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# A Note on the Approximation by Continued Fractions under an Extra Condition

**Karma Dajani and Cor Kraaikamp**

Abstract. In this note the distribution of the approximation coefficients $\Theta_n$, associated with the regular continued fraction expansion of numbers $x \in [0,1)$, is given under extra conditions on the numerators and denominators of the convergents $p_n/q_n$. Similar results are also obtained for $S$-expansions. Further, a Gauss-Kusmin type theorem is derived for the regular continued fraction expansion under these extra conditions.

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## 1. Introduction

A classical result by Hurwitz states that for every irrational number $x$ there exist infinitely many pairs of (co-prime) integers $p$ and $q$, $q > 0$, such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}} \frac{1}{q^2}\,. \tag{1}$$

In the past century a great number of papers appeared, aimed at reproving, refining or generalizing Hurwitz' result (1). Here we mention a theorem by Koksma [Kok], which in itself was a refinement of a result by Hartman [H].

**Theorem 1.** (Koksma) *Let $x$ be an irrational real number, and let $m \geq 1$, $a$ and $b$ be integers. Then for every $\epsilon > 0$ there exist infinitely many pairs of integers $p$ and $q$, $q > 0$, such that*

$$\left| x - \frac{p}{q} \right| < \frac{m^2(1+\epsilon)}{\sqrt{5}} \frac{1}{q^2} \quad \text{and} \quad p \equiv a (\mathrm{mod}\ m),\ q \equiv b (\mathrm{mod}\ m). \tag{2}$$

*The constant $\sqrt{5}$ is best possible, i.e., it can not be replaced by a larger one.*

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The proof of this theorem, and also that of related results by Descombes and Poitou [DP], rests on the strong approximation properties of the regular continued fraction.

In this note we will determine for almost all[1] $x$ the asymtotic density of those regular continued fraction convergents $p_n/q_n$ of $x$ satisfying (2). To be more precise, let $A(x,c,N)$ be the cardinality of the set $\mathcal{A}(x,c,N)$, defined as

$$\{n\,;\,1\le n\le N,\ q_n|q_nx-p_n|<c,\ p_n\equiv a(\mathrm{mod}\ m)\ \text{and}\ q_n\equiv b(\mathrm{mod}\ m)\}\ .$$

If $(a,b,m)=1$, $1\le a,\ b\le m$, we show that for almost all $x$ and for all $c\ge 0$ one has

$$\lim_{n\to\infty}\frac{1}{N}A(x,c,N)\ =\ \frac{F(c)}{J(m)}. \tag{3}$$

Here $J$ is *Jordan's arithmetical totient function*, defined by

$$J(m)\ :=\ m^2\prod_{p|m}(1-\frac{1}{p^2})\,,$$

(the product is taken over all the primes $p$ for which $p|m$) and $F$ is a distribution function - the so-called *Lenstra curve* (see also [BJW]) - given by

$$F(z)\ =\ \begin{cases}\frac{z}{\log 2}, & 0\le z\le\frac{1}{2},\\[2ex] \frac{1}{\log 2}(-z+\log 2z+1), & \frac{1}{2}\le z\le 1.\end{cases}$$

The above result (and several others) will be obtained from a suitable natural extension of a skew product which was introduced and studied by H. Jager and P. Liardet [JL]. In the last section we will extend these results to $S$-expansions, which form a very large class of continued fraction expansions. In fact we will see, that the results obtained by Jager and Liardet in [JL] on the numerators and denominators of convergents hold for any $S$-expansion.

## 2. A Natural Extension of a Skew Product by Jager and Liardet

Let the regular continued fraction expansion (RCF) of $x\in[0,1)$ be given by

$$x\ =\ [\,0;\,B_1,\,\ldots\,,\,B_n,\,\ldots]\,. \tag{4}$$

This expansion is finite if and only if $x$ is rational. Finite truncation in (4) yields the sequence of RCF-convergents $p_n/q_n\ =\ [\,0;\,B_1,\,\ldots\,,\,B_n]$, $n\ge 1$. Underlying the RCF is the operator $T\,:\,[0,1)\to[0,1)$, defined by

$$T0:=0\ \ \text{and}\ \ Tx\ :=\ \frac{1}{x}-\lfloor\frac{1}{x}\rfloor\,,\ x\ne 0,$$

where $\lfloor\xi\rfloor$ is the *floor* (or *entier*) value of $\xi$. Let $\mu$ be the so-called *Gauss measure* on [0,1), i.e., $\mu$ is a probability measure on [0,1) with density

$$\frac{1}{\log 2}\,\frac{1}{1+x}\,.$$

It is well-known that the dynamical system $([0,1)\backslash\mathbf{Q},\mu,T)$ is ergodic, see e.g. [R-N].

[1] All *almost all* statements will be with respect to Lebesgue measure.

For $m \in \mathbf{Z}$, $m \geq 2$, let $G(m)$ be the group of $2 \times 2$ matrices with entries from $\mathbf{Z}/m\mathbf{Z}$ and determinant $\pm 1$, i.e.,

$$G(m) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} ; \alpha,\, \beta,\, \gamma,\, \delta \in \mathbf{Z}/m\mathbf{Z},\, \alpha\delta - \beta\gamma = \pm 1 \right\}.$$

In [JL] it was shown that

$$\operatorname{card} G(m) = \begin{cases} 2mJ(m), & m \geq 3, \\ mJ(m) = 6, & m = 2. \end{cases}$$

Setting

$$A_n := \begin{pmatrix} 0 & 1 \\ 1 & B_n \end{pmatrix}; \quad M_n := A_1 \cdots A_n, \;\; n \geq 1,$$

one has, see e.g. [K],

$$M_n := \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}, \;\; n \geq 1.$$

Notice that the well-known recursion relations for the $p_n$'s and $q_n$'s at once follow from $M_{n+1} = M_n A_{n+1}$.

