(\xi,\eta,\cdot) \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d\xi A_{g,\xi}\hat{v}_j(\xi,\eta,\cdot), \end{aligned}$$

where

$$(6.14)\qquad \hat{v}_j(\xi,\eta,\omega) = -i\left[\xi_j + \sum_{k=1}^d \xi_k \Psi_{k,j}(\xi,\eta,\omega)\right]\hat{G}_{\mathbf{a}}(\xi,\eta)\hat{f}(\xi).$$

Now we put,

$$v_j(x,\eta,\omega) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d\xi e^{-ix\cdot\xi}\hat{v}_j(\xi,\eta,\omega).$$

It is clear that $v_j(x,\eta,\omega)$, regarded as a function of (x,ω), is in $L^2(\mathbb{R}^d\times\Omega)$, whence $v_j(x,\eta,\tau_x\omega)$ is also in $L^2(\mathbb{R}^d\times\Omega)$. It follows now from (6.13) that the function $\nabla_j u(x,\eta,\omega) = v_j(x,\eta,\tau_x\omega)$ is in $L^2(\mathbb{R}^d)$ with probability 1 in ω and is the distributional derivative of $u(x,\eta,\omega)$.

Next we wish to show that with probability 1, $u(x,\eta,\cdot)$ is a weak solution of the equation (6.10). To do that we need to observe that for any $\varphi\in\mathcal{H}(\Omega)$ and C^∞ function $g:\mathbb{R}^d\to\mathbb{C}$ of compact support, one has

$$(6.15)\qquad \sum_{j,k'=1}^d A_{\nabla_j g,\xi}T_{j,k'\eta,\xi}\varphi_{k'} = -\eta A_{g,\xi}S_{\eta,\xi}\varphi + \sum_{k'=1}^d A_{\nabla_{k'}g,\xi}\varphi_{k'}.$$

We have now, for any C^∞ function $g:\mathbb{R}^d\to\mathbb{R}$ with compact support,

(6.16)

$$\begin{aligned} \int_{\mathbb{R}^d} dx \sum_{i,j=1}^d \nabla_i g(x) a_{i,j}(\tau_x\cdot)\nabla_j u(x,\eta,\cdot) &= \int_{\mathbb{R}^d} dx \sum_{i,j=1}^d \nabla_i g(x) a_{i,j}(\tau_x\cdot) v_j(x,\eta,\tau_x\cdot) \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d\xi \sum_{i,j=1}^d A_{\nabla_i g,\xi}\left[a_{i,j}(\cdot)\hat{v}_j(\xi,\eta,\cdot)\right]. \end{aligned}$$

Observe next that for any $k,\ 1\le k\le d$,

$$\begin{aligned} \sum_{i,j=1}^d A_{\nabla_i g,\xi}\left[a_{i,j}(\cdot)\Psi_{k,j}(\xi,\eta,\cdot)\right] &= \Lambda\sum_{i=1}^d A_{\nabla_i g,\xi}\Psi_{k,i}(\xi,\eta,\cdot) \\ &\quad -\Lambda\sum_{i=1}^d A_{\nabla_i g,\xi}\left[b_{i,j}(\cdot)\Psi_{k,j}(\xi,\eta,\cdot)\right]. \end{aligned}$$

If we use the fact that $\Psi_k = T_{\eta/\Lambda,\xi}\ \Psi_k^*$ and (6.15), then we see that

$$\sum_{i,j=1}^{d} A_{\nabla_i g,\xi}\Psi_{k,i}(\xi,\eta,\cdot) = -\frac{\eta}{\Lambda}A_{g,\xi}S_{\eta/\Lambda,\xi}\Psi_k^*(\xi,\eta,\cdot) + \sum_{i=1}^{d} A_{\nabla_i g,\xi}\Psi_{k,i}^*(\xi,\eta,\cdot).$$

