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Characterizing Mildly Mixing Actions by Orbit Equivalence of Products

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Abstract. We characterize mildly mixing group actions of a noncompact, locally compact, second countable group G using orbit equivalence. We show an amenable action Φ of G is mildly mixing if and only if G is amenable and for any nonsingular ergodic G-action Ψ, the product G-action $\Phi \times \Psi$ is orbit equivalent to Ψ. We extend the result to the case of finite measure preserving noninvertible endomorphisms, i.e., when $G = \mathbb{N}$, and show that the theorem cannot be extended to include nonsingular mildly mixing endomorphisms.

Contents

1. Introduction.

The main purpose of this paper is to present a new characterization of mildly mixing group actions. Mild mixing was introduced by Furstenberg and Weiss in [11] and characterized in terms of Cartesian products with an ergodic infinite measure preserving transformation. It was later discussed in the context of nonsingular transformations by Aaronson, Lin and Weiss in [2], and generalized to nonsingular actions of locally compact groups by Schmidt and Walters in [22]. The related notion of rigid factors and their absence was discussed under a different name by Walters in [26]. For the case of amenable group actions, we characterize the

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property of mild mixing in terms of orbit equivalence. If the action is not amenable, or in the case of finite measure preserving endomorphisms, we characterize the property of mild mixing in terms of the ratio sets only. We also show that the characterization in terms of orbit equivalence does not extend to mildly mixing nonsingular endomorphisms.

The motivation for this characterization of mildly mixing, in fact for the paper, is to provide a definitive answer to a question that has arisen in the literature about determining the ratio set of a product transformation when the ratio set of each factor in the product is known. A discussion about the difficulties of defining ratio sets for nonivertible maps and of product transformations appears in Section 4. In particular, we illustrate an obstruction to testing for a ratio set value only on a dense subalgebra (like rectangles in a product space). Using a mildly mixing multiplier allows one to avoid this problem.

We assume throughout this paper that all groups are noncompact, locally compact, second countable, and that spaces are nonatomic standard Borel spaces; sometimes for convenience we complete the measure and work with nonatomic Lebesgue probability spaces. A nonsingular action Φ of a group G on a space $(X, \mathcal{B}, \mu)$ consists of an action of G on X such that the map $\Phi : G \times X \to X$ is measurable and for each $g \in G$ the map $\phi_g(x) = \Phi(g, x)$ is a nonsingular automorphism of X, i.e., ϕ_g is an invertible measurable transformation and for any $B \in \mathcal{B}$, $\mu(B) = 0$ if and only if $\mu(\phi_g^{-1}B) = \mu(\phi_{g^{-1}}B) = 0$. An action is *ergodic* if whenever $\phi_g(A) = A$ for all $g \in G$ then $\mu(A) = 0$ or 1. If an action is ergodic and μ is not concentrated on a single orbit $\{\phi_g x : g \in G\}$) we say it is *properly ergodic*. We will work only with properly ergodic actions. When no confusion arises, we will often write $\phi_g(x) \equiv g(x)$ for simplicity of notation and $G(x)$ will be used (instead of $\Phi(x)$) to denote the entire orbit of the point x under the action of G. All group actions are assumed to be free.

A G-action Φ is defined to be *mildly mixing* if, for every $B \in \mathcal{B}$ with $0 < \mu(B) < 1$,

$$\liminf_{g \to \infty} \mu(B \triangle gB) > 0.$$

We also say that the G-action *has no rigid factors* in this case. We recall the following theorem which was first proved by Furstenberg and Weiss [11] and in the generality we use here, by Schmidt and Walters [22].

Theorem 1.1. [22] *A nonsingular properly ergodic G-action Φ is mildly mixing if and only if for every nonsingular properly ergodic action Ψ of G on a space $(Y, \mathcal{F}, \nu)$, the product action $\Phi \times \Psi$ on $(X \times Y, \mathcal{B} \times \mathcal{F}, \mu \times \nu)$ given by $g(x, y) = (\phi_g(x), \psi_g(y))$ is $\mu \times \nu$ ergodic.*

Let G_1 and G_2 be two groups with actions Φ_1 on $(X_1, \mathcal{B}_1, \mu_1)$ and Φ_2 on $(X_2, \mathcal{B}_2, \mu_2)$ respectively. We say that Φ_1 is *orbit equivalent* (or *Dye equivalent*) to Φ_2 if there exists a bimeasurable, nonsingular invertible map $\zeta : (X_1, \mathcal{B}_1, \mu_1) \to (X_2, \mathcal{B}_2, \mu_2)$ such that $\zeta(G_1(x)) = G_2(\zeta x)$ for μ_1 a.e. $x \in X_1$. We will define the notion of ratio set in Section 3. Our main theorem is the following.

Main Theorem. *Assume that G is a noncompact, locally compact, second countable group. If Φ is any amenable properly ergodic nonsingular action of G on a standard Borel space $(X, \mathcal{B}, \mu)$, then the following are equivalent:*

1. *Φ is mildly mixing (and hence preserves a finite measure $\nu \cong \mu$ [22]);*

2. *For every nonsingular properly ergodic action Ψ of G on $(Y, \mathcal{F}, \nu)$, the product action $\Phi \times \Psi$ on $(X \times Y, \mathcal{B} \times \mathcal{F}, \mu \times \nu)$ given by $g(x,y) = (\phi_g(x), \psi_g(y))$ is orbit equivalent to Ψ.*

   If G is countable (i.e., discrete), then 1 and 2 are equivalent to:
3. *For every nonsingular properly ergodic action Ψ of G on $(Y, \mathcal{F}, \nu)$ such the product action $\Phi \times \Psi$ on $(X \times Y, \mathcal{B} \times \mathcal{F}, \mu \times \nu)$ given by $g(x,y) = (\phi_g(x), \psi_g(y))$ is ergodic, we have $r(\Phi \times \Psi) = r(\Psi)$, where $r(\Psi)$ denotes the Krieger ratio set of the action Ψ.*

We also prove versions of the Main Theorem for nonamenable groups and for mildly mixing finite measure preserving endomorphisms, i.e., when G is the semigroup $\mathbb{N}$.

Section 2 reduces the theorem to the case of countable amenable groups, and in Section 3 we present a proof of the theorem in this setting and its extension to continuous groups.

In Section 4 we discuss ratio sets in more detail and present an example that illustrates the difficulty of computing the ratio set of a Cartesian product and correct some gaps in the literature on this point.

Finally, Section 5 is devoted to extending these results to the case of endomorphisms and proving a version of the main theorem for this case. We show that while the main theorem holds for measure preserving mildly mixing endomorphisms, it cannot be extended to include all nonsingular mildly mixing endomorphisms.