Denoting by $h_m$ Haar measure on $G(m)$, and setting $|G(m)| := \operatorname{card} G(m)$, H. Jager and P. Liardet [JL] obtained the following theorem.

**Theorem 2.** (Jager & Liardet) *Let* $m \in \mathbf{Z}$, $m \geq 2$. *For* $x \in (0,1)$, *let* $B_1(x) = \lfloor \frac{1}{x} \rfloor$ *and denote* $a \pmod m$ *by* $\overline{a}$. *Then the skew product*

$$\Gamma := ([0,1) \setminus \mathbf{Q} \times G(m),\, \mu \otimes h_m,\, \mathcal{L}) \,, \tag{5}$$

*where the transformation* $\mathcal{L}$ *is given by*

$$\mathcal{L}(x,g) := \left( \frac{1}{x} - B_1(x),\, g \begin{pmatrix} 0 & 1 \\ 1 & \overline{B_1(x)} \end{pmatrix} \right), \;\; (x,g) \in \Gamma\,,$$

*is ergodic.*

*Furthermore, for almost all* $x \in [0,1)$ *the sequence of matrices*

$$n \mapsto \begin{pmatrix} \overline{p_{n-1}} & \overline{p_n} \\ \overline{q_{n-1}} & \overline{q_n} \end{pmatrix}, \;\; n \geq 1,$$

*is uniformly distributed over* $G(m)$.

From Theorem 2 and the Ergodic Theorem, Jager and Liardet [JL] were able to draw a number of corollaries, some of which were previously obtained (in a completely different way) by R. Moeckel [M]. Here we mention the follwing result.

**Proposition 1.** (Moeckel; Jager & Liardet) *Let* $p$, $q$ *and* $m$ *be three integers, such that* $m \geq 2$ *and* $(p,q,m) = 1$. *Then for almost all* $x$ *one has*

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ n\,;\, 1 \leq n \leq N,\, \begin{pmatrix} p_n \\ q_n \end{pmatrix} \equiv \begin{pmatrix} p \\ q \end{pmatrix} \pmod m \right\} = \frac{1}{J(m)}\,.$$

Further interesting applications of Theorem 2 were obtained by V. Nolte in [No].

In order to show that for almost every $x$ the limit in (3) exists and equals $F(c)/J(m)$, the above skew product is not sufficient. To see this, put

$$\Theta_n = \Theta_n(x) := q_n|q_n x - p_n|,\, n \geq 0,$$

and

$$(T_n, V_n) := (T^n x,\, \frac{q_{n-1}}{q_n}),\, n \geq 0.$$

Then one easily shows (see e.g. [K]), that

$$\Theta_{n-1} = \frac{V_n}{1+T_nV_n}\,; \quad \Theta_n = \frac{T_n}{1+T_nV_n}\,. \tag{6}$$

Since $q_{n-1}/q_n = [\,0;\, B_n, \ldots\,,\, B_1]$, we see that $\Theta_n$ depends on both 'the future' (i.e., $T_n$) and 'the past' ($V_n$) of $x$. In order to study these $\Theta$'s, W. Bosma, H. Jager and F. Wiedijk used in [BJW] a natural extension of $([0,1)\setminus\mathbf{Q},\, \mu,\, T)$, which was first studied by H. Nakada, S. Ito and S. Tanaka in [NIT], see also [Na]. Here we will study the skew product of this natural extension with $G(m)$, and we will show that this new skew product is actually the natural extension of the skew product used by Jager and Liardet. We first recall the definition of natural extension, see also [R] or [Br].

**Definition 1.** *Let $T$ and $S$ be two measure preserving transformations of $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ respectively. The transformation $S$ is said to be a factor map of $T$ if there exists a measurable map $\pi : (X,\mathcal{B},\mu) \to (Y,\mathcal{C},\nu)$ such that*

$$\text{(i) } \nu = \mu\circ\pi^{-1}; \qquad \text{(ii) } \pi\circ T = S\circ\pi\,.$$

**Definition 2.** *An invertible, measure preserving transformation $T$ on $(X,\mathcal{B},\mu)$ is said to be the natural extension of the measure preserving transformation $S$ on $(Y,\mathcal{C},\nu)$ if there exists a factor map $\pi : (X,\mathcal{B},\mu) \to (Y,\mathcal{C},\nu)$ such that $\mathcal{B} = \vee_{n=0}^{\infty} T^n(\pi^{-1}\mathcal{C})$ ($\mu$ mod 0).*

We have the following theorem.

