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Convergence of Moving Averages of Multiparameter Superadditive Processes

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Abstract. It is shown that moving averages sequences are good in the mean for multiparameter strongly superadditive processes in L_1, and good in the p-mean for multiparameter admissible superadditive processes in L_p, $1 \leq p < \infty$. Also, using a decomposition theorem in L_p-spaces, a.e. convergence of the moving averages of multiparameter superadditive processes with respect to positive L_p-contractions, $1 < p < \infty$, is obtained.

Contents

1. Introduction

Beginning with the moving averages theorem of Bellow, Jones and Rosenblatt [BJR_1], determining the conditions that ensure a.e. convergence (or divergence) of moving averages of various processes has been a subject of intensive study. Subsequently, a.e. convergence of moving averages has been obtained in several different settings [AD, $\mathrm{Ç}_2$, ÇF, JO_1, JO_2]. In [JO_1, JO_2] the moving averages theorem has been extended to the operator setting. Generalization of this theorem to (multiparameter) superadditive processes relative to measure preserving transformations (MPTs) is due to Ferrando [F]. Recently, the a.e. convergence of the moving averages of superadditive processes relative to positive L_p-contractions, $1 < p < \infty$, has been obtained [$\mathrm{Ç}_2$].

In proving the a.e. convergence of moving averages, a condition on the sequence, called the *cone condition*, plays a vital role. It has been observed that the class of

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sequences satisfying the cone condition is the same as the class of B-sequences, introduced by Akcoglu and Déniel [AD]. Multiparameter B-sequences are introduced in [F]. It turns out, however, that for the norm convergence of moving averages of the additive processes this cone condition is not necessary.

In this article we will investigate the a.e. and norm convergence of the moving averages of multiparameter superadditive processes relative to positive L_p-contractions. Our results will generalize some results in [Ç$_2$, F] to the multiparameter operator theory setting, and the result in [JO$_1$] and the norm convergence result in [BJR$_2$] to the superadditive setting.

Let (X, F, μ) be a σ-finite measure space, and let T and S be commuting positive linear $L_p(X)$-contractions, $1 \leq p < \infty$ fixed. A family of real-valued functions $F = \{F_{(m,n)}\}_{m \geq 0, n \geq 0} \subset L_p$ with $F_{(0,0)} = F_{(1,0)} = F_{(0,1)}$ is called a (two parameter) (T, S)-*superadditive process* if, for all $m \geq 0,\ n \geq 0$,

$$F_{(m+k,n)} \geq F_{(m,n)} + T^m F_{(k,n)} \text{ if } k > 0, \text{ and } F_{(m,n+l)} \geq F_{(m,n)} + S^n F_{(m,l)} \text{ if } l > 0.$$

F is called a *strongly* (T, S)-*superadditive* process, if, for all $0 < k < m,\ 0 < l < n$,

$$F_{(m+k,n+l)} \geq F_{(m,n)} + T^m F_{(k,n+l)} + S^n F_{(m+k,l)} - T^m S^n F_{(k,l)}.$$

When both $\{F_{(m,n)}\}$ and $\{-F_{(m,n)}\}$ are (T, S)-superadditive, then $\{F_{(m,n)}\}$ is called (T, S)-*additive* process. Clearly, (T, S)-additive processes are necessarily of the form $\{\sum_{i,j=0}^{m-1,n-1} T^i S^j F_{(1,1)}\}$. If there exists $g \in L_p$ such that, for all $m, n > 0$, $F_{(m,n)} \leq \sum_{i,j=0}^{m-1,n-1} T^i S^j g$, then F is called *dominated,* and the function g is called a *dominant* for F . If $\sup_{m,n \geq 1} \frac{1}{mn} \|F_{(m,n)}\|_p < \infty$, the process is called *bounded.* It is well known that, when $p = 1$, any bounded superadditive process relative to MPTs has a dominant g that satisfies $\int g = \gamma_F := \sup_{m,n \geq 1} \frac{1}{mn} \|F_{(m,n)}\|_1 < \infty$, called an *exact dominant* [AS$_3$, S].

Remark 1. Any strongly superadditive process satisfying $F_{(m,0)} = F_{(0,n)} \equiv 0$, for all $m, n \geq 0$, is a superadditive process (which will be assumed throughout this article). In one parameter case, strong superadditivity and superadditivity coincide. Furthermore, if F is a superadditive process with $F_{(1,1)} \geq 0$, then $F_{(m,n)} \geq 0$, for all $m > 0,\ n > 0$.

Remark 2. If F is a (T, S)-superadditive process, then

$$G = \{G_{(m,n)}\} = \sum_{i,j=0}^{m-1,n-1} T^i S^j F_{(1,1)}$$

is a (T, S)-additive process, and hence $F'_{(m,n)} = F_{(m,n)} - G_{(m,n)}$ is a *positive* superadditive process. Therefore, we can always assume that a superadditive process is positive. Also, $F' = \{F'_{(m,n)}\}$ is dominated if F is dominated.