We also have from (6.5) that

$$\begin{aligned}\mathbf{b}(\omega)\Psi_k(\xi,\eta,\omega) &= \mathbf{b}(\omega)T_{\eta/\Lambda,\xi}\Psi_k^*(\xi,\eta,\omega)\\ &= PT^*_{\mathbf{b},\eta/\Lambda,\ \xi}\Psi_k^*(\xi,\eta,\omega) + \langle \mathbf{b}(\cdot)\Psi_k(\xi,\eta,\cdot)\rangle\\ &= \Psi_k^*(\xi,\eta,\omega) + \frac{1}{\Lambda}[a_k(\omega) - \langle a_k(\cdot)\rangle] + \langle \mathbf{b}(\cdot)\Psi_k(\xi,\eta,\cdot)\rangle\\ &= \Psi_k^*(\xi,\eta,\omega) + \frac{1}{\Lambda}a_k(\omega) - \frac{1}{\Lambda}\left\langle a_k(\cdot) + \mathbf{a}(\cdot)\Psi_k(\xi,\eta,\cdot)\right\rangle.\end{aligned}$$

It follows now from the last three equations that

$$\begin{aligned}\sum_{i,j=1}^{d} A_{\nabla_i g,\xi}\left[a_{i,j}(\cdot)\Psi_{k,j}(\xi,\eta,\cdot)\right] = -\eta A_{g,\xi}\psi_k(\xi,\eta,\cdot) - \sum_{j=1}^{d} A_{\nabla_j g,\xi}a_{k,j}(\cdot)\\ + i\overline{\hat{g}(\xi)}\sum_{j=1}^{d}\xi_j q_{j,k}(\xi,\eta).\end{aligned}$$

Hence from (6.14), (6.16) we have that

$$\begin{aligned}&\int_{\mathbb{R}^d} dx \sum_{i,j=1}^{d} \nabla_i g(x)a_{i,j}(\tau_x\cdot)\nabla_j u(x,\eta,\cdot)\\ &= -\frac{\eta}{(2\pi)^d}\int_{\mathbb{R}^d} d\xi A_{g,\xi}\hat{v}(\xi,\eta,\cdot) + \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d\xi[\eta + \xi q(\xi,\eta)\xi]\hat{G}_{\mathbf{a}}(\xi,\eta)\hat{f}(\xi)\overline{\hat{g}(\xi)}\\ &= -\eta\int_{\mathbb{R}^d} dx g(x)u(x,\eta,\cdot) + \int_{\mathbb{R}^d} dx g(x)f(x),\end{aligned}$$

where we have used (6.9). The result follows from this last equation. □

Next, let $G_{\mathbf{a}}(x,y,\eta,\cdot),\ x,y\in\mathbb{R}^d$ be the Green's function for the equation (6.10). It follows easily now from Lemma 6.3 that if $G_{\mathbf{a}}(x,\eta)$ is the Fourier inverse of the function $\hat{G}_{\mathbf{a}}(\xi,\eta)$ of (6.9), then

$$G_{\mathbf{a}}(x-y,\eta) = \Big\langle G_{\mathbf{a}}(x,y,\eta,\cdot)\Big\rangle.$$

We can now use the methods of Section 3 to estimate $G_{\mathbf{a}}(x,\eta)$. We shall restrict ourselves to the case $d=3$ since the method generalizes to all $d\ge 3$. Evidently one can generalize Lemma 3.6 to obtain:

Lemma 6.4. *Let $d=3,\ \eta>0,\ 1\le k,k'\le d$. Then $q_{k,k'}(\xi,\eta)$ is a C^∞ function of $\xi\in\mathbb{R}^d$ and for any $i,j,\ 1\le i,j\le d$, the function $\partial q_{k,k'}/\partial\xi_i\in L^3_w(\mathbb{R}^d)$ and $\partial^2 q_{k,k'}/\partial\xi_i\partial\xi_j\in L^{3/2}_w(\mathbb{R}^d)$. Further, there is a constant $C_{\lambda,\Lambda}$, depending only on λ,Λ such that*

$$\|\partial q_{k,k'}/\partial\xi_i\|_{3,w}\le C_{\lambda,\Lambda},\ \ \|\partial^2 q_{k,k'}/\partial\xi_i\partial\xi_j\|_{3/2,\ w}\le C_{\lambda,\Lambda}\ .$$