**Theorem 3.** *Let $m$ be an integer, $m \geq 2$, and let $B_1(x)$ and $\overline{B_1(x)}$ be defined as before. Let $\Omega := [0,1)\setminus\mathbf{Q}\times[0,1]$, and let $\overline{\mu}$ be the probability measure on $\Omega$, with density*

$$d(x,y) := \frac{1}{\log 2}\frac{1}{(1+xy)^2}\ ,\ (x,y)\in\Omega\,.$$

*Finally, let $\mathcal{T} : \Omega\times G(m)\to\Omega\times G(m)$ be defined by* [2]

$$\mathcal{T}(x,y,g) := \left(\frac{1}{x}-B_1,\ \frac{1}{B_1+y},\ g\begin{pmatrix} 0 & 1\\ 1 & B_1\end{pmatrix}\right)\ ,\ (x,y,g)\in\Omega\times G(m)\,.$$

*Then the skew product*

$$(\Omega\times G(m),\, \overline{\mu}\otimes h_m,\, \mathcal{T}) \tag{7}$$

*is the natural extension of the skew product from (5), and is therefore ergodic.*

**Remarks**

1. In fact, the skew product (7) has mixing properties far stronger than ergodicity. In [L], Liardet showed that $(\Gamma,\, \mu\otimes h_m,\, \mathcal{L})$ is exact, and therefore it follows, see also [Br], p. 39, that $(\Omega\times G(m),\, \overline{\mu}\otimes h_m,\, \mathcal{T})$ is a K-system.
2. The projection $\tilde{\pi} : \Omega\times G(m)\to\Omega$ given by $\tilde{\pi}(x,y,g) := (x,y)$ yields the aforementioned natural extension $(\Omega,\overline{\mu},\mathcal{N})$ by Nakada-Ito-Tanaka of $([0,1),\mu,T)$. Clearly $\mathcal{N}(x,y) = (Tx,\,(B_1(x)+y)^{-1})$ for $(x,y)\in\Omega$.

**Proof of Theorem 3.** Let $\pi : \Omega\times G(m)\to[0,1)\setminus\mathbf{Q}\times G(m)$ denote the projection $\pi(x,y,g) := (x,g)$. Then clearly one has

[2] We will suppress the dependence of $B_1$ on $x$ whenever possible.

(i) $\pi\mathcal{T} = \mathcal{L}\pi$ ;
(ii) $(\overline{\mu} \otimes h_m) \circ \pi^{-1} = \mu \otimes h_m$ ;
(iii) $\vee_{n\geq 0}\mathcal{T}^n(\pi^{-1}(\mathcal{B}\times\mathcal{F})) = \overline{\mathcal{B}}\times\mathcal{F}$ ;

where $\mathcal{B}$ is the Borel $\sigma$-algebra on $[0,1)\setminus\mathbf{Q}$, $\overline{\mathcal{B}}$ is the Borel $\sigma$-algebra on $\Omega$ and $\mathcal{F}$ is the Borel $\sigma$-algebra on $G(m)$. Notice that (iii) is an immediate consequence of Remark 2 with $\mathcal{F}$ the power set of $G(m)$. Thus $\pi$ satisfies the conditions of Definition 2, and therefore $\mathcal{T}$ is the natural extension of $\mathcal{L}$ as given in (5). As is well-known, see e.g. [R] or [Br], the natural extension $\mathcal{T}$ inherits all mixing properties from $S$. Since Jager and Liardet showed that $S$ is ergodic, the result follows. $\square$

**Lemma 1.** *For almost every $x\in[0,1)$ and every $g\in G(m)$ the sequence*

$$(\mathcal{T}^n(x,0,g))_{n\geq 0}$$

*is distributed over $\Omega\times G(m)$ according to the probability measure $\overline{\mu}\otimes h_m$, i.e., for every Borel set $D$ in $\Omega\times G(m)$, for almost every $x\in[0,1)$ and for every $g\in G(m)$ one has*

$$\lim_{n\to\infty}\frac{1}{N}\#\{n;\,1\leq n\leq N,\,\mathcal{T}^n(x,0,g)\in D\} = \overline{\mu}\otimes h_m(D)\,.$$

**Proof.** Notice, that if the sequence $(\mathcal{N}^n(x,y))_{n\geq 0}$ is distributed over $\Omega$ according to the density $d$, i.e., if for every Borel set $B\subset\Omega$ one has

$$\lim_{n\to\infty}\frac{1}{N}\#\{n;\,1\leq n\leq N,\,\mathcal{N}^n(x,y)\in B\} = \overline{\mu}(B)\,,$$

then it follows from

$$\overline{\mu}\otimes h_m(B\times\{g\}) = \frac{\overline{\mu}(B)}{|G(m)|},$$

where $B\subset\Omega$ is a Borel set and $g\in G(m)$, that for every $g\in G(m)$ the sequence $(\mathcal{T}^n(x,y,g))_{n\geq 0}$ is distributed over $\Omega\times G(m)$ according to $\overline{\mu}\otimes h_m$.

Next observe, that for all $(x,y)\in\Omega$ one has

$$|\mathcal{N}^n(x,y)-\mathcal{N}^n(x,0)| \leq \frac{1}{\mathcal{F}_n\mathcal{F}_{n+1}}\;,$$

where $(\mathcal{F}_n)_{n\geq 0}$ is the Fibonacci-sequence $1,1,2,3,5,8,\cdots$ . Hence for all irrational $x$ and all $y\in[0,1]$

$$\lim_{n\to\infty}|\mathcal{N}^n(x,y)-\mathcal{N}^n(x,0)| = 0,$$

and the convergence is uniform. But then for all $y\in[0,1]$ the sequence $(\mathcal{N}^n(x,y))_{n\geq 0}$ has the same distribution as $\mathcal{N}^n(x,0))_{n\geq 0}$. Thus, also the sequences $\mathcal{T}^n(x,0,g))_{n\geq 0}$ and $\mathcal{T}^n(x,y,g))_{n\geq 0}$ have the same distribution for all $y\in[0,1]$ and all $g\in G(m)$.