Throughout this article, any sequence of the form $\mathbf{w} = \{(a_n, r_n)\}$, with $a_n > 0,\ r_n > 0$ for all n, and $r_n \to \infty$, will be called a moving average sequence (MAS). A (two parameter) MAS is a sequence $\mathbf{w} = \{(\mathbf{a}_n, \mathbf{r}_n)\}$, where $\mathbf{a}_n = (a_n^1, a_n^2)$, $\mathbf{r}_n = (r_n^1, r_n^2)$, with $a_n^i > 0,\ r_n^i > 0$ for all n, and $r_n^i \to \infty$. We will call the MASs $\mathbf{w}^1 = \{(a_n^1, r_n^1)\}$ and $\mathbf{w}^2 = \{(a_n^2, r_n^2)\}$ as the *components* of $\mathbf{w}$, each of which are MASs themselves.

If F is a T-superadditive process and $\mathbf{w} = \{(a_n, r_n)\}$ is a one parameter MAS, we define the averages of F along $\mathbf{w}$ by $\frac{1}{r_n} T^{a_n} F_{r_n}$. (Hence, if F is T-additive, the averages along $\mathbf{w}$ will be $A_{\mathbf{w},n}(T)F_1 := \frac{1}{r_n}T^{a_n}\sum_{i=0}^{r_n-1}T^iF_1$.) Similarly, if F is a (T,S)-superadditive process and $\mathbf{w} = \{(\mathbf{a}_n,\mathbf{r}_n)\}$ is a (two parameter) MAS, then we define the averages of F along $\mathbf{w}$ by $\frac{1}{|\mathbf{r}_n|}T^{a_n^1}S^{a_n^2}F_{(r_n^1,r_n^2)}$, where $|\mathbf{r}_n| = r_n^1r_n^2$. (So, for (T,S)-additive processes the averages along $\mathbf{w}$ will be $A_{\mathbf{w},n}(T,S)F_{(1,1)} = \frac{1}{|\mathbf{r}_n|}T^{a_n^1}S^{a_n^2}\sum_{i=0}^{r_n^1-1}\ \sum_{j=0}^{r_n^2-1}T^iS^jF_{(1,1)}$.) It should be noted here that, in the superadditive setting, it is possible to give alternative definitions of moving averages, however, such averages may fail to converge a.e. and in the mean as shown in [ÇF, F].

Let T be a positive linear operator on L_p. A MAS $\mathbf{w}$ is called *good in the p-mean for* T if, for every $f\in L_p$, $\lim_n A_{\mathbf{w},n}(T)f$ exists in the L_p-norm. $\mathbf{w}$ is called *good in the p-mean* if it is good in the p-mean for all (operators induced by) MPTs. Similarly, a MAS $\mathbf{w}$ is called *good a.e. for* T if, for every $f\in L_p$, $\lim_n A_{\mathbf{w},n}(T)f$ exists a.e., and $\mathbf{w}$ is called *good a.e.* if it is good a.e. for all (operators induced by) MPTs. A (two parameter) MAS $\mathbf{w}$ is called *good in the p-mean* (*good a.e.*) *for linear operators* T *and* S if $\lim_n A_{\mathbf{w},n}(T,S)f$ exists in the L_p-norm (a.e.) for all $f\in L_p$. A MAS $\mathbf{w}$ is called *good in the p-mean* (*good a.e.*) *for a* (T,S)*-superadditive process* F if $\lim_{n\to\infty}\frac{1}{|\mathbf{r}_n|}T^{a_n^1}S^{a_n^2}F_{(r_n^1,r_n^2)}$ exists in L_p-norm (a.e.).

In what follows, for practicality, all the propositions that require the cone condition will be stated in terms of B-sequences. Of course, one can easily restate them using the cone condition. Also, we will use the notation $\mathbf{n}=(n,n)$, for any integer $n\geq 0$. We will assume that $\mathbb{Z}_+^2$ is ordered in the usual manner, that is, $(m,n)\leq(u,v)$ if $m\leq u,\ n\leq v$, and $(m,n)<(u,v)$ if $(m,n)\leq(u,v)$ and $(m,n)\neq(u,v)$.

2. Convergence in the p-Mean

Moving averages of additive processes converge a.e. if and only if the MAS satisfy the cone condition. An example of MAS for which a.e. convergence fails is given in [AdJ]. On the other hand, the situation is different for the norm convergence. Indeed, by a criterion of Bellow Jones and Rosenblatt [BJR$_2$, Corollary 1.8], if $\mathbf{w}=\{(a_n,r_n)\}$ is a MAS and $\nu_nf(x)=\frac{1}{r_n}\sum_{i=0}^{r_n-1}T^{a_n+i}f(x)$, then

$$\hat{\nu}_n(\gamma)=\frac{1}{r_n}(\gamma^{a_n}\frac{1-\gamma^{r_n}}{1-\gamma})\to 0 \ \text{ for all } \ |\gamma|=1,\ \gamma\neq 1.$$

Hence *any* MAS is good in the p-mean (with a T-invariant limit). Recently, in [ÇF] the following is proved:

Theorem A. *Let T be a positive L_p-contraction, $1<p<\infty$, or a positive Dunford-Schwartz operator on L_1. If $\{n_k\}$ is a sequence of positive integers which is good in the p-mean for a class of* (*super*)*additive processes relative to MPTs, then it is good in the p-mean for T-*(*super*)*additive processes of the same class.*

Theorem A implies that a MAS is good in the p-mean for positive L_p-contractions, when $1<p<\infty$, or for L_1-L_∞-contractions (i.e. Dunford-Schwartz operators). Naturally, one asks if the same is valid for superadditive processes. Since the tool employed in [BJR$_2$] is inherently applicable to additive processes, one needs other techniques to answer this question. Also, there are sequences (not MAS) which are good in the mean for additive processes but not so for superadditive processess [ÇF]. In this section we will answer this question for both one-parameter and multiparameter (super)additive processes.