We can use Lemma 6.4 to estimate $G_{\mathbf{a}}(x,\eta)$. To do this suppose $\varphi:\mathbb{R}^d\to\mathbb{R}$ is a C^∞ function of compact support satisfying

$$\int_{\mathbb{R}^d}\varphi(y)dy=1.$$

For $R>0$ we put $\varphi_R(y)=R^d\varphi(Ry)$. Evidently the Fourier transform of φ_R is a rapidly decreasing function and $\hat{\varphi}_R(\xi)=\hat{\varphi}(\xi/R)$, where $\hat{\varphi}(0)=1$. We define now the function $G_{\mathbf{a}}(x,\eta)$ by

$$G_{\mathbf{a}}(x,\eta)=\lim_{R\to\infty}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}d\xi\frac{\hat{\varphi}_R(\xi)e^{-ix\cdot\xi}}{[\eta+\xi q(\xi,\eta)\xi]}. \tag{6.17}$$

Lemma 6.5. *Let $d=3$. The function $G_{\mathbf{a}}(x,\eta)$ defined by* (6.17) *is continuous in (x,η), $x\in\mathbb{R}^d\backslash\{0\}$, $\eta>0$. The limit $G_{\mathbf{a}}(x)=\lim_{\eta\to 0}G_{\mathbf{a}}(x,\eta)$ exists for all $x\in\mathbb{R}^d\backslash\{0\}$ and $G_{\mathbf{a}}(x)$ is a continuous function of x, $x\neq 0$. Further, there is a constant $C_{\lambda,\Lambda}$ depending only on λ,Λ such that*

$$0\le G_{\mathbf{a}}(x,\eta)\le C_{\lambda,\Lambda}/|x|,\quad x\in\mathbb{R}^d\backslash\{0\},\quad \eta>0. \tag{6.18}$$

Proof. We argue as in Proposition 3.1. Thus for γ satisfying $1\le\gamma\le 2$, we write

$$G_{\mathbf{a}}(x,\eta)=\lim_{R\to\infty}\int_{|\xi|<\gamma/|x|}+\lim_{R\to\infty}\int_{|\xi|>\gamma/|x|}\ . \tag{6.19}$$

It is clear that

$$\lim_{R\to\infty}\int_{|\xi|<\gamma/|x|}=\frac{1}{(2\pi)^d}\int_{|\xi|<\gamma/|x|}d\xi\frac{e^{-ix\cdot\xi}}{[\eta+\xi q(\xi,\eta)\xi]}\ .$$

To evaluate the limit as $R\to\infty$ in the second integral on the RHS of (6.19) we integrate by parts. Thus for fixed $R>0$, assuming $x_1\neq 0$,

$$\begin{aligned}\int_{|\xi|>\gamma/|x|}&=\frac{1}{ix_1}\frac{1}{(2\pi)^d}\int_{|\xi|>\gamma/|x|}d\xi e^{-ix\cdot\xi}\frac{\partial}{\partial\xi_1}\left[\frac{\hat{\varphi}(\xi/R)}{\eta+\xi q(\xi,\eta)\xi}\right].\\ &+\frac{1}{ix_1}\frac{1}{(2\pi)^d}\int_{|\xi|=\gamma/|x|}d\xi\frac{e^{-ix\cdot\xi}\hat{\varphi}(\xi/R)\xi_1}{[\eta+\xi q(\xi,\eta)\xi]|\xi|}\ .\end{aligned} \tag{6.20}$$

Evidently for the surface integral in the last expression one has

$$\lim_{R\to\infty}\int_{|\xi|=\gamma/|x|}=\frac{1}{ix_1}\frac{1}{(2\pi)^d}\int_{|\xi|=\gamma/|x|}d\xi\frac{e^{-ix\cdot\xi}\xi_1}{[\eta+\xi q(\xi,\eta)\xi]|\xi|}\ .$$

To evaluate the limit of the volume integral in (6.20) as $R\to\infty$, we need to integrate by parts again. Thus, for the integral over $\{|\xi|>\gamma/|x|\}$ on the RHS of (6.20) one has

$$\begin{aligned}\int_{|\xi|>\gamma/|x|}&=\frac{-1}{x_1^2}\frac{1}{(2\pi)^d}\int_{|\xi|>\gamma/|x|}d\xi e^{-ix\cdot\xi}\frac{\partial^2}{\partial\xi_1^2}\left[\frac{\hat{\varphi}(\xi/R)}{\eta+\xi q(\xi,\eta)\xi}\right]\\ &-\frac{1}{x_1^2}\frac{1}{(2\pi)^d}\int_{|\xi|=\gamma/|x|}d\xi e^{-ix\cdot\xi}\frac{\partial}{\partial\xi_1}\left[\frac{\hat{\varphi}(\xi/R)}{\eta+\xi q(\xi,\eta)\xi}\right]\frac{\xi_1}{|\xi|}\ .\end{aligned} \tag{6.21}$$