Now let $E\subset[0,1)\setminus\mathbf{Q}$ be the set of those irrational $x\in[0,1)$, for which $(\mathcal{N}^n(x,0))_{n\geq 0}$ is **not** distributed over $\Omega$ according to $\overline{\mu}$, then $\overline{E} := E\times[0,1]$ is the set of points $(x,y)\in\Omega$ for which $(\mathcal{N}^n(x,y))_{n\geq 0}$ is **not** distributed over $\Omega$ according to $\overline{\mu}$. Now if $E$ had, as subset of $[0,1)$, positive Lebesgue measure, so would $\overline{E}$ as a subset of $\Omega$. However, this is impossible since $(\Omega,\overline{\mu},\mathcal{N})$ is ergodic.

Thus we see that $E$ is a null-set, and therefore we have for almost all $x$ and for all $g\in G(m)$ that the sequence $(\mathcal{T}^n(x,0,g))_{n\geq 0}$ is distributed according to $\overline{\mu}\otimes h_m$. $\square$

From Theorem 3, (6) and Lemma 1 we have the following results, (3) being one of them.

**Corollary 1.** *Let $a$ and $m$ be two integers, $m \geq 2$, and let $(c_1, c_2) \in [0,1]^2$. Furthermore, let $A(x, c_1, c_2, N)$ be the cardinality of the set $\mathcal{A}(x, c_1, c_2, N)$, given by*

$$\{n;\, 1 \leq n \leq N,\, \Theta_{n-1} < c_1,\, \Theta_n < c_2,\, p_n \equiv a (\mathrm{mod}\ m) \text{ and } q_n \equiv b (\mathrm{mod}\ m)\}.$$

*Then for almost all $x$ one has*

$$\lim_{N\to\infty} \frac{1}{N} A(x, c_1, c_2, N) = \frac{G(c_1, c_2)}{J(m)},$$

*where $G(c_1, c_2)$ is a distribution function with density $g$, given by*

$$g(\xi, \eta) = \begin{cases} \frac{1}{\log 2} \frac{1}{\sqrt{1-4\xi\eta}}, & \xi > 0,\, \eta > 0 \text{ and } \xi + \eta < 1, \\ \\ 0, & \text{otherwise.} \end{cases}$$

**Remarks**

1. Taking $c_1$ equal to 1 in Corollary 1 at once yields (3).
2. Using some of the lemmas in [JL] on the number of elements of certain sets, obtained from $G(m)$ by imposing extra conditions on $G(m)$, at once yield several other corollaries, e.g., from Theorem 3, (6), Lemma 1 and Lemma (3.10) from [JL] one has the following corollary.

**Corollary 2.** *Let $a$ and $m$ be two integers, $m \geq 2$, and let $(c_1, c_2) \in [0,1]^2$. Furthermore, let $B(x, c_1, c_2, N)$ be the cardinality of the set $\mathcal{B}(x, c_1, c_2, N)$, given by*

$$\{n;\, 1 \leq n \leq N,\, \Theta_{n-1} < c_1,\, \Theta_n < c_2,\, q_n \equiv a (\mathrm{mod}\ m)\}.$$

*Then for almost all $x$ one has*

$$\lim_{N\to\infty} \frac{1}{N} \mathcal{B}(x, c_1, c_2, N) = G(c_1, c_2) \frac{m}{J(m)} \frac{\varphi((a,m))}{(a,m)}.$$

*Here $\varphi(n)$ denotes the Euler $\varphi$-function.*

In 1959, P. Erdös [E] showed, that for each $z \geq 0$ and for almost all $x$ one has

$$\lim_{N\to\infty} \frac{1}{\log N} \#\{(q,p);\, q|qx - p| < z,\, (q,p) = 1,\, q \leq N\} = \frac{\pi^2}{12} z.$$

Using the method from [BJW], Ito and Nakada showed in [IN], that for $0 \leq z \leq \frac{1}{2}$ this result is an easy consequence of the fact that $(\Omega, \overline{\mu}, \mathcal{N})$ is an ergodic system, and from classical theorems by Legendre and Lévy. Their method can be extended to $0 \leq z \leq 1$, but then $(\Omega, \overline{\mu}, \mathcal{N})$ should be replaced by a 'suitable' ergodic system, see also [I]. The reason for this is, that there exist rational numbers $p/q$ with $(q,p) = 1$ and $\frac{1}{2} < q|qx - p| < 1$ which are not 'picked up' as convergents of $x$ by the RCF. This also 'explains' why the Lenstra-curve is not linear between 0 and 1, but only between 0 and $\frac{1}{2}$.

Replacing $(\Omega, \overline{\mu}, \mathcal{N})$ by $(\Omega \times G(m), \overline{\mu} \otimes h_m, \mathcal{T})$ yields the following proposition.