The following result, that can be proved in an arbitrary Banach space setting, tells us that for the mean convergence of *additive* processes, we can consider one parameter case only:

Proposition 2.1. *Let T and S be commuting contractions on a Banach space X. If $\mathbf{w} = \{(\mathbf{a}_n, \mathbf{r}_n)\}$, $r_n^i \to \infty$, is a MAS whose components are good in the p-mean for T and S, respectively, then $\mathbf{w}$ is good in the p-mean for T and S.*

Proof. Observe first that, the norm limit is (T and S) invariant. Hence

$$\lim_n A_{\mathbf{w}^1,n}(T)x = E_1 x, \quad \text{and} \quad \lim_n A_{\mathbf{w}^2,n}(S)x = E_2 x,$$

for some projections E_1 and E_2. Since T and S commute, so do the projections. Then

$$\|A_{\mathbf{w},n}(T,S)x - E_1E_2x\|_p \leq \|A_{\mathbf{w},n}(T,S)x - A_{\mathbf{w}^2,n}(S)E_1x\|_p + \|A_{\mathbf{w}^2,n}(S)E_1x - E_1E_2x\|_p$$

and, by assumption, both the terms on the right hand side tend to zero. $\square$

Since MASs are good in the p-mean for additive processes relative to MPTs, Theorem A and Proposition 2.1 imply that MASs are good in the p-mean for additive processes relative to positive L_p-contractions, when $1 < p < \infty$, or for Dunford-Schwartz operators on L_1. Hence, we obtain:

Corollary 2.2. *Multiparameter MASs are good in the p-mean for positive L_p-contractions (when $1 < p < \infty$) or Dunford-Schwartz operators on L_1.*

For the rest of this section, unless stated otherwise, we will consider *superadditive processes* relative to MPTs only. The solution to the problem for superadditive processes will be studied in two cases.

Case 1. $p = 1$. The following is a two parameter version of a lemma of Akcoglu and Sucheston [AS$_3$], which is proved similarly. So, we omit the proof.

Lemma 2.3. *Let F be a positive (T,S)-superadditive process, where T and S are positive L_p-contractions, $1 \leq p < \infty$. If $h_k = \frac{1}{k^2}F_{\mathbf{k}}$, $k > 1$, then (with the convention that sums over void sets are zero) $F_{\mathbf{n}} \geq \sum_{i=0}^{n-k-1}\sum_{j=0}^{n-k-1} T^iS^jh_k$.*

If $F \subset L_1$ is a bounded positive strongly (T,S)-superadditive process with an exact dominant $\delta \in L_1$, then, together with Lemma 2.3, this yields that, for all $n > k \geq 1$,

$$H^k_{\mathbf{n}-\mathbf{k}} \leq F_{\mathbf{n}} \leq G_{\mathbf{n}},$$

where $G_{(m,n)} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} T^iS^j\delta$, and $H^k_{(m,n)} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} T^iS^jh_k$. So, for a MAS $\mathbf{w} = \{(a_n, r_n)\}$, if n is large enough, we have

$$0 \leq \frac{1}{|\mathbf{r}_n|}(F_{\mathbf{r}_n} - H^k_{\mathbf{r}_n-\mathbf{k}}) \leq \frac{1}{|\mathbf{r}_n|}(G_{\mathbf{r}_n} - H^k_{\mathbf{r}_n-\mathbf{k}}).$$

Both the processes $\{G_{(m,n)}\}$ and $\{H^k_{(m,n)}\}$ are positive (T,S)-additive processes. Thus, by Corollary 2.2, the averages $\frac{1}{|\mathbf{r}_n|}T^{a_n^1}S^{a_n^2}G_{\mathbf{r}_n}$ and $\frac{1}{|\mathbf{r}_n|}T^{a_n^1}S^{a_n^2}H^k_{\mathbf{r}_n}$ converge in the L_1-norm. Observe that

$$L_1 - \lim_n \frac{1}{|\mathbf{r}_n|}T^{a_n^1}S^{a_n^2}H^k_{\mathbf{r}_n} = L_1 - \lim_n \frac{1}{|\mathbf{r}_n|}T^{a_n^1}S^{a_n^2}H^k_{\mathbf{r}_n-\mathbf{k}}.$$

Hence,

$$\lim_n \frac{1}{|\mathbf{r}_n|} T^{a_n^1} S^{a_n^2} (G_{\mathbf{r}_n} - H^k_{\mathbf{r}_n - \mathbf{k}}) = \lim_n A_{\mathbf{w},n}(T,S)(\delta - h_k) = \delta^* - g_k$$

exists in the L_1-norm, where $\delta^* = L_1 - \lim A_{\mathbf{w},n}(T,S)\delta$ and $g_k = L_1 - \lim_n A_{\mathbf{w},n}(T,S)h_k$. Consequently

$$(1) \qquad 0 \le L_1 - \lim \frac{1}{|\mathbf{r}_n|} T^{a_n^1} S^{a_n^2} (F_{\mathbf{r}_n} - H^k_{\mathbf{r}_n - \mathbf{k}}) \le \delta^* - g_k.$$