In view of Lemma 6.4 it follows that the limit of the volume integral on the RHS of (6.21) is given by

$$\lim_{R\to\infty}\int_{|\xi|>\gamma/|x|}=\frac{-1}{x_1^2}\frac{1}{(2\pi)^d}\int_{|\xi|>\gamma/|x|}d\xi e^{-ix\cdot\xi}\frac{\partial^2}{\partial\xi_1^2}\left[\frac{1}{\eta+\xi q(\xi,\eta)\xi}\right].$$

We can similarly see that the limit of the average of the surface integral on the RHS of (6.21) is given by

$$\lim_{R\to\infty} \operatorname*{Av}_{1\le\gamma\le 2} \int_{|\xi|=\gamma/|x|} =$$

$$\operatorname*{Av}_{1\le\gamma\le 2} \left\{ \frac{-1}{x_1^2}\frac{1}{(2\pi)^d}\int_{|\xi|=\gamma/|x|} d\xi e^{-ix\cdot\xi}\frac{\partial}{\partial\xi_1}\left[\frac{1}{\eta+\xi q(\xi,\eta)\xi}\right]\frac{\xi_1}{|\xi|}\right\}.$$

We have therefore established a formula for the function $G_{\mathbf{a}}(x,\eta)$. It easily follows from this that $G_{\mathbf{a}}(x,\eta)$ is continuous in (x,η), $x\in\mathbb{R}^3\backslash\{0\}$, $\eta>0$. To show that the function $G_{\mathbf{a}}(x)=\lim_{\eta\to 0} G_{\mathbf{a}}(x,\eta)$ exists we observe that $q(\xi,\eta)$ converges as $\eta\to 0$ to a function $q(\xi,0)$. This follows from the fact that the operators $T_{k,k',\eta,\xi}$ of (6.1) converge strongly as $\eta\to 0$ to bounded operators $T_{k,k',0,\xi}$ on $L^2(\Omega)$. One can prove this last fact by using Bochner's Theorem. Suppose p satisfies $1<p<3$. Then $\partial q_{k,k'}(\xi,\eta)/\partial\xi_i$ can be written as a sum,

$$\partial q_{k,k'}(\xi,\eta)/\partial\xi_i = f_{k,k',i}(\xi,\eta)+g_{k,k',i}(\xi,\eta).$$

The function $f_{k,k',i}(\xi,\eta)\in L^3_w(\mathbb{R}^d)$ and converges in $L^3_w(\mathbb{R}^d)$ as $\eta\to 0$ to the distributional derivative $\partial q_{k,k'}(\xi,0)/\partial\xi_i$ of the function $\partial q_{k,k'}(\xi,0)$. The function $g_{k,k',i}(\xi,\eta)\in L^p(\mathbb{R}^d)$ and converges as $\eta\to 0$ in $L^p(\mathbb{R}^d)$ to 0. This follows by writing

$$\eta[|\xi|^2+\eta]^{-1} = f(\xi,\eta)+g(\xi,\eta),$$

where $f\in L^\infty(\mathbb{R}^d)$ $\|f(\cdot,\eta)\|_\infty\le\eta^{1/2}$ and $g\in L^\infty(\mathbb{R}^d)$ $\|g(\cdot,\eta)\|_\infty\le 1$, $g(\xi,\eta)=0$ if $|\xi|>\eta^{1/4}$. One can also make a similar statement about convergence of the derivative $\partial^2 q_{k,k'}(\xi,\eta)/\partial\xi_i\partial\xi_j$ as $\eta\to 0$. We conclude that one can take the limit as $\eta\to 0$ in the integral formula we have established for $G_{\mathbf{a}}(x,\eta)$. In view of Lemmas 6.2 and 6.4 the limiting function $G_{\mathbf{a}}(x)$ is also continuous for $x\neq 0$. Finally the inequality (6.18) follows by exactly the same argument as we used in Proposition 3.1. $\square$

We can complete the proof of Theorem 1.2 by applying the argument for the proof of Theorem 1.5 at the end of Section 3.