**Proposition 2.** *Let $a$, $b$ and $m$ be three integers, such that $m \geq 2$ and $(a, b, m) = 1$. Then for almost all $x$ and $0 \leq z \leq \frac{1}{2}$ one has that the limit*

$$\lim_{N\to\infty} \frac{\#\{(q,p); q|qx - p| < z, (q,p) = 1, q \leq N, \begin{pmatrix} p \\ q \end{pmatrix} \equiv \begin{pmatrix} a \\ b \end{pmatrix} (\mathrm{mod}\ m)\}}{\log N}$$

*equals $\frac{\pi^2}{12 J(m)} z$.*

The above results all deal with the pointwise convergence of ergodic averages. Classically such results for continued fractions are like one face of a coin, the other face being weak convergence of probability measures with a given speed of convergence. To conclude this section, we will show that also in the present setting such results are easily obtained.

Let $K \subset \Omega$ be a simply connected subset of $\Omega$, such that $\partial K = \ell_1 \cup \ldots \cup \ell_k$, where $k \in \mathbf{N}$ and each $\ell_i$ is either a vertical line segment

$$\ell_i = \{(A_i, \eta);\, C_i \le \eta \le D_i\}$$

(where $A_i \in [0,1]$ and $0 \le C_i < D_i \le 1$), or by

$$\ell_i = \{(\xi, f_i(\xi));\, A_i \le \xi \le B_i\}$$

(where $0 \le A_i < B_i \le 1$ and $f_i : [A_i, B_i] \to [0,1]$ is monotone and continuous), $i = 1, \ldots, k$. Then setting

$$E_n(K) := \{\xi \in [0,1);\, (T_n, V_n) \in K\},$$

it was shown in [DK] that

$$\lambda(E_n(K)) = \overline{\mu}(K) + \mathcal{O}(g^n),$$

where the constant in the big $\mathcal{O}$-symbol is uniform and where $g = \frac{\sqrt{5}-1}{2} = 0.61\cdots$ is the so-called *golden mean.*

Now let $L \subset G(m)$ be some subset of $G(m)$, and put

$$\overline{E}_n(K \times L) := \{(x,g) \in \Gamma;\, \mathcal{T}^n(x,0,g) \in K \times L\}. \tag{8}$$

In case the continued fraction expansion of $x$ is given by (4) and the matrix $M_n(x)$ is given — as before — by

$$M_n(x) = \begin{pmatrix} 0 & 1 \\ 1 & B_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & B_n \end{pmatrix},$$

one clearly has

$$\overline{E}_n(K \times L) = \{(x,g) \in \Gamma;\, \mathcal{N}^n(x,0) \in K \text{ and } gM_n(x) \in L\}.$$

Consequently one has, that

$$\begin{aligned}
(\lambda \otimes h_m)(\overline{E}_n(K \times L)) &= \int_{E_n(K)} \int_{L\, M_n^{-1}(x)} dh_m d\lambda(x) \\
&= \int_{E_n(K)} h_m(L\, M_n^{-1}(x)) d\lambda(x) = \int_{E_n(K)} h_m(L) d\lambda(x) \\
&= \lambda(E_n(K)) h_m(L) = \frac{|L|}{|G(m)|}\overline{\mu}(K) + \mathcal{O}(g^n).
\end{aligned}$$

Thus we find the following theorem.

**Theorem 4.** *Let $K \subset \Omega$ and $L \subset G(m)$ be as before. Furthermore, let $\overline{E}_n(K \times L)$ be defined as in* (8). *Then*

$$(\lambda \otimes h_m)(\overline{E}_n(K \times L)) = \frac{|L|}{|G(m)|}\overline{\mu}(K) + \mathcal{O}(g^n),$$

*where the constant in the big $\mathcal{O}$-symbol is uniform.*

Several corollaries can be obtained easily. We mention here only one.

**Corollary 3.** *Let $a$, $b$ and $m$ be three integers, such that $(a,b,m)=1$ and $m\geq 2$. Furthermore, let for $0\leq z_i\leq 1$ (where $i=1,2$) the set $K_n(z_1,z_2,m)$ be defined by*

$$\{(x,g)\in\Gamma;\ \Theta_{n-1}(x)\leq z_1,\ \Theta_n(x)\leq z_2 \text{ and } p_n\equiv a(\operatorname{mod} m),\ q_n\equiv b(\operatorname{mod} m)\}\,.$$

*Then*

$$\lambda(K_n(z_1,z_2,m)) \;=\; \frac{G(z_1,z_2)}{J(m)}+\mathcal{O}(g^n)\,,$$

*where the constant in the big $\mathcal{O}$-symbol is uniform.*

## 3. $S$-expansions

In [Ba], D. Barbolosi showed that the method of Jager and Liardet [JL] can be extended to the *continued fraction with odd partial quotients* (OddCF). Essential in [Ba] is, that in case $m$ is even one should replace $G(m)$ by $G(m)'$, which is a subgroup of $G(m)$ of index 2, generated by all matrices of the type

$$\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{\varepsilon} & \overline{a} \end{pmatrix}, \tag{9}$$

where $\varepsilon\in\{-1,+1\}$ and $a$ is an odd integer. We denote the set of all such matrices from (9) by $H'$. In case $m$ is odd one has that $G(m)=G(m)'$. Heuristically Barbolosi's result can be understood as follows. In [JL] the following lemma was obtained for the RCF.