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2. Definitions and Reduction to Countable Amenable Groups.

In this section we reduce the statement of the main theorem by applying a sequence of results about group actions from the literature. Many of the statements below are well-known. However, since not all these results appear in print, we provide complete statements of each result needed in this paper.

We first use the following theorem of Schmidt and Walters which allows us to assume from now on that all mildly mixing group actions preserve the given measure.

Theorem 2.1. [22] *Let G be a locally compact second countable group, and let $\Phi(g,x) = \phi_g(x)$ be a nonsingular properly ergodic action of G on the standard probability space $(X, \mathcal{B}, \mu)$. If μ is not equivalent to any Φ-invariant probability measure on $(X, \mathcal{B})$, then the action of G is not mildly mixing.*

In order to obtain the full strength of our main theorem, we use the fact that orbit equivalence classes of group actions have a complete classification when the group action is amenable. To avoid unnecessary technicalities about amenable actions of nonamenable groups, we state our second simplifying theorem. We recall that a group G is *amenable* if for every continuous action Φ of G on a compact metrizable space Ω, there is a Φ-invariant measure on Ω.

Theorem 2.2. [27] *If Φ is an amenable action of G on $(X, \mathcal{B}, \mu)$ and Φ preserves μ, then G is amenable.*

Corollary 2.1. *If G has a mildly mixing amenable action on $(X,\mathcal{B},\mu)$, then G is an amenable group.*

Remark 2.1. In [5], Connes, Feldman, and Weiss united the concepts of amenability and orbit equivalence by showing that a free properly ergodic action of a countable group is amenable if and only if it is orbit equivalent to a $\mathbb{Z}$-action, and a free properly ergodic action of a continuous group is amenable if and only if it is orbit equivalent to an $\mathbb{R}$-action.

Assume that G is uncountable and locally compact, and has a nonsingular free action on X. We use a countable cross section to reduce the classification problem to that of a countable orbit structure, a procedure similar to finding a cross section of a flow to represent it as a flow built under a function. Studying the orbits of the base transformation gives information about the orbits of the flow. The ideas outlined below have been written about in detail by Feldman [9].

Definition 2.1. If K is precompact with nonempty interior in G, and B is measurable in X, then B is called a *K-base* if the map $\Phi|_{K\times B}$ is one-to-one and the set KB has positive measure. The set KB is called a *K-tower*.

The existence of cross sections was shown by Forrest [10]; in particular if K is a compact subset of G acting on $(X,\mathcal{B},\mu)$, and V is any open subset of G, then for any measurable set $S\subset X$ of positive measure, there is a K-base $B\subset X$ such that $\mu(VB\cap S)>0$.

The usefulness of a K-base is to change a continuous G-orbit into a countable orbit. We need a more general notion of orbit to accomplish this.

Definition 2.2. A *discrete equivalence relation* R on $(X,\mathcal{B})$ is an equivalence relation, which, as a subset of $X\times X$ is product measurable, and each equivalence class $R(x)$ is countable. Any measure on X gives rise to a natural measure for R, and μ is said to be *nonsingular* for R if $\mu(A)=0 \iff \mu(R(A))=0$. Notions of ergodic and properly ergodic carry over analogously to the relation R. We always assume that R is nonsingular with respect to the given measure μ.

Definition 2.3. Two discrete equivalence relations R_1 on $(X_1,\mathcal{B}_1,\mu_1)$ and R_2 on $(X_2,\mathcal{B}_2,\mu_2)$ are *isomorphic* if there is a one-to-one measurable map $\psi: X_1\to X_2$ with $\mu_1\circ\psi^{-1}\sim\mu_2$, and for μ a.e. $x\in X_1$, $R_2(\psi x)=R_1(x)$. We write $R_1\cong R_2$.

Every example of a discrete equivalence relation is isomorphic to one obtained by taking a countable group G (which, in our setting, is equivalent to a discrete group G) with a nonsingular action on X and defining $R_G(x)=\{Gx\}$: i.e., the orbit relation (cf. [9]).

Given a K-base B for an uncountable action of G, we define a countable equivalence relation on B which is isomorphic to the orbit equivalence relation of a countable amenable group H. Define $R=\{(gx,x): gx,x\in B, g\in G\}$. This is a measurable subset of $X\times X$, and an equivalence class is: $R(x)=\{y\in B: y=gx$ for some $y\in B, g\in G\}$. One can show that $R(x)$ is countable for each $x\in B$. Also we have a measure for the relation R defined by $\mu_B(A)\equiv\mu(KA)$ for each measurable set $A\subset B$; with respect to this measure, R is nonsingular and ergodic if and only if the original action Φ is nonsingular and ergodic with respect to μ.

We adapt these results to our setting by proving the following result.

Proposition 2.1. *Suppose that G is a continuous amenable group, and has the following properly ergodic actions: a finite measure preserving action Φ on $(X, \mathcal{B}, \mu)$, and a nonsingular action Ψ on $(Y, \mathcal{F}, \nu)$. Then the product action $\Phi \times \Psi$ on $(X \times Y, \mathcal{B} \times \mathcal{F}, \mu \times \nu)$ given by $g(x, y) = (\phi_g(x), \psi_g(y))$ is orbit equivalent to Ψ if and only if there exists a countable group H acting on a standard space $(Z, \mathcal{D}, \rho)$, and K-bases $B \subset Y$ for Ψ, and $C \subset X \times Y$ for $\Phi \times \Psi$ such that $R_B \cong R_C \cong R_H$. Furthermore, given a compact $K \subset G$, and K-bases $B_1 \subset X$ for Φ and $B_2 \subset Y$ for Ψ, the set $C = B_1 \times B_2$ is a K- base for $\Phi \times \Psi$.*

Proof. ($\Longleftarrow$) This implication follows from the fact that if G_i, $i = 1, 2$ are uncountable groups, and each G_i action on $(X_i, \mathcal{B}_i, \mu_i)$ is almost free and properly ergodic, then the two actions are orbit equivalent if and only if they have they have bases (B_i, μ_{B_i}) on which the corresponding R_i are isomorphic; equivalently, if and only if for any bases (B_i, μ_{B_i}), either R_1 is isomorphic to some restriction of R_2 to a subset $C_2 \subset B_2$, or vice versa (cf. [9]).

($\Longrightarrow$) If Ψ is orbit equivalent to $\Phi \times \Psi$, then if Ψ preserves a measure equivalent to ν, we can choose any type *II* relation R for the R_H (cf. Definition 3.2). If Ψ is type *III*, then every R obtained will be in the same isomorphism class and again we can find a single R_H in that isomorphism class.