Lemma 2.4. *If F is a positive (T,S)-superadditive process, where T and S are positive L_p-contractions, then for every $k \ge 1$, $g_k \le g_{2k}$ and $\|g_k\|_1 \le \gamma_F$.*

Proof. By superadditivity, $F_{\mathbf{2k}} \ge F_{\mathbf{k}} + T^k F_{\mathbf{k}} + S^k F_{\mathbf{k}} + T^k S^k F_{\mathbf{k}}$, for all $k \ge 1$. Therefore,

$$\begin{aligned} g_{2k} &= L_1 - \lim_n \frac{1}{|\mathbf{r}_n|} T^{a_n^1} S^{a_n^2} H^{2k}_{\mathbf{r}_n} \\ &\ge L_1 - \lim_n \frac{1}{4|\mathbf{r}_n|} T^{a_n^1} S^{a_n^2} (H^k_{\mathbf{r}_n} + T^k H^k_{\mathbf{r}_n} + S^k H^k_{\mathbf{r}_n} + T^k S^k H^k_{\mathbf{r}_n}) \\ &= \frac{1}{4}[L_1 - \lim_n \frac{1}{|\mathbf{r}_n|} (T^{a_n^1} S^{a_n^2} H^k_{\mathbf{r}_n} + T^{a_n^1 + k} H^k_{\mathbf{r}_n} + S^{a_n^2 + k} H^k_{\mathbf{r}_n} + T^{a_n^1 + k} S^{a_n^2 + k} H^k_{\mathbf{r}_n})] = g_k. \end{aligned}$$

On the other hand, since $\|\frac{1}{|\mathbf{r}_n|} T^{a_n^1} S^{a_n^2} H^k_{\mathbf{r}_n}\|_1 \le \|\frac{1}{k^2} F_{\mathbf{k}}\|_1 \le \gamma_F$, for every k, the second assertion also follows. □

Theorem 2.5. *Let T and S be commuting MPTs and $\mathbf{w} = \{(\mathbf{a}_n, \mathbf{r}_n)\}$ be a MAS. Then $\mathbf{w}$ is good in the 1-mean for bounded strongly (T,S)-superadditive processes and the limit is T-invariant.*

Proof. Let F be a bounded strongly (T,S)-superadditive process. Then there exists an exact dominant δ for F. Since $\mathbf{w}$ is good in the 1-mean for additive processes, we can assume that F is positive. Given $\epsilon > 0$, pick k such that $\|h_k\|_1 = \|\frac{1}{k^2} F_{\mathbf{k}}\|_1 > \gamma_F - \epsilon/2$. By (1), $0 \le L_1 - \lim_n \frac{1}{|\mathbf{r}_n|} T^{a_n^1} S^{a_n^2} [F_{\mathbf{r}_n} - H^k_{\mathbf{r}_n}] \le \delta^* - g_k$ (where g_k and δ^* are as defined above). It follows from the measure preserving property that $\int g_k d\mu > \gamma_F - \frac{\epsilon}{2}$. For the same reason, and since δ is an exact dominant, we also have $\gamma_F = \int \delta = \int \delta^*$. By Lemma 2.4, $\{g_{2^i k}\}_i$ is an increasing sequence of L_1-functions, hence there exists $g \in L_1$ such that $g_k \uparrow g$ in L_1-norm. So, there exists a k (can be assumed equal to the previous one) such that $\|g_k - g\|_1 < \frac{\epsilon}{2}$. Then

$$\|\delta^* - g\|_1 \le \|\delta^* - g_k\|_1 + \|g_k - g\|_1 < [\int \delta^* - \int g_k] + \frac{\epsilon}{2} < (\gamma_F - \gamma_F + \frac{\epsilon}{2}) + \frac{\epsilon}{2} = \epsilon,$$

since $\int g_k = \int h_k > \gamma_F - \frac{\epsilon}{2}$. Arbitrariness of ϵ implies that $L_1 - \lim_n \frac{1}{|\mathbf{r}_n|} T^{a_n^1} S^{a_n^2} F_{\mathbf{r}_n}$ exists (and is equal to g). The invariance of the limit follows from the invariance of g_k's and of δ^*. □

Theorem A yields to the extension of one parameter version of Theorem 2.5 to the operator setting when T is a positive Dunford-Schwartz operator:

Corollary 2.6. *Let T be a positive Dunford-Schwartz operator on L_1 and $\mathbf{w} = \{(a_n, r_n)\}$ be a MAS. Then $\mathbf{w}$ is good in the 1-mean for bounded T-superadditive processes and the limit is T-invariant.*

Remark 3. Since the method of proof in [ÇF] is valid only in *one parameter case*, extending Theorem 2.5 to *operator* setting requires a different approach (see the discussion in Section 4). Also, in this approach one has to use a technique that does not depend on the existence of an (exact) dominant for F, since in the multiparameter operator setting no result on the existence of an exact dominant for superadditive processes is known.

Case 2. $\mathbf{1 < p < \infty}$. The solution to the problem in this case will be considered for one parameter processes first. In [DK], Derriennic and Krengel constructed an example of a positive superadditive process in L_2, satisfying $\sup_n \|\frac{1}{n}F_n\|_2 < \infty$, whose averages do not converge in the L_2-norm. Hence for the convergence in the p-mean for superadditive processes one needs more than *boundedness* condition on the process. One possibility is the condition $(*)$ considered in the next section (for a.e. convergence). Although it leads to a.e. convergence for (multiparameter) superadditive processes, the tools available do not yield to a conclusive result for the norm convergence if $(*)$ is assumed. In [ÇF] it has been observed that for a more restrictive class of superadditive processes, namely *Chacon admissible processes*, one can obtain affirmative results both for a.e. and norm convergence. That is why, for the rest of this section we will work with such processes.