**Lemma 2.** *Let*

$$H \;:=\; \left\{\begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix};\ \alpha\in\mathbf{Z}/m\mathbf{Z}\right\}\,,\ m\geq 2.$$

*Then the group $G(m)$ is generated by $H$.*

In case $m$ is even $H'$ does not contain those matrices in $H$ for which $a$ is even; the remaining matrices generate the subgroup $G(m)'$ of $G(m)$ instead of $G(m)$.

Apart from the OddCF there are several other — classical and new — continued fraction algorithms. To mention a few: the *nearest integer continued fraction* (NICF), Hurwitz' *singular continued fraction* (SCF), Mikowski's *diagonal continued fraction* (DCF), and more recently Hitoshi Nakada's $\alpha$-expansions and Wieb Bosma's *optimal continued fraction* (OCF). All these expansions are all examples of a much larger class of continued fraction expansions, the so-called $S$-expansion, which we will now briefly describe (for proofs and details, see [K]).

Each $S$-expansion is an example of a *semi-regular continued fraction* (SRCF) expansion. In general a SRCF is a finite or infinite fraction

$$b_0+\cfrac{\varepsilon_1}{b_1+\cfrac{\varepsilon_2}{b_2+\ddots+\cfrac{\varepsilon_n}{b_n+\ddots}}} = [\,b_0;\,\varepsilon_1 b_1,\,\varepsilon_2 b_2,\,\cdots,\,\varepsilon_n b_n,\,\cdots]\,, \tag{10}$$

with $\varepsilon_n=\pm 1$; $b_0\in\mathbf{Z}$; $b_n\in\mathbf{N}$, for $n\geq 1$, subject to the condition

$$\varepsilon_{n+1}+b_n\geq 1,\ \text{for } n\geq 1,$$

and with the restriction that in the infinite case

$$\varepsilon_{n+1}+b_n\geq 2,\ \text{infinitely often.}$$

Moreover we demand that $\varepsilon_n + b_n \geq 1$ for $n \geq 1$.

Taking finite truncations in (10) yields a finite or infinite sequence of rational numbers $r_n/s_n$, $n \geq 1$, the convergents of (10). An SRCF-expansion (10) is an SRCF-expansion of $x$ if $\lim_{n\to\infty} r_n/s_n = x$.

Let $x$ be an irrational number, and let (10) be some SRCF-expansion of $x$. Suppose that we have for a certain $k \geq 0$ : $b_{k+1} = 1$ , $\varepsilon_{k+2} = 1$ . The operation by which the continued fraction (10) is replaced by[3]

$$[\,b_0;\, \varepsilon_1 b_1,\, \dots\, ,\, \varepsilon_{k-1} b_{k-1},\, \varepsilon_k(b_k + \varepsilon_{k+1}),\, -\varepsilon_{k+1}(b_{k+2}+1),\, \varepsilon_{k+3} b_{k+3},\, \dots]\ ,$$

which again is a SRCF-expansion of $x$, with convergents, say, $(c_n/d_n)_{n\geq -1}$, is called *the singularization of the partial quotient* $b_{k+1}$ *equal to* 1. As in case of the RCF, setting

$$\tilde{A}_0 := \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix};\ \tilde{A}_n := \begin{pmatrix} 0 & \varepsilon_n \\ 1 & b_n \end{pmatrix};\, n \geq 1,$$

and $\tilde{M}_n := A_0 A_1 \cdots A_n$, $n \geq 0$, yields that

$$\tilde{M}_n = \begin{pmatrix} r_{n-1} & r_n \\ s_{n-1} & s_n \end{pmatrix},\ n \geq 0.$$

Similarly one has for the new sequence of convergents $(c_n/d_n)_{n\geq -1}$

$$\hat{M}_n = \begin{pmatrix} c_{n-1} & c_n \\ d_{n-1} & d_n \end{pmatrix},\ n \geq 0.$$

In [K], Section 2, it was shown that

$$\hat{M}_n = \tilde{M}_n \text{ for } n = 1, \dots , k-1;\ \hat{M}_n = \tilde{M}_{n+1} \text{ for } n = k+1, \dots$$

and

$$\hat{M}_k = \tilde{M}_{k+1} \begin{pmatrix} -\varepsilon_{k+1} & 0 \\ \varepsilon_{k+1} & 1 \end{pmatrix}.$$

From this it follows that $(c_n/d_n)_{n\geq -1}$ is obtained from $(r_n/s_n)_{n\geq -1}$ by skipping the term $r_k/s_k$. See also [K], Sections 2 and 4.

A simple way to derive a strategy for singularization is given by a *singularization area* $S$.

**Definition 3.** *A subset $S$ from $\Omega$ is called a singularisation area if it satisfies*

$$\text{(i) } S \in \mathcal{B} \text{ and } \mu(\partial S) = 0;\ \text{(ii) } S \subset ([\tfrac{1}{2}, 1) \setminus \mathbf{Q}) \times [0,1];\ \text{(iii) } \mathcal{N}(S) \cap S = \emptyset\,.$$

**Definition 4.** *Let $S$ be a singularisation area and let $x$ be a real irrational number. The $S$-expansion of $x$ is that semi-regular continued fraction expansion converging to $x$, which is obtained from the RCF-expansion of $x$ by singularizing $B_{n+1}$ if and only if $\mathcal{N}^n(x,0) \in S$, $n \geq 0$.*

From these two definitions a whole theory of $S$-expansions can be developed. See [K], where also several examples (as the aforementioned NICF, SCF etc.) are