The last statement follows since the set $K(B_1 \times B_2)$ has positive $\mu \times \nu$ measure, and the map $(g, x, y) \longmapsto (\phi_g(x), \psi_g(y))$ from $K \times B_1 \times B_2$ to $K(B_1 \times B_2) \subset X \times Y$ is one-to-one. Suppose that $(\phi_g(x), \psi_g(y)) = (\phi_h(w), \psi_h(v))$. Then $g = h$, $x = w$, and $y = v$, by our assumptions. This concludes the proof. $\square$

3. The Orbit Equivalent Multiplier Theorem for Countable Amenable Groups.

We now let G denote an arbitrary countable group. In this section we characterize mildly mixing actions of countable amenable groups G. At the end of this section we extend the characterization to continuous groups by applying the results from the previous section. We assume that Φ denotes a properly ergodic almost free action of G on $(X, \mathcal{B}, \mu)$.

3.1. **Orbit equivalence theory for countable amenable groups.** The notion of ratio set for a countable group of ergodic automorphisms was introduced by Krieger as an invariant under orbit equivalence of nonsingular automorphisms [18, 19]. In this section it is convenient to assume that $(X, \mathcal{B}, \mu)$ is a Lebesgue probability space. Since Φ is a nonsingular action, for each $g \in G$ the measure $\mu\phi_g(A) \equiv \mu(\phi_g A)$ is equivalent to μ and the Radon-Nikodym derivative $\frac{d\mu\phi_g}{d\mu}$ exists and is positive a.e.

Definition 3.1. We denote by $r_\mu(\Phi)$ the set of nonnegative numbers λ satisfying: for any $\varepsilon > 0$ and any set A with $\mu(A) > 0$, there exists a ϕ_g such that:

$$\mu(A \cap \phi_{-g} A \cap \{x : |\frac{d\mu\phi_g}{d\mu}(x) - \lambda| < \varepsilon\}) > 0.$$

Many properties of $r_\mu(\Phi)$ are proved in [19] and [13]. In particular, $r_\mu(\Phi)$ depends only on the measure class of μ; we will therefore denote $r_\mu(\Phi)$ by $r(\Phi)$, and call it the *ratio set of* Φ. It is an invariant of orbit equivalence, but not a complete invariant unless $r(\Phi) = \{\lambda^n : n \in \mathbf{Z}\}$ for some $\lambda \in (0, 1)$ or $r(\Phi) = \mathbb{R}^+ \cup \{0\}$.

Definition 3.2. The action Φ is defined to be of type *II* if $r(\Phi) = \{1\}$; this case occurs if and only if Φ admits a σ-finite invariant measure $\nu \sim \mu$. If $\nu(X) = 1$, then we say Φ is type II_1; if $\nu(X) = \infty$, then we say Φ is of type II_∞. Otherwise, $0 \in r(\Phi)$, and we say Φ is of type *III*.

All ergodic type II_1 countable amenable G-actions are orbit equivalent; this was proved for abelian groups by H. Dye [7], and extended to this generality in [5]. Also, all type II_∞ form a single (distinct) orbit equivalence class as well [7, 5]. It was over a decade later that the rich structure of orbit equivalence classes of hyperfinite type *III* group actions was discovered by Krieger [19] and extended to include all amenable actions in [5].

Poincaré flows for G-actions. For each $g \in G$, we consider the automorphism ϕ_g given by the action Φ, and we define a related automorphism ϕ_g^* on the product space $(X \times \mathbb{R}, \mathcal{B} \times \mathcal{B}_{\mathbb{R}}, \mu \times e^t dt)$ by $\phi_g^*(x,t) = (\phi_g x, t - \log(\frac{d\mu\phi_g}{d\mu}(x)))$ for all $(x,t) \in X \times \mathbb{R}$. We denote by Φ^* the action of all ϕ_g^*, $g \in G$. Let $\zeta(\Phi)$ denote a measurable partition of $X \times \mathbb{R}$ which generates all Φ^*- invariant sets, and let π_Φ denote the natural surjection from $X \times \mathbb{R}$ onto the Lebesgue space $(Z, \mathcal{S}, \rho) \cong X \times \mathbb{R}/\zeta(\Phi)$; that is, π_Φ is a factor map with respect to the ergodic decomposition of Φ^*. We define a flow on $X \times \mathbb{R}$ by $F_s(x,t) = (x, t+s), s \in \mathbb{R}$. Since Φ^* commutes with F_s, the image under π_Φ of F_s is a flow defined by

$$U_s(\pi_\Phi(x,t)) = \pi_\Phi(F_s(x,t)).$$

Using this procedure we obtain a measurable decomposition of $\mu \times e^t dt$ into measures $\{q_z : z \in Z\}$ such that for ρ-a.e. $z \in Z$, q_z is an ergodic invariant measure (infinite but σ-finite) for the G-action given by Φ^*. Krieger proved that orbit equivalent ergodic actions give rise to isomorphic flows, and every ergodic, nonsingular aperiodic flow arises in this way. We will call the flow U_s on Z *the Poincaré flow of* Φ.

Theorem 3.1. [19] *Let G_1 and G_2 be two countable amenable groups with ergodic type III actions Φ_1 on $(X_1, \mathcal{B}_1, \mu_1)$ and Φ_2 on $(X_2, \mathcal{B}_2, \mu_2)$ respectively. Then Φ_1 and Φ_2 are orbit equivalent if and only if their Poincaré flows are isomorphic.*

When T denotes a nonsingular automorphism of $(X, \mathcal{B}, \mu)$, we write (X^*, μ^*, T^*) for the infinite measure preserving skew product defined above. This is called the Maharam skew product [20].

3.2. **Using orbit equivalence to characterize mildly mixing actions.** We first prove the main theorem for countable groups G and then extend the result to continuous groups. Our assumptions on G imply that G is countable if and only if G is discrete.

Theorem 3.2 (Countable Orbit Equivalence Multiplier Theorem)**.** *Let Φ be any amenable, nonsingular, properly ergodic action of a countable group G on $(X, \mathcal{B}, \mu)$. Then the following are equivalent*:

1. *Φ is mildly mixing (and therefore of type II_1);*
2. *G is amenable and for every nonsingular properly ergodic action Ψ of G on $(Y, \mathcal{F}, \nu)$, the product G-action $\Phi \times \Psi$ on $(X \times Y, \mathcal{B} \times \mathcal{F}, \mu \times \nu)$ given by $g(x,y) = (\phi_g(x), \psi_g(y))$ is orbit equivalent to Ψ;*

3. *For every nonsingular properly ergodic action Ψ of G on $(Y, \mathcal{F}, \nu)$, the product action $\Phi \times \Psi$ on $(X \times Y, \mathcal{B} \times \mathcal{F}, \mu \times \nu)$ is ergodic and $r(\Psi) = r(\Phi \times \Psi)$.*

Proof. By Theorem 1.1 it is clear that $2 \Longrightarrow 1$ since the property of ergodicity is invariant under orbit equivalence.