Definition 1. A family of functions $\{f_n\} \subset L_p$ is called a *Chacon T-admissible family* (or simply *admissible family*) if $Tf_i \leq f_{i+1}$ for all $i \geq 1$.

If $\{f_n\}$ is T-admissible, then the process $F = \{F_n\}$, where $F_n = \sum_{i=0}^{n-1} f_i$, is a T-superadditive process, called an *admissible process*. The following is an important property of admissible processes:

Proposition 2.7. *Let $F \subset L_p$, $1 < p < \infty$, be a positive T-admissible process, where T is a MPT. If F is bounded, then* $\sup_k \|f_k\|_p < \infty$.

Proof. By the measure preserving property of T and by the admissibility,

$$\|f_k\|_p^p = \|T^j f_k\|_p^p = \frac{1}{r}\sum_{j=0}^{r-1} \|T^j f_k\|_p^p \leq \frac{1}{r}\sum_{j=0}^{r-1} \|f_{k+j}\|_p^p = \frac{1}{r}\sum_{j=0}^{r-1} \|F_{k+j+1} - F_{k+j}\|_p^p.$$

First let $2 \leq p < \infty$. In that case, by Clarkson's inequality for $2 \leq p < \infty$ and superadditivity, we have

$$\begin{aligned}\|F_{k+j+1} - F_{k+j}\|_p^p &\leq 2^{p-1}(\|F_{k+j+1}\|_p^p + \|F_{k+j}\|_p^p) - \|F_{k+j+1} + F_{k+j}\|_p^p \\ &\leq 2^p[\|F_{k+j+1}\|_p^p - \|F_{k+j}\|_p^p].\end{aligned}$$

Thus,

$$\begin{aligned}\|f_k\|_p^p &\leq \frac{1}{r}\sum_{j=0}^{r-1} \|F_{k+j+1} - F_{k+j}\|_p^p \leq \frac{2^p}{r}\sum_{j=0}^{r-1} \|F_{k+j+1}\|_p^p - \|F_{k+j}\|_p^p \\ &= \frac{2^p}{r}[\|F_{k+r}\|_p^p - \|F_k\|_p^p] \leq \frac{2^p}{r}\|F_{k+r}\|_p^p \leq \frac{2^p(k+r)}{r}\sup_{n\geq 1}\frac{1}{n}\|F_n\|_p^p.\end{aligned}$$

Therefore, taking the limit $r \to \infty$, we have $\|f_k\|_p^p \leq 2^p \sup_n \frac{1}{n}\|F_n\|_p^p < \infty$, proving the assertion. When $1 < p < 2$, again applying Clarkson's inequality for this case, we have

$$\|f_k\|_p^q \le \frac{1}{r}\sum_{j=0}^{r-1}\|F_{k+j+1}-F_{k+j}\|_p^q \le \frac{1}{r}\sum_{j=0}^{r-1}2^q[\|F_{k+j+1}\|_p^q-\|F_{k+j}\|_p^q] = \frac{2^q}{r}\|F_{k+r}\|_p^q.$$

Now, the assertion follows as before. □

Theorem 2.8. *Let T be a MPT and F be a bounded T-admissible process. Then any MAS* $\mathbf{w}$ *is good in the p-mean for F, $1<p<\infty$, and the limit is T-invariant.*

Proof. Since $\{f_i\}\subset L_p$ is an admissible family, $\sum_{j=0}^{n-1}T^jf_0\le F_n$, and hence we can assume that $f_i\ge 0$, $i\ge 0$. For convenience, define $P_i=f_i-Tf_{i-1}$, $i\ge 1$, where we set $P_0=f_0$. Observe that, by Clarkson's inequalities and Proposition 2.7,

$$(2)\quad \int P_r^p=\int(f_r-Tf_{r-1})^p\le C_p\Big[\int f_r^p-\int f_{r-1}^p\Big]<\infty\quad (C_p=2^{p-1}\ \text{ or }\ 2^{q-1}).$$

Now we will use a technique employed in [ÇF]: for a fixed positive integer k, define

$$g_n^k(x)=\begin{cases} f_k(T^{n-k}x) & \text{for } n>k\\ f_n(x) & \text{for } 0\le n\le k.\end{cases}$$

Then, it follows that

$$(3)\qquad f_n(x)-g_n^k(x)=\begin{cases}0 & \text{if } 0\le n\le k\\ \sum_{i=1}^m P_{k+i}(T^{m-i}x) & \text{for } n>k, \text{ where } m=n-k.\end{cases}$$