Lemma 6.6. *Let $d=3$ and $G_{\mathbf{a}}(x)$ be the function defined in Lemma 6.5. Then $G_{\mathbf{a}}(x)$ is a C^1 function for $x\neq 0$ and there is a constant $C_{\lambda,\Lambda}$, depending only on λ,Λ such that*

$$\left|\frac{\partial G_{\mathbf{a}}(x)}{\partial x_i}\right| \le \frac{C_{\lambda,\Lambda}}{|x|^2},\quad x\neq 0,\quad i=1,2,3. \tag{6.22}$$

Proof. Let $r > 0$ and suppose $x \in \mathbb{R}^3$ satisfies $10r < |x| < 20r$. From Lemma 6.5 we have that

$$
\begin{aligned}
(6.23) \qquad G_{\mathbf{a}}(x) = {} & \frac{1}{(2\pi)^3}\int_1^2 d\gamma \int_{|\xi|<\gamma/r} d\xi \frac{e^{-ix\cdot\xi}}{\xi q(\xi,0)\xi} \\
& + \frac{1}{ix_1}\frac{1}{(2\pi)^3}\int_1^2 d\gamma \int_{|\xi|=\gamma/r} d\xi \frac{e^{-ix\cdot\xi}}{\xi q(\xi,0)\xi}\frac{\xi_1}{|\xi|} \\
& - \frac{1}{x_1^2}\frac{1}{(2\pi)^3}\int_1^2 d\gamma \int_{|\xi|=\gamma/r} d\xi e^{-ix\cdot\xi}\frac{\partial}{\partial\xi_1}\left[\frac{1}{\xi q(\xi,0)\xi}\right]\frac{\xi_1}{|\xi|} \\
& - \frac{1}{x_1^2}\frac{1}{(2\pi)^3}\int_1^2 d\gamma \int_{|\xi|>\gamma/r} d\xi e^{-ix\cdot\xi}\frac{\partial^2}{\partial\xi_1^2}\left[\frac{1}{\xi q(\xi,0)\xi}\right],
\end{aligned}
$$

where $q(\xi, 0)$ is defined in Lemma 6.5. Let $H_{\mathbf{a}}(x)$ be the final integral on the RHS of (6.23). Then it follows from Lemmas 6.2 and 6.4 that if $x_1 \sim |x|$ then $G_{\mathbf{a}}(x) - H_{\mathbf{a}}(x)$ is a C^1 function and

$$
\left|\frac{\partial}{\partial x_j}[G_{\mathbf{a}}(x) - H_{\mathbf{a}}(x)]\right| \le \frac{C_{\lambda,\Lambda}}{|x|^2}, \quad x \neq 0, \quad j = 1, 2, 3.
$$

To show the differentiability of $H_{\mathbf{a}}(x)$ we expand

$$
(6.24) \qquad \frac{\partial^2}{\partial\xi_1^2}\left[\frac{1}{\xi q(\xi,0)\xi}\right] = \frac{-2q_{1,1}(\xi,0)}{[\xi q(\xi,0)\xi]^2} + \frac{8}{[\xi q(\xi,0)\xi]^3}\left[\mathrm{Re}\sum_{j=1}^3 q_{1,j}(\xi,0)\xi_j\right]^2,
$$

plus terms involving derivatives of $q(\xi, 0)$. The contribution of the first term on the RHS of (6.24) to $H_{\mathbf{a}}(x)$ is given by

$$
\begin{aligned}
(6.25) \quad & \frac{2}{x_1^2}\frac{1}{(2\pi)^3}\int_1^2 d\gamma \int_{|\xi|>\gamma/r} d\xi e^{-ix\cdot\xi}\frac{q_{1,1}(\xi,0)}{[\xi q(\xi,0)\xi]^2} \\
& = \frac{2}{ix_1^3}\frac{1}{(2\pi)^3}\int_1^2 d\gamma \int_{|\xi|=\gamma/r} d\xi e^{-ix\cdot\xi}\frac{q_{1,1}(\xi,0)}{[\xi q(\xi,0)\xi]^2}\frac{\xi_1}{|\xi|} \\
& \quad + \frac{2}{ix_1^3}\frac{1}{(2\pi)^3}\int_1^2 d\gamma \int_{|\xi|>\gamma/r} d\xi e^{-ix\cdot\xi}\frac{\partial}{\partial\xi_1}\left[\frac{q_{1,1}(\xi,0)}{(\xi q(\xi,0)\xi)^2}\right].
\end{aligned}
$$