It is also clear that $2 \Longrightarrow 3$, since the ratio set is invariant under orbit equivalence.

It is trivial that $3 \Longrightarrow 1$.

Now we show that $1 \Longrightarrow 2$. By Theorem 2.1 we can assume that the action given by Φ preserves μ (by replacing μ by an equivalent probability measure if necessary.) We assume first that Ψ is of type *III*, and denote by U_s its Poincaré flow on $(Z, \mathcal{S}, \rho) \cong Y \times \mathbb{R} \ /\zeta(\Psi)$. Therefore, Ψ^* is an ergodic type II_∞ transformation with respect to the measure q_z for ρ-a.e. $z \in Z$. (Note that if Ψ is of type III_1, then Ψ^* is an ergodic type II_∞ transformation with respect to the measure $\nu \times e^t dt$ and Z is a single point.) We now use the G action given by Ψ^* on $(Y \times \mathbb{R}, \mathcal{S} \times B_{\mathbb{R}}, \nu \times e^t dt)$ as the multiplier, and our hypothesis and Theorem 1.1 imply that $\Phi \times \Psi^*$ is ergodic with respect to $\mu \times q_z$ for ρ-a.e. $z \in Z$. So we obtain an ergodic decomposition of $\Phi \times \Psi^*$ with respect to the measure $\mu \times \nu$ which is indexed by points in Z with the measure ρ. By the uniqueness of ergodic decompositions, we have shown that the ergodic decomposition of $\Phi \times \Psi^*$ with respect to the measure $\mu \times \nu$ is isomorphic to that of Ψ^* with respect to ν. We now consider the Maharam skew product G action on $(X \times Y \times \mathbb{R}, \mathcal{B} \times S \times B_{\mathbb{R}}, \mu \times \nu \times e^t dt)$ given by $(\Phi \times \Psi)^*$. Since $(\Phi \times \Psi)^* (x, y, s) = \Phi \times \Psi^*(x, y, s)$, these two actions clearly have the same ergodic decomposition. Therefore applying the above result, the ergodic decompositions of $(\Phi \times \Psi)^*$ and Ψ^* are the same, so the resulting Poincaré flows are isomorphic. By Theorem 3.1, this implies that $\Phi \times \Psi$ is orbit equivalent to Ψ.

It remains to show that if Ψ is a type II_1 action then so is $\Phi \times \Psi$, and if Ψ is type II_∞ then so is $\Phi \times \Psi$. This follows immediately since Φ preserves μ, so $\Phi \times \Psi$ will be ergodic finite measure preserving as well. By the results of [7] and [5] discussed in 3.1, we have that all type II_1 actions of a countable amenable group are orbit equivalent. The type II_∞ case follows for the same reason. □

Example 3.1. Suppose that T is a type *III* ergodic automorphism of a Lebesgue probability space $(X, \mathcal{B}, \mu)$, and let R_α denote rotation by α on the circle. Then for a generic value of α, the product automorphism (T, R_α) is orbit equivalent to T [4]. However since R_α is not mildly mixing, the theorem shows that for each α there will always be some ergodic automorphism T for which the product cannot be orbit equivalent to T.

Using the idea of the proof in Theorem 3.2, we obtain the following corollary for countable nonamenable groups G. In this generality we cannot draw conclusions about the oribt equivalence of mildly mixing actions, only about their ratio sets and Poincaré flows.

Corollary 3.1. *If G is any countable group, and Φ is any nonsingular, properly ergodic action of G on $(X, \mathcal{B}, \mu)$, Then the following are equivalent:*

1. *Φ is mildly mixing (and therefore of type II_1);*
2. *For every nonsingular properly ergodic action Ψ of G on $(Y, \mathcal{F}, \nu)$, the product G-action $\Phi \times \Psi$ on $(X \times Y, \mathcal{B} \times \mathcal{F}, \mu \times \nu)$ is ergodic and $r(\Psi) = r(\Phi \times \Psi)$;*

3. *For every nonsingular properly ergodic action Ψ of G on $(Y, \mathcal{F}, \nu)$ with Poincaré flow U_s, the product action $\Phi \times \Psi$ on $(X \times Y, \mathcal{B} \times \mathcal{F}, \mu \times \nu)$ is ergodic and has Poincaré flow isomorphic to U_s.*

Proof. $3 \implies 2$ and $2 \implies 1$ are obvious (using Theorem 1.1). To show that $1 \implies 2$, we consider any nonsingular properly ergodic action Ψ, and we use the proof from Theorem 3.2, $1 \implies 2$, verbatim to conclude that the Poincaré flows of Ψ and $\Phi \times \Psi$ are isomorphic; this concludes the proof. □

We now prove the main theorem for continuous groups.

Theorem 3.3 (Continuous Orbit Equivalence Multiplier Theorem). *Assume G is a noncompact, continuous, locally compact, second countable group and Φ is any properly ergodic nonsingular amenable action of G on $(X, \mathcal{B}, \mu)$, then the following are equivalent:*

1. *Φ is mildly mixing (and hence of type II_1);*
2. *For every nonsingular properly ergodic action Ψ of G on $(Y, \mathcal{F}, \nu)$, the product action $\Phi \times \Psi$ on $(X \times Y, \mathcal{B} \times \mathcal{F}, \mu \times \nu)$ given by $g(x, y) = (\phi_g(x), \psi_g(y))$ is orbit equivalent to Ψ.*

Proof. We have that $2 \implies 1$ since ergodicity is invariant under orbit equivalence, so the ergodicity of Ψ will force the ergodicity of $\Phi \times \Psi$, which, in turn implies mild mixing of Φ, using Theorem 1.1.