Define $D_i(x)=\sum_{n=0}^{r_i-1}f_n(x)-g_n^k(x)$. By making use of (3) we estimate that

$$D_i(x)\le\sum_{n=0}^{r_i-1}\sum_{r=k+1}^{n}P_r(T^{n-r}x).$$

Next, if we let

$$b_{k,t}(x)=\sum_{r=k+1}^{t}P_r(T^rx)\qquad\text{and}\qquad b_k(x)=\lim_{t\to\infty}b_{k,t}(x),$$

then $b_{k,t}\ge 0$, and $b_k\ge 0$. Using the Lebesgue monotone convergence theorem, we obtain that

$$(4)\qquad \int_X b_k^p d\mu=\lim_{t\to\infty}\int_X b_{k,t}^p d\mu\le\sum_{r=k+1}^{\infty}\int P_r^p\le C_p\lim_{r\to\infty}\int f_r^p<\infty,$$

by Proposition 2.7 and (2). Because $b_{k,t}\uparrow b_k$ and $b_k\in L_p$ we conclude that $T^jb_{k,t}\uparrow T^jb_k$ in L_p, for all j, since T is strongly continuous. Therefore, (4) implies that

$$T^{a_i}D_i\le\sum_{n=0}^{r_i-1}T^{a_i+n}b_k.$$

On the other hand, observe that, since T is measure preserving, as $k\to\infty$,

$$\|\frac{1}{r_i}\sum_{n=0}^{r_i-1}T^{a_i+n}b_k\|_p^p\le\|b_k\|_p^p\le\sum_{i=k+1}^{\infty}\int P_i^p\downarrow 0.$$

By assumption $G_k := L_p - \lim_{i\to\infty} \frac{1}{r_i} T^{a_i} \sum_{n=0}^{r_i-1} g_n^k$ exists and is T-invariant. Since, for all $n \geq 1$, $g_n^k \leq g_n^{k+1}$, we also have $G^k \leq G^{k+1}$. Therefore, $\{G^k\}$ is a monotone increasing sequence of functions in L_p, and consequently, $G = \lim_{k\to\infty} G^k$ exists in L_p and is T-invariant. Now, given $\epsilon > 0$, find a positive integer K such that for $k \geq K$, $\|b_k\|_p^p < \epsilon/3$, $\|\frac{1}{r_i} T^{a_i} \sum_{n=0}^{r_i-1} g_n^k - G^k\|_p^p < \epsilon/3$, and $\|G - G^k\|_p^p < \epsilon/3$. Then,

$$\|\frac{1}{r_i} T^{a_i} \sum_{n=0}^{r_i-1} f_n - G\|_p^p \leq \|\frac{1}{r_i} T^{a_i} \sum_{n=0}^{r_i-1} (f_n - g_n^k)\|_p^p + \|\frac{1}{r_i} T^{a_i} \sum_{n=0}^{r_i-1} g_n^k - G^k\|_p^p + \|G - G^k\|_p^p < \epsilon,$$

proving the assertion. $\square$

Now, as an immediate consequence of Theorem 2.8 and Theorem A, we have:

Corollary 2.9. *Let T be a positive L_p-contraction, $1 < p < \infty$. If F is a bounded T-admissible process, then any MAS $\mathbf{w}$ is good in the p-mean for F, and the limit is T-invariant.*

In order to obtain the two-parameter version of Theorem 2.8, we define two-parameter (T,S)-admissible processes as a family $\{f_{i,j}\} \subset L_p$ such that $Tf_{i,j} \leq f_{i+1,j}$, and $Sf_{i,j} \leq f_{i,j+1}$, for all $i \geq 0$, $j \geq 0$. As before, any such (T,S)-admissible family defines a (T,S)-superadditive processes $\{F_{(m,n)}\}$, where $F_{(m,n)} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f_{i,j}$.

Theorem 2.10. *Let T and S be commuting MPTs and $F = \{F_{(m,n)}\}$ be a bounded (T,S)-admissible process. Then any MAS $\mathbf{w}$ is good in the p-mean for F, $1 < p < \infty$.*

Proof. Let $\mathbf{w}^1 = \{(a_n^1, r_n^1)\}$ and $\mathbf{w}^2 = \{(a_n^2, r_n^2)\}$ be the components of $\mathbf{w}$. Define, for $i \geq 0$, $g_i = L_p - \lim_n \frac{1}{r_n^2} S^{a_n^2} \sum_{j=0}^{r_n^2-1} f_{i,j}$. By Theorem 2.8, these functions $g_i \in L_p$ are well defined. Observe that, by (T,S)-admissiblity of F and strong continuity of T,

$$Tg_i = L_p - \lim_n \frac{1}{r_n^2} TS^{a_n^2} \sum_{j=0}^{r_n^2-1} f_{i,j} \leq L_p - \lim_n \frac{1}{r_n^2} S^{a_n^2} \sum_{j=0}^{r_n^2-1} f_{i+1,j} = g_{i+1},$$

implying that $\{g_i\}$ is a T-admissible process. Furthermore, boundedness of F implies that this process is bounded also. Hence $g := L_p - \lim \frac{1}{r_n^1} T^{a_n^1} \sum_{i=0}^{r_n^1-1} g_i$ exists by Theorem 2.8. Now, the inequality

$$\|\frac{1}{|\mathbf{r}_n|} T^{a_n^1} S^{a_n^2} F_{(r_n^1, r_n^2)} - g\|_p \leq \|\frac{1}{|\mathbf{r}_n|} T^{a_n^1} S^{a_n^2} F_{(r_n^1, r_n^2)} - \frac{1}{r_n^1} T^{a_n^1} \sum_{i=0}^{r_n^1-1} g_i\|_p + \|\frac{1}{r_n^1} T^{a_n^1} \sum_{i=0}^{r_n^1-1} g_i - g\|_p$$

proves the theorem. $\square$

Remark 4. In the proofs of Theorem 2.8 and Theorem 2.10 only superadditivity is needed as opposed to strong superadditivity (in Theorem 2.5.) Hence, the assertion of Theorem 2.10 is valid for any n-parameter bounded admissible process, $n \geq 1$.