[3] In case $k = 0$ this comes down to replacing (10) by the SRCF $[\,b_0 + \varepsilon_{k+1};\, -\varepsilon_{k+1}(b_2 + 1),\, \varepsilon_3 b_3,\, \varepsilon_4 b_4,\, \dots]$.

discussed. Essential in the theory of $S$-expansions is, that if we denote $\Omega \setminus S$ by $\Delta$, and if we define the map $\mathcal{I} : \Delta \to \Delta$ by

$$\mathcal{I}(x,y) := \begin{cases} \mathcal{N}(x,y), & \mathcal{N}(x,y) \notin S, \\ \\ \mathcal{N}^2(x,y), & \mathcal{N}(x,y) \in S, \end{cases}$$

that — since $\mathcal{I}$ is an induced transformation with return time bounded by 2 — the system

$$(\Delta, \mu_\Delta, \mathcal{I})$$

is ergodic. Here $\mu_\Delta$ is the probability measure on $\Delta$ with density

$$\frac{1}{\overline{\mu}(\Delta)\log 2}\,\frac{1}{(1+xy)^2}\,,\ (x,y) \in \Delta.$$

From this, using elementary properties of the Nakada-Ito-Tanaka natural extension of the RCF, one finds the two-dimensional ergodic system underlying every $S$-expansion. To be more precise, let $M : \Delta \to \mathbf{R}^2$ be defined by

$$M(x,y) := \begin{cases} (-x/(1+x), 1-y), & (x,y) \in \Delta^- := \mathcal{N}(S), \\ \\ (x,y), & (x,y) \in \Delta^+ := \Delta \setminus \Delta^-, \end{cases}$$

let $\Omega_S := M(\Delta)$ and let the operator $\tau_S : \Omega_S \to \Omega_S$ be defined by $\tau_S(t,v) := M\mathcal{I}M^{-1}(t,v)$ for $(t,v) \in \Omega_S$. Then one has the following theorem.

**Theorem 5.** ([K]) *Let $\rho$ be the probability measure on $\Omega_S$ with density*

$$\frac{1}{(1-\overline{\mu}(S))\log 2}\,\frac{1}{(1+tv)^2}\,,\ (t,v) \in \Omega_S.$$

*Then $(\Omega_S, \rho, \tau_S)$ forms an ergodic system. Furthermore, if $b : \Omega_S \to \mathbf{N}$ is given by*

$$b(t,v) = \begin{cases} B(t), & \text{if } \operatorname{sgn}(t) = 1, \mathcal{N}(t,v) \notin S, \\ B(t)+1, & \text{if } \operatorname{sgn}(t) = 1, \mathcal{N}(t,v) \in S, \\ B(-t/(1+t))+1, & \text{if } \operatorname{sgn}(t) = -1, \mathcal{N}(M^{-1}(t,v)) \notin S, \\ B(-t/(1+t))+2, & \text{if } \operatorname{sgn}(t) = -1, \mathcal{N}(M^{-1}(t,v)) \in S, \end{cases} \tag{11}$$

*then*

$$\tau_S(t,v) = \left( \left|\frac{1}{t}\right| - b(t,v),\ \frac{1}{b(t,v) + \operatorname{sgn}(t)\cdot v} \right)\,,\ (t,v) \in \Omega_S.$$

In view of (10) and Theorem 5 one has the following result.

**Theorem 6.** *Let $S \subset \Omega$ be a singularization area, let $m \geq 2$ be an integer, and let $\Omega_S$ and $\rho_S$ be defined as before. Furthermore, let $\mathcal{T}_S : \Omega_S \times G(m) \to \Omega_S \times G(m)$ be defined by*

$$\mathcal{T}_S(t,v,g) := \left( \left|\frac{1}{t}\right| - b,\ \frac{1}{b+\varepsilon\cdot v},\ g\begin{pmatrix} 0 & \varepsilon \\ 1 & b \end{pmatrix} \right)\,,$$

*where $b = b(t,v)$ is defined as in (11) and $\varepsilon = \operatorname{sgn}(t)$. Then*

$$(\Omega_S \times G(m),\ \rho_S \otimes h_m,\ \mathcal{T}_S)$$

*forms an ergodic system.*

**Proof.** Define the map $\tilde{\mathcal{I}} : \Delta \times G(m) \to \Delta \times G(m)$ by

$$\tilde{\mathcal{I}}(x,y,g) := \begin{cases} \mathcal{T}(x,y,g), & \mathcal{N}(x,y) \notin S, \\ \\ \mathcal{T}^2(x,y,g), & \mathcal{N}(x,y) \in S. \end{cases}$$

Then $(\Delta \times G(m), \overline{\mu} \otimes h_m, \tilde{\mathcal{I}})$ forms an ergodic system, which is — in view of (10), Theorem 5 and the fact that in case of the RCF always $\varepsilon_n = +1$ — metrically isomorphic to $(\Omega_S \times G(m),\ \rho_S \otimes h_m,\ \mathcal{T}_S)$ via the map $\mathcal{M} : \Delta \times G(m) \to \Delta \times G(m)$, given by

$$\mathcal{M}(x,y,g) := \begin{cases} (x,y,g), & (x,y) \in \Delta^+, \\ \\ (-x/(1+x), 1-y, g\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}), & (x,y) \in \Delta^-. \end{cases}$$