To show that $1 \implies 2$, we assume that Φ is mild mixing. Then we can fix a compact set K of G and obtain a K-base B_1 and a countable amenable group H, whose orbits generate R_Φ. We can assume by by Remark 2.1 that $H = \mathbb{Z}$, so is generated by a single automorphism T. By Proposition 2.1 the action generated by T is mildly mixing if and only if the original G-action is. Given the action Ψ, we similarly obtain a K-base B_2 with a nonsingular ergodic automorphism S (i.e., a $\mathbb{Z}$-action) generating R_Ψ. Since $C = B_1 \times B_2$ is a K-base, the ergodicity of R_Ψ, R_Φ, and $R_{\Phi \times \Psi}$ will follow since T is mildly mixing, and will give the result. □

4. Ratio Sets for Group Actions and Endomorphisms.

This paper was motivated by the question of when a finite measure preserving endomorphism T preserves the ratio set of its multiplier S in the product $T \times S$. More generally, when can one compute the ratio set of a transformation by testing the defining condition (only) on a dense sub-σ-algebra of sets? A partial answer to this question appears in [21].

In order to correct some incomplete proofs in the literature on ratio sets of Cartesian products of transformations ([3, Lemma 3.2 and Theorem 3.3] and [15, Theorem 3.6]), we include a short discussion here showing that the proofs are incomplete since only rectangles were checked in the product spaces. The results in the last section of this paper complete those results.

4.1. **Ratio sets of countable group actions.** In general it is important to determine whether or not a particular value λ is in the ratio set of a given action Φ of a countable group G. Simplifications of the defining condition are usually necessary in order to calculate the ratio set of a countable group of automorphisms. One such result, using the full group of an invertible action, appears in [21].

The following example, similar to one given in [4], shows that in order to guarantee that the value λ be in the ratio set of a transformation it is not sufficient to check the defining condition on a countable dense subalgebra. In particular, when computing a value in the ratio set of a Cartesian product it is not enough to verify the property on product sets. We write $r(G)$ for the ratio set throughout this section because the action of G will not vary.

Example 4.1. We fix $X = \prod_{j=1}^{\infty}\{0,1\}_j$, and we give this compact space the σ-algebra $\mathcal{B}$ of Borel sets. We define Γ to be the group of transformations on X generated by $\delta_k(x) = x_j + 1 \pmod 2$ if $j = k$, and x_k if $j \neq k$. If we put any nonsingular measure for Γ on X, it is well-known that the orbits of the Γ-action are identical to the orbits of the usual adding machine or odometer (add 1 and carry). In fact, every countable amenable group action is orbit equivalent to this action of Γ with respect to some nonsingular μ [19]. We use the Γ notation in order to see precisely which coordinates change under the group action.

We define a product measure $\mu = \prod_{j=1}^{\infty}\mu_j$ as follows. We fix $\lambda \in (0,1)$. For each $j = 2q$, $q \in \mathbb{N}$, we define $\mu_j(0) = \frac{1}{2} = \mu_j(1)$. For each $j = 4q-1$, $q \in \mathbb{N}$, we define

$$\mu_j(0) = 1 - \frac{1}{j^2} = 1 - \frac{1}{(4q-1)^2}, \mu_j(1) = \frac{1}{j^2} = \frac{1}{(4q-1)^2};$$

for each $j = 4q-3$, $q \in \mathbb{N}$, we define

$$\mu_j(0) = 1 - \frac{\lambda}{(4q-1)^2}; \mu_j(1) = \frac{\lambda}{(4q-1)^2}.$$

In other words, for all even j, we have the $(\frac{1}{2},\frac{1}{2})$ measure, and for odd indices j we have two measures which give a ratio of λ.

We can show that $r(\Gamma) = \{1\}$; to do this, it is enough to produce a measurable set C, $\mu(C) > 0$, satisfying the condition

$$\mu(C \cap \gamma^{-1}C) > 0, \gamma \in \Gamma \Rightarrow \frac{d\mu\gamma}{d\mu}(x) = 1$$

for all points x in $C \cap \gamma^{-1}C$. This will imply that $\beta \notin r(\Gamma)$ for any $\beta \neq 1$. We claim that the set $C = \{x : x_{2q+1} = 0; q \in \mathbb{N}\}$ has these properties. Clearly whenever C gets mapped back onto itself, only even coordinates can change, so the Radon-Nikodym derivative condition holds. It remains to show that $\mu(C) > 0$. This follows from the fact that $\prod_{q=1}^{\infty}(1 - \frac{1}{(4q-1)^2}) > 0$.

If we now consider the countable dense subalgebra generated by: $B_j^i = \{x : x_j = i, j \in \mathbb{N}, i \in \{0,1\}\}$, then clearly we obtain the countable dense subalgebra of $\mathcal{B}$, call it $\mathcal{B}_o$ consisting of the usual cylinder sets. We claim that on cylinder sets we appear to see the value λ in the ratio set, in the sense that the defining property holds.

First we remark that

$$\lim_{q\to\infty} \frac{\lambda \cdot (q^2-1)}{q^2 - \lambda} = \lambda;$$

we now consider any set of the form: $C_p = \{x : x_{j_1} = i_1, \ldots, x_{j_p} = i_p\} \in \mathcal{B}_o$. Given any $\varepsilon > 0$, we first find Q large enough so that $|\frac{\lambda\cdot(q^2-1)}{q^2-\lambda} - \lambda| < \varepsilon$ for all $q > Q$. We now choose $j = 4q-1$ and such that $j > \max\{j_p + 2, Q+2\}$. We define an element $\gamma \in \Gamma$ as follows:

Consider the set

$$C^{01}_{j-2,j} = \{x \in C_p : x_j = 0, x_{j-2} = x_{4q-3} = 1\}.$$

Define $\gamma = \delta_j \cdot \delta_{j-2}$. Then

$$\gamma(C^{01}_{j-2,j}) = \{x \in C_p : x_j = 1, x_{j-2} = 0\} = C^{10}_{j-2,j} \subset C_p,$$

and

$$\frac{d\mu\gamma}{d\mu}(x) = \frac{\lambda \cdot (j^2 - 1)}{j^2 - \lambda},$$

which is within ε of λ.

Therefore,

$$\mu(C_p \cap \gamma^{-1} C_p \cap \{x : |\frac{d\mu\gamma}{d\mu}(x) - \lambda| < \varepsilon\}) > 0,$$

and this holds for any cylinder C_p, even though λ is not in the ratio set $r(\Gamma)$.