Using the fact that $\partial q_{1,1}/\partial\xi_1 \in L^3_w(\mathbb{R}^3)$, it is easy to see that the RHS of (6.25) is a C^1 function of x, $x \neq 0$, and its derivative is bounded by the RHS of (6.22). The same argument can be used to estimate the contribution to $H_{\mathbf{a}}(x)$ from all terms on the RHS of (6.24) except the term involving the second derivative of $q(\xi, 0)$. The contribution to $H_{\mathbf{a}}(x)$ from this term is given by $K_{\mathbf{a}}(x)/x_1^2$, where

$$
K_{\mathbf{a}}(x) = \frac{1}{(2\pi)^3}\int_1^2 d\gamma \int_{|\xi|>\gamma/r} d\xi e^{-ix\cdot\xi}\frac{1}{[\xi q(\xi,0)\xi]^2}\frac{\xi\partial^2 q(\xi,0)\xi}{\partial\xi_1^2}.
$$

We can see now just as in Lemma 3.10 that for any $\rho \in \mathbb{R}^3$, the function $[\partial^2 q(\xi+\rho,0)/\partial\xi_1^2 - \partial^2 q(\xi,0)/\partial\xi_1^2]/|\rho|^{1-\varepsilon}$ is in $L_w^{3/(3-\varepsilon)}(\mathbb{R}^3)$ and

$$
(6.26) \qquad \|[\partial^2 q(\xi+\rho,0)/\partial\xi_1^2 - \partial^2 q(\xi,0)/\partial\xi^2]/|\rho|^{1-\varepsilon}\|_{3/(3-\varepsilon),w} \le C_{\lambda,\Lambda,\varepsilon},
$$

where ε is any number satisfying $0 < \varepsilon < 1$. For $h \in \mathbb{R}^3$ we write

$$K_{\mathbf{a}}(x+h) - K_{\mathbf{a}}(x) = \frac{1}{(2\pi)^3}\int_1^2 d\gamma \int_{|\xi|>\gamma/r} d\xi e^{-ix\cdot\xi}[g_h(\xi) - g_h(\xi+\rho)] + \int_{1/4r<|\xi|<4/r} d\xi e^{-ix\cdot\xi}[e^{-ih\cdot\xi} - 1]\frac{f(\xi)}{|\xi|^2}, \tag{6.27}$$

where

$$g(\xi) = \frac{1}{2[\xi q(\xi,0)\xi]^2}\frac{\xi\partial^2 q(\xi,0)\xi}{\partial\xi_1^2}, \quad g_h(\xi) = [e^{-ih\cdot\xi} - 1]g(\xi),$$

$f \in L_w^{3/2}(\mathbb{R}^3)$ and $\rho \in \mathbb{R}^3$ has the property that $x \cdot \rho = \pi$. In view of (6.26) and the fact that $f \in L_w^{3/2}(\mathbb{R}^3)$ it follows from the Dominated Convergence Theorem that one can take the limit in (6.27) as $h \to 0$ to obtain that $K_{\mathbf{a}}(x)$ is differentiable in x and

$$\frac{\partial K_{\mathbf{a}}(x)}{\partial x_j} = \frac{-i}{(2\pi)^3}\int_1^2 d\gamma \int_{|\xi|>\gamma/r} d\xi e^{-ix\cdot\xi}\{\xi_j[g(\xi) - g(\xi+\rho)] - \rho_j g(\xi+\rho)\} - i\int_{1/4r<|\xi|<4/r} d\xi e^{-ix\cdot\xi}\xi_j\frac{f(\xi)}{|\xi|^2}.$$

We can see from this last expression that $\partial K_{\mathbf{a}}(x)/\partial x_j$ is continuous in x and also $|\partial K_{\mathbf{a}}(x)/\partial x_j| \le C_{\lambda,\Lambda}$. □

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