□

In case $S \subset \Omega$ is a singularization area, one has (see also [K], Section 4)

$$0 \leq \overline{\mu}(S) \leq 1 - \frac{\log(g+1)}{\log 2} = 0.30575\cdots .$$

Conversely, for every $s \in (0,\ 0.30575\cdots]$ there exist infinitely many singularization areas $S \subset \Omega$ such that $\overline{\mu}(S) = s$. For every irrational number $x$ and every singularization area $S$ define the monotonically increasing arithmetical function $n_S(k) = n_S(k,x)$ by $r_k/s_k = p_{n_S(k)}/q_{n_S(k)}$. Then for almost all $x$

$$\lim_{k\to\infty} \frac{n_S(k)}{k} = \frac{1}{1-\overline{\mu}(S)} ,$$

see also [K], Theorem (4.13).

In spite of this, it follows from Theorem 6 that for any $S$-expansion and for almost every $x$ the sequence of numerators $(r_n)_{n\geq -1}$ and denominators $(s_n)_{n\geq -1}$ of the $S$-convergents $(r_n/s_n)_{n\geq -1}$ of $x$ have — mod $m$ — the same asymptotic behaviour as the sequence of numerators $(p_n)_{n\geq -1}$ and denominators $(q_n)_{n\geq -1}$ of the RCF-convergents of $x$.

To be more precise, we have the following corollary.

**Corollary 4.** *Let $r$, $s$ and $m$ be three integers, such that $m \geq 2$ and $(r,s,m) = 1$. Then for almost all $x$ one has*

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ n\,;\, 1 \leq n \leq N,\ \begin{pmatrix} r_n \\ s_n \end{pmatrix} \equiv \begin{pmatrix} p \\ q \end{pmatrix} \pmod{m} \right\} = \frac{1}{J(m)} .$$

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## References

[Ba] D. Barbolosi, *Sur le développement en fractions continues a quotients partiels impairs*, Mh. Math., **109** (1990), 25–37.

[Br] J. R. Brown, *Ergodic Theory and Topological Dynamics*, Academic Press, New York, San Francisco, London, 1976.

[BJW] W. Bosma, H. Jager, and F. Wiedijk, *Some metrical observations on the approximation by continued fractions*, Indag. Math., **45** (1983), 281–299.

[DK] K. Dajani and C. Kraaikamp *Generalization of a theorem of Kusmin*, Monatsh. Math., **118** (1994), 55–73.

[DP] R. Descombes and G. Poitou, *Sur certains problèmes d'approximation*, C. R. Acad Sci. Paris, **234** (1952), 581–583 and II, 1522–1524.

[E] P. Erdös, *Some results on Diophantine approximation*, Acta Arith., **V** (1959), 359–369.

[H] S. Hartman, *Sur une condition supplémentaire dans les approximations diophantiques*, Coll. Math., **2** (1949), 48–51.

[IN] Sh. Ito and H. Nakada, *On natural extensions of transformations related to Diophantine approximations*, Proceedings of the conference on Number Theory and Combinatorics, World Scientific Publ. Co., Singapore (1985), 185–202.

[I] Sh. Ito, *Algorithms with mediant convergents and their metrical theory*, Osaka J. Math., **26** (1989), 557–578.

[JL] H. Jager and P. Liardet, *Distributions arithmétiques des demoninateurs de convergents de fractions continues*, Indag. Math., **50** (1988), 181–197.

[Kok] J. Koksma, *Sur l'approximation des nombres irrationels sous une condition supplémentaire*, Simon Stevin **28** (1951), 199–202.

[K] C. Kraaikamp, *A new class of continued fraction expansions*, Acta Arithm., **LVII** (1991), 1–39.

[L] P. Liardet, *Distributions Arithmétiques des Demoninateurs de Convergents de Fractions Continues*, Sem. Th. des Nombres Bordeaux 1986–1987, Exposé N$^o$ 36, Université de Bordeaux, 1987.

[M] R. Moeckel, *Geodesics on modular surfaces and continued fractions*, Erg. Th. Dyn. Sys., **2** (1982), 69–83.

[NIT] H. Nakada, Sh. Ito, and S. Tanaka, S. *On the invariant measure for the transformation associated with some real continued fraction*, Keio Eng. Rep., **30** (1977), 159–175.

[Na] H. Nakada, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math., **4** (1981), 399–426.

[No] V. N. Nolte, *Some probabilistic results on the convergents of continued fractions*, Indag. Math. (N.S.) **1** (1990), 381–389.

[R] V. A. Rohlin, *Exact endomorphisms of a Lebesgue space*, Izv. Akad. Naik SSSR, Ser. Mat., **24** (1960); English AMS translation, Series **2**, **39** (1969), 1–36.

[R-N] C. Ryll-Nardzewski, *On the ergodic theorems* (II) (*Ergodic theory of continued fractions*), Studia Math., **12** (1951), 74–79.

Universiteit Utrecht, Department of Mathematics, Budapestlaan 6, P.O.Box 80.000, 3508 TA Utrecht, the Netherlands
dajani@math.ruu.nl

Thomas Stieltjes Institute for Mathematics and Technische Universiteit Delft, Faculty ITS, Department of Mathematics (SSOR), Mekelweg 4, 2628 CD Delft, the Netherlands
c.kraaikamp@twi.tudelft.nl

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