Remark 5. The arguments employed in the proofs of Theorem 2.8 and Theorem 2.10 can be repeated almost verbatim (in fact, more simply) when $p = 1$, with the same conclusions. Therefore: *if* $F \subset L_1$ *is an* n*-parameter bounded* admissible (*superadditive*) *process,* $n \geq 1$*, then any MAS* $\mathbf{w}$ *is good in the 1-mean for* F*.*

As remarked earlier, it is possible to define averages of a superadditive process along a MAS in an alternative fashion. More precisely, if $\mathbf{w} = \{(a_n, r_n)\}$ is a MAS, the averages of a superadditive process F along $\mathbf{w}$ can be defined as

$$(\dagger) \qquad \frac{1}{r_n}(F_{a_n + r_n} - F_{r_n}).$$

It is observed in [ÇF] that the averages (†) may fail to converge in the mean (and a.e.) There, when $p = 1$, it is shown that if the process is T-admissible and $\mathbf{w}$ is a B-sequence, then such averages *do* converge a.e. and in the L_1-norm. The argument used there is very similar to the proof of Theorem 2.8. Hence, the same proof works for the convergence in the p-mean if the averages of admissible processes along $\mathbf{w}$ is defined by (†).

3. Almost Everywhere Convergence

In this section we study the a.e. convergence of moving averages of multiparameter superadditive processes relative to positive L_p-contractions, $1 < p < \infty$. The averages we will consider are those averages along sequence of *B-cubes*, which are multiparameter B-sequences $\mathbf{w} = \{(\mathbf{a}_n, \mathbf{r}_n)\}$, with $r_n := r_n^1 = r_n^2$. (See [F, JO_1] for definitions.) The components of all the B-cubes form one parameter B-sequences. When $p = 1$, a.e. convergence of moving averages of bounded superadditive processes (relative to MPTs) along B-sequences is proved in [F].

We will assume that all the processes under study will satisfy the condition

$$(*) \qquad \liminf_{v \to \infty} \Big\| \frac{1}{v^2} \sum_{i,j=0}^{v-1} F_{(i,j)} - TF_{(i-1,j)} - SF_{(i,j-1)} + TSF_{(i-1,j-1)} \Big\|_p < \infty.$$

This condition was introduced and used in [EH] to obtain the a.e. convergence of multiparameter superadditive processes with respect to positive L_p-contractions. If a positive (T, S)-superadditive process F satisfies the condition $(*)$, then the sequence $\{\phi_v\} \subset L_p$ is a bounded set in L_p, where

$$\phi_v = \frac{1}{v^2} \sum_{i,j=0}^{v-1} F_{(i,j)} - TF_{(i-1,j)} - SF_{(i,j-1)} + TSF_{(i-1,j-1)}.$$

Hence, $\{\phi_v\}$ is weakly sequentially compact. Consequently, along a subsequence $\{v_i\}$, the weak limit of $\{\phi_v\}$ exists, that is, there is a function $\phi \in L_p$ such that, $\phi = w - \lim_{i \to \infty} \phi_{v_i}$.

It is known that, if F is a positive strongly (T, S)-superadditive process, then, for any $v > 1$ and for any $1 \leq n \leq v$, $(1 - \frac{n}{v})^2 F_{\mathbf{n}} \leq \sum_{i,j=0}^{n-1} T^i S^j \phi_v$ [$\mathrm{Ç}_1$, EH]. Thus, for a positive strongly (T, S)-superadditive process F satisfying $(*)$, by the strong continuity of T and S, we have

$$F_{\mathbf{n}} \leq \sum_{i,j=0}^{n-1} T^i S^j \phi,$$

where $\phi = w - \lim_{i\to\infty} \phi_{v_i}$. This shows that:

Lemma 3.1. *Let F be a positive strongly (T,S)-superadditive process satisfying $(*)$, then it has a dominant $\phi \in L_p$.*

The following result of Akcoglu and Sucheston will be very instrumental in proving the a.e. convergence. We state it here for easy reference.

Theorem B. [AS$_1$] *Let U be a positive L_p-contraction, $1 < p < \infty$ fixed. Then there is a unique decomposition of X into sets E and E^c such that*

(i) *E is the support of a U-invariant function $h \in L_p$, and the support of each U-invariant function is contained in E.*
(ii) *The subspaces $L_p(E)$ and $L_p(E^c)$ are both invariant under U.*

Now, as in Section 2 (case $p = 1$), if $H^k_{(m,n)} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} T^i S^j h_k$, for $k \geq 1$, then Lemma 2.3, together with Lemma 3.1, imply that, for a positive strongly (T,S)-superadditive process F, if $n > k$, $H^k_{\mathbf{n}-\mathbf{k}} \leq F_{\mathbf{n}} \leq G_{\mathbf{n}}$, where in this case $G_{(m,n)} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} T^i S^j \phi$. Hence, if $\mathbf{w} = \{(\mathbf{a}_n, \mathbf{r}_n)\}$ is a B-sequence, for n large enough, we have

$$0 \leq \frac{1}{n^2}(F_{\mathbf{n}} - H^k_{\mathbf{n}-\mathbf{k}}) \leq \frac{1}{n^2}(G_{\mathbf{n}} - H^k_{\mathbf{n}-\mathbf{k}}).$$