4.2. **Endomorphisms and ratio sets.** A dichotomy occurs when one considers a noninvertible, nonsingular, ergodic conservative endomorphism T of a standard probability space $(X, \mathcal{B}, \mu)$. We will denote by ω_μ the Radon-Nikodym derivative of T (which is not a priori uniquely defined), and consider only the unique $T^{-1}\mathcal{B}$ measurable function satisfying:

$$\int f \circ T \cdot \omega_\mu d\mu = \int f d\mu$$

for every nonnegative integrable function f. The Radon-Nikodym derivative determines an $\mathbb{R}^+$-valued cocycle for the $\mathbb{N}$ action given by: for all $n > 0$,

$$\omega_\mu(n, x) = \prod_{i=0}^{n-1} \omega_\mu(T^i x).$$

With respect to the finite measure μ, either $\sum_{i=0}^{\infty} \omega_\mu(i, x) = \infty$ for a.e. x, in which case we say that μ is a *recurrent measure* for T, or $\sum_{i=0}^{\infty} \omega_\mu(i, x) < \infty$ for a.e. x, in which case we say that μ is a *nonrecurrent measure* for T. This notion was introduced in [23] and studied further in [24, 8, 14].

In [14] the authors defined the concept of ratio set for endomorphisms exactly as in Definition 3.1 above and showed that $r_\mu(T)$ *does* depend on the representative in the measure class of μ, and that $r_\mu(T) \cap \mathbb{R}^+$ is a closed subgroup of $\mathbb{R}^+$ if and only if μ is recurrent. In [24] it was shown that if μ is a recurrent measure for T, then T admits a σ-finite measure $\nu \cong \mu$ if and only if ω_μ is a coboundary; i.e.,

$$\omega_\mu = \frac{f \circ T}{f}$$

for some positive measurable function f. It follows then that this is also equivalent to saying $r_\mu(T) = \{1\}$ [15].

In the next section we extend our results to the case of endomorphisms, where we must take into account the special nature of ratio sets for noninvertible maps.

5. The Orbit Equivalent Multiplier Theorem for Endomorphisms.

In this section we extend the orbit equivalence characterization theorem to finite measure preserving endomorphisms. We also show that the main theorem does not extend to arbitrary nonsingular mildly mixing endomorphisms.

The equivalent characterizations of mildly mixing group actions given in Theorem 3.2 are no longer equivalent in the noninvertible setting. We generalize the definition presented in our introduction and compare it to another generalization that has been studied [1, 2]. Definition 5.1, a condition on sets, leads to simpler proofs of our main results; in the measure preserving and invertible case Definitions 5.1 and 5.2 are the same. We show most theorems hold for either definition though it is not known if they are equivalent.

In this section we will always work with nonatomic Lebesgue probability spaces. Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a *nonsingular endomorphism*; i.e., T is measurable and $\mu(A) = 0$ if and only if $\mu(T^{-1}(A)) = 0$. We recall that T is *ergodic* if for all $A \in \mathcal{B}$ with $T^{-1}A = A$, $\mu(A) = 0$ or 1. T is *conservative* if for every A with $0 < \mu(A) \leq 1$ there is an integer $n > 0$ such that $\mu(A \cap T^{-n}(A)) > 0$.

It follows then that T is conservative ergodic if and only if for all measurable sets A and B with $0 < \mu(A) \leq 1$, $0 < \mu(B) \leq 1$ there is an integer $n > 0$ such that $\mu(B \cap T^{-n}(A)) > 0$. This is equivalent to the condition that for all measurable A with $0 < \mu(A) < 1$ there is an integer $n > 0$ such that $\mu(A^c \cap T^{-n}(A)) > 0$ (where A^c denotes the complement of A).

When $T^{-1}\mathcal{B} = \mathcal{B}(\mu \bmod 0)$, then T is called an *automorphism* and a measurable inverse T^{-1} exists. If $T^{-1}\mathcal{B} \neq \mathcal{B}(\mu \bmod 0)$, then we call T *noninvertible*.

Definition 5.1. A nonsingular endomorphism T is *mildly mixing on sets* if

$$\liminf_{n\to\infty} \mu(A \triangle T^{-n}(A)) > 0$$

for all sets A with $0 < \mu(A) < 1$.

It is obvious from the definition that mildly mixing on sets implies ergodic.

Definition 5.2. [1] A nonsingular endomorphism T is *mildly mixing* if for all $f \in L^\infty$, $n_k \to \infty$, $f_{n_k} = f \circ T^{n_k} \to f$ weak-$*$ in L^∞ implies f is constant μ a.e.

Proposition 5.1. *Let T be a nonsingular endomorphism on $(X, \mathcal{B}, \mu)$.*

1. *If T is mildly mixing, then T is mildly mixing on sets.*
2. *If T is mildly mixing on sets, then for any $f = \chi_A \in L^\infty$, $n_k \to \infty$, if $f_{n_k} = f \circ T^{n_k} \to f$ weak-$*$ in L^∞ then f is constant a.e.*

Proof. (1): If T it is not mildly mixing on sets, then there exists a measurable set A, $0 < \mu(A) < 1$, and a subsequence n_k such that $n_k \to \infty$ as $k \to \infty$, and

$$\lim_{k\to\infty} ||\chi_A - \chi_{T^{-n_k}A}||_1 = 0.$$

It is straightforward to show that convergence of f_{n_k} to f, $f \in L^\infty$, in the L^1 norm implies convergence weak-$*$ in L^∞. So $\chi_A \to \chi_{T^{-n_k}A}$ weak-$*$ in L^∞ which is a contradiction, since the assumption implies that χ_A is constant a.e.; i.e., $\mu(A) = 0$ or 1.

(2): Suppose T is mildly mixing on sets and there is a measurable function of the form $f = \chi_A$, and a subsequence n_k such that $f \circ T^{n_k} \to f$ weak-$*$ in L^∞.

Then for all $g \in L^1$,

$$\int g \cdot \chi_{T^{-n_k}A} d\mu \to \int g \cdot \chi_A d\mu,$$

or,

$$|\int g \cdot \chi_{T^{-n_k}A} d\mu - \int g \cdot \chi_A d\mu| \to 0,$$

or,

$$|\int g \cdot (\chi_{T^{-n_k}A} - \chi_A) d\mu| \to 0.$$

Choosing $g = \chi_A$, we have

$$|\int \chi_A \cdot (\chi_{T^{-n_k}A} d\mu - \chi_A) d\mu| \to 0,$$

so

$$\int (\chi_A - \chi_{A \cap T^{-n_k}A}) d\mu \to 0,$$

and we have

$$\mu(A \cap T^{-n_k}A) \to \mu(A)$$

and therefore

$$\mu(A \setminus A \cap T^{-n_k}A) \to 0.$$

Similarly, using $g = \chi_{A^c}$,

$$\int \chi_{A^c} \cdot (\chi_{T^{-n_k}A} - \chi_A) d\mu \to 0,$$

which gives us that

$$\mu(T^{-n_k}A \setminus A) \to 0.$$

Therefore by our hypothesis, $\mu(A) = 0$ or 1. □

Remark 5.1.