Both $\{G_{(m,n)}\}$ and $\{H^k_{(m,n)})\}$ are positive (T,S)-additive processes, so, by the Theorem of Jones and Olsen [JO$_1$, Theorem 2.1], the averages $\frac{1}{r_n^2}T^{a_n^1}S^{a_n^2}G_{\mathbf{r}_n}$ and $\frac{1}{r_n^2}T^{a_n^1}S^{a_n^2}H^k_{\mathbf{r}_n}$ converge a.e. (and also, $\lim_n \frac{1}{r_n^2}T^{a_n^1}S^{a_n^2}H^k_{\mathbf{r}_n} = \lim_n \frac{1}{r_n^2}T^{a_n^1}S^{a_n^2}H^k_{\mathbf{r}_n-\mathbf{k}}$ a.e.) Hence,

$$\lim_n \frac{1}{r_n^2}T^{a_n^1}S^{a_n^2}(G_{\mathbf{r}_n} - H^k_{\mathbf{r}_n-\mathbf{k}}) = \lim_n A_{\mathbf{w},n}(T,S)(\phi - h_k) \text{ exists a.e.}$$

Since,

$$\|\sum_{n=0}^{\infty}\frac{1}{r_n^2}(T^{a_n^1+r_n} - T^{a_n^1})\sum_{j=a_n^2}^{a_n^2+r_n-1} S^j(\phi - h_k)\|_2^2 \leq 2\|\phi - h_k\|_2^2\sum_{n=0}^{\infty}\frac{1}{r_n^2} < \infty,$$

it follows that this limit is T-invariant. Similarily, the limit is also S-invariant. Hence, if E is the maximal support of a non-negative (T,S)-invariant function h, then this limit is zero on the set E^c by Theorem B. Furthermore, the process $1_E F$ is strongly (T_E, S_E)-superadditive process, where T_E and S_E are restrictions of T and S to $L_p(E)$, respectively. (We will denote them with T and S for simplicity.)

Now, let $m = h^p.\mu$ be a new (finite) measure on X, and consider the operators $\hat{T}f = h^{-1}T(fh)$ and $\hat{S}f = h^{-1}S(fh)$, $f \in L_p(m)$. Then they are positive $L_p(m)$-contractions with $\hat{T}1 = 1$ and $\hat{S}1 = 1$, and $\int \hat{T}f dm = \int f dm = \int \hat{S}f dm$ [Ç$_2$, DK]. Therefore, they can be extended to Markovian operators on $L_1(m)$, which will still be denoted by $\hat{T}$ and $\hat{S}$. Furthermore, $A_{\mathbf{w},n}(T,S)g$, $g \in L_p(\mu)$, converge μ-a.e. if and only if $A_{\mathbf{w},n}(\hat{T},\hat{S})(h^{-1}g)$ converge m-a.e.

Theorem 3.2. *Let T and S be commuting positive linear L_p-contractions and F be a strongly (T,S)-superadditive process satisfying $(*)$. If $\{(\mathbf{a}_n, \mathbf{r}_n)\}$ is a sequence of B-cubes, then it is good a.e. for F.*

Proof. Since (T,S)-additive processes converge a.e. by the theorem of Jones and Olsen [JO$_1$], we can assume F be nonnegative. Then, by the observations above we can assume for the rest of the proof that $X = E$. Let $m = h^p.\mu$, where h is a (T,S)-invariant function given by Theorem B. Consider the operators $\hat{T}f = h^{-1}T(fh)$ and $\hat{S}f = h^{-1}S(fh)$, $f \in L_p(m)$, which are positive $L_p(m)$-contractions with $\hat{T}1 = 1$ and $\hat{S}1 = 1$, and are Markovian on $L_1(m)$. If $F' = \{h^{-1}F_{(m,n)}\}$, then $F' \subset L_1(m)$ is a strongly $(\hat{T},\hat{S})$-superadditive bounded process with $\gamma_{F'} = \sup_{m,n\geq 1}\int \frac{1}{mn}h^{-1}F_{(m,n)}dm$. Therefore, $\gamma_{F'} = \lim_{m,n\to\infty}\frac{1}{mn}\int h^{-1}F_{(m,n)}dm$ [Ç$_1$, EH]. Now given $\epsilon > 0$, find $k \geq 1$ such that $\frac{1}{k^2}\int h^{-1}F_{\mathbf{k}}dm > \gamma_{F'} - \epsilon$. Define $H_{(m,n)} = H^k_{(m,n)} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}T^iS^jh_k$, where $h_k = \frac{1}{k^2}F_{\mathbf{k}}$, and $G_{(m,n)} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}T^iS^j\phi$. Also, define $H'_{(m,n)} = h^{-1}H_{(m,n)}$, and $G'_{(m,n)} = h^{-1}G_{(m,n)}$. Then $\{H_{(m,n)}\}$ is a positive (T,S)-additive process, and so $\{H'_{(m,n)}\} \subset L_1(m)$ is a $(\hat{T},\hat{S})$-additive process. Hence,

$$0 \leq \int \limsup_n \frac{1}{r_n^2}\hat{T}^{a_n^1}\hat{S}^{a_n^2}(F'_{\mathbf{r}_n} - H'_{\mathbf{r}_n})dm \leq \int \limsup_n \frac{1}{r_n^2}\hat{T