1. When T is type II_1 or invertible the proof of Propositon 5.1 can be extended to show that the two definitions coincide. This also follows from the orbit equivalence characterization which is proved later in this section.
2. If T is exact (i.e., $T^{-n} \circ T^n(A) = A$ a.e. for all $n \in \mathbb{N}$ implies $\mu(A) = 0$ or 1), then T is clearly mildly mixing on sets. This is true for mildly mixing as well (Proposition 2.7.4, [1]).
3. There exist non-conservative endomorphisms which are mildly mixing on sets. One can easily construct an exact non-conservative transformation and the above remark implies this will be mildly mixing on sets.

In the next proposition, we show that the Countable Orbit Equivalence Multiplier Theorem (Theorem 3.2) cannot be extended to the case of nonsingular endomorphisms; however in Theorem 5.2 we prove it for finite measure preserving ones.

Proposition 5.2. *There exists an ergodic conservative nonsingular type III endomorphism T on $(X, \mathcal{B}, \mu)$, with μ recurrent, such that:*

1. *T is mildly mixing (and hence mildly mixing on sets); and*
2. *for any finite measure preserving weak mixing automorphism S of $(Y, \mathcal{F}, \nu)$, $T \times S$ is ergodic but $r_{\mu\times\nu}(T \times S) \neq r_\nu(S)$.*

Proof. Let T_H be the type *III* conservative ergodic nonsingular automorphism with an independent generator constructed by Hamachi in [12]. It is known that T_H is the natural extension of an exact conservative endomorphism T and that T is type *III* with respect to a recurrent measure μ (see [6] or [25]). Since T is exact, by the remark above T is mildly mixing. If S is any type II_1 weak mixing automorphism then by [2] and [15], $T \times S$ is ergodic and type *III*, thus has a different ratio set from that of S. $\square$

The next theorem was proved by Aaronson, Lin, and Weiss for nonsingular mildly mixing endomorphisms (cf. [1, Theorem 2.7.6]). We include it as it holds for mildly mixing on sets as well. By Remark 3 above, it cannot be strengthened to include conservativity, but Proposition 5.4 below gives the related result which yields conservative products.

Theorem 5.1. *If T is a nonsingular endomorphism which is mildly mixing on sets, then for any ergodic nonsingular automorphism S, $T \times S$ is ergodic.*

Proof. We assume that A is an invariant set for $T \times S$, and let $f = \chi_A$. Then we use the proof exactly as in [1], applying Proposition 5.1(2) above to conclude that f must be constant a.e. $\square$

For the remainder of this section we restrict to finite measure preserving mildly mixing endomorphisms where the two definitions coincide. Therefore the terminology mildly mixing is used unambiguously. We characterize mildly mixing in the finite measure preserving case, again using a set condition.

Proposition 5.3. *Let T be a nonsingular endomorphism. T is type II_1 mildly mixing if and only if*

$$\liminf_{n\to\infty} \mu(A^c \cap T^{-n}(A)) > 0$$

for all sets A with $0 < \mu(A) < 1$.

Proof. ($\Longleftarrow$) We recall the well-known fact (cf. [17], p.143) that a nonsingular endomorphism T is type II_1 if and only if for all sets A with $\mu(A) > 0$, $\inf_{n>0} \mu(T^{-n}(A)) > 0$. Now $\liminf \mu(T^{-n}(A)) \geq \liminf \mu(A^c \cap T^{-n}(A)) > 0$ for all sets A with $1 > \mu(A) > 0$ and so T is type II_1. Since

$$A \triangle T^{-n}(A) = [A^c \cap T^{-n}A] \cup [A \cap T^{-n}A^c],$$

it follows that T is mildy mixing.

($\Longrightarrow$) Assume that T is a type II_1 mildly mixing endomorphism and that μ is the invariant measure. Assume that there exists a sequence $\{n_i\}$ tending to ∞ such that

$$\lim_{i\to\infty} \mu(A^c \cap T^{-n_i}A) = 0.$$

Note that

$$\mu(A) = \mu(T^{-n_i}A) = \mu(A^c \cap T^{-n_i}A) + \mu(A \cap T^{-n_i}A).$$

The assumption implies that

$$\lim_{i\to\infty} \mu(A \cap T^{-n_i}A) = \mu(A).$$

But $\mu(A) = \mu(A \cap T^{-n_i} A) + \mu(A \cap T^{-n_i} A^c)$, therefore $\lim_{i\to\infty} \mu(A \cap T^{-n_i} A^c) = 0$, and so

$$\lim_{i\to\infty} \mu(A \triangle T^{-n_i} A) = 0,$$

which is a contradiction. □

We use Proposition 5.3 to generalize the Furstenberg-Weiss result [11], Theorem 1.1 for automorphisms, to our noninvertible setting. In addition we strengthen it by proving it for any nonsingular noninvertible multiplier; a related result is stated in [1].

Our proof is based on Definition 5.1 and an idea used by King [16, Theorem 4].

Proposition 5.4. *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a mildly mixing finite measure preserving endomorphism. If $S : (Y, \mathcal{F}, \nu) \to (Y, \mathcal{F}, \nu)$ is a conservative ergodic nonsingular endomorphism then $T \times S$ is conservative ergodic.*

Proof. Let $A \in \mathcal{B} \times \mathcal{F}$ with $0 < (\mu \times \nu)(A) < 1$. It is enough to show that there exists an $n > 0$ such that:

$$\mu \times \nu(A^c \cap (T \times S)^{-n}(A)) > 0 \tag{1}$$

First suppose A is, up to a set of $\mu \times \nu$ measure 0, of the form $A = X \times A_1$. Then since S is conservative ergodic, it is obvious that condition (1) holds.

Now suppose that A is not of the above form. We let $A_y = \{x \in X : (x, y) \in A\}$ denote the cross section of A over y. Using a vector-valued version of Lusin's Theorem, continuity of the map $y \mapsto \chi_{A_y}$ on arbitrarily large sets implies the existence of a set $B \in \mathcal{B}$ with $\mu(B) > 0$ so that for all $\delta > 0$, if $M(\delta) = \{y : \mu(B \triangle A_y) < \delta\}$ then $\nu(M(\delta)) > 0$. Since B can be chosen to be A_y for ν a.e. y, we can find B such that $0 < \mu(B) < 1$.

Now since T is mildly mixing we can find $N > 0$ and $\delta > 0$ such that for all $n > N$,

$$\mu(B^c \cap T^{-n} B) > 3\delta.$$

Using that

$$(A^c \cap (T \times S)^{-n} A)_y = (A^c)_y \cap T^{-n}(A_{S^n y}),$$

that $B \triangle A_y = B^c \triangle (A_y)^c = B^c \triangle (A^c)_y$ and the definition of $M(\delta)$ we obtain,

$$\begin{aligned} \mu \times \nu(A^c \cap (T \times S)^{-n} A) &= \int_Y \mu((A^c)_y \cap T^{-n}(A_{S^n y})) d\nu(y) \\ &\geq \int_{M(\delta)} (\mu(B^c \cap T^{-n} A_{S^n y}) - \delta) d\nu(y) \\ &> \int_{M(\delta) \cap S^{-n} M(\delta)} (\mu(B^c \cap T^{-n} B) - 2\delta) d\nu(y), \end{aligned}$$

since T preserves μ and $S^n y \in M(\delta)$ if and only if $y \in S^{-n} M(\delta)$. Thus

$$\mu \times \nu(A^c \cap (T \times S)^{-n} A) \geq \int_{M(\delta) \cap S^{-n} M(\delta)} \delta d\nu(y).$$

Since $\nu(M(\delta) \cap S^{-n} M(\delta)) > 0$ for infinitely many n (by the conservativity of S), this last integral is positive for some $n > N$, and thus (1) is satsified and so $T \times S$ is conservative ergodic. □

Every finite measure preserving endomorphism admits an invertible natural extension, and the following well-known result allows us to apply group action results obtained earlier. The authors include a short proof for completeness.

Proposition 5.5. *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a finite measure preserving endomorphism and $\tilde{T} : (\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu}) \to (\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$ its natural extension. Then T is mildly mixing if and only if $\tilde{T}$ is mildly mixing.*

Proof. There is a measure preserving map $\phi : \tilde{X} \to X$ such that $\phi \circ \tilde{T} = T \circ \phi$ a.e., and the sigma-algebra $\mathcal{G} = \phi^{-1}\mathcal{B}$ is such that $\mathcal{G} \subset \tilde{T}^{-1}\mathcal{G} \subset \tilde{T}^{-2}\mathcal{G} \subset \cdots$ and $\cup_{n>0}\tilde{T}^n\mathcal{G}$ generates $\tilde{\mathcal{B}}$ (this corresponds to the fact that the decomposition on $\tilde{X}$ that gives the endomorphism is exhaustive).

($\Longrightarrow$): Suppose there exists a set A with $0 < \tilde{\mu}(A) < 1$ and

$$\liminf_{n\to\infty} \tilde{\mu}(A \triangle \tilde{T}^{-n}(A)) = 0.$$

Let $\varepsilon > 0$.

Since A is measurable there exists k and $E \in \tilde{T}^k\mathcal{G}$ such that $\tilde{\mu}(A \triangle E) < \varepsilon$. Given k there exists $n > k$ such that $\tilde{\mu}(A \triangle \tilde{T}^{-n}A) < \varepsilon$. Since E is also in $\tilde{T}^n\mathcal{G}$, there exists $G \in \mathcal{G}$ such that $\tilde{\mu}(A \triangle \tilde{T}^n G) < \varepsilon$. Thus $\tilde{\mu}(\tilde{T}^{-n}A \triangle G) < \varepsilon$, which implies $\tilde{\mu}(A \triangle G) < 2\varepsilon$. Therefore $A \in \mathcal{G}, (\tilde{\mu} \mod 0)$. Thus $A = \phi^{-1}B$ for some $B \in \mathcal{B}$. Now $\tilde{\mu}(A \triangle \tilde{T}^{-n}A) = \tilde{\mu}(\phi^{-1}B \triangle \tilde{T}^{-n}\phi^{-1}B) = \tilde{\mu} \circ \phi^{-1}(B \triangle T^{-n}B) = \mu(B \triangle T^{-n}B)$. Therefore T cannot be mildly mixing, and the contradiction implies the theorem.

The converse is clear. □

The converse of Proposition 5.4 now follows easily from Theorem 1.1, and the fact that $T \times S$ is ergodic if and only if $\tilde{T} \times S$ is ergodic (cf. [25]).

We conclude with the main theorem of this section which extends the invertible theorems to the noninvertible setting.

Theorem 5.2. (Orbit Equivalence Multiplier Theorem for Finite Measure Preserving Endomorphisms) *Let T be a finite measure preserving endomorphism. The following statements are equivalent*:

1. *T is mildly mixing*;
2. *For every conservative ergodic nonsingular automorphism S, $r_{(\mu\times\nu)}(T \times S) = r_\nu(S)$ (so $T \times S$ is ergodic)*;
3. *For every conservative ergodic nonsingular automorphism S, $\tilde{T} \times S$ is orbit equivalent to S, where $\tilde{T}$ is the natural extension of T.*

Proof. The implications $1 \Longleftrightarrow 3$ and $2 \Longrightarrow 1$ follow immediately.

$3 \Longrightarrow 2$: Given a conservative ergodic nonsingular automorphism S, let S^* denote its Maharam skew product. Then (3) together with Theorem 3.2 imply that $r_{(\mu\times\nu)}(\tilde{T} \times S) = r_\nu(S)$; more precisely, the ergodic decomposition of S^* is isomorphic to the ergodic decomposition of $(\tilde{T} \times S)^*$. However the hypothesis on T implies that $(\tilde{T} \times S)^* = \tilde{T} \times S^*$, and $(T \times S)^* = T \times S^*$. Taking natural extensions, we have:

$$(\widetilde{T \times S})^* = (\widetilde{T \times S^*}) = \tilde{T} \times S^*.$$

We claim that the ergodic decompositions of $T \times S^*$ and $(\widetilde{T \times S^*})$ (with respect to the obvious measures) are isomorphic; we note that these are both infinite measure preserving transformations. It is known that a measure for an infinite measure preserving endomorphism is ergodic if and only if the lifted measure is ergodic for the natural extension. This and the uniqueness of ergodic decompositions proves the claim.

It then follows immediately that the ergodic decomposition of $(T \times S)^*$ is isomorphic to that of S^*, and this decomposition completely determines the ratio set, so (2) follows immediately. $\square$

We have the following corollary to the proof just given. For the definition and proof of existence of the natural extension of T when T is a conservative nonsingular endomorphism with a recurrent measure, see [23].

Corollary 5.1. *If T is a conservative nonsingular endomorphism of $(X, \mathcal{B}, \mu)$ and μ is a recurrent measure for T, then $r_\mu(T) = r(\tilde{T})$ where $\tilde{T}$ on $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$ is the natural extension.*

Proof. As above, we show that the ergodic decomposition of $(\tilde{T})^*$ is identical to that of $(\tilde{T^*})$ from which the result follows immediately. $\square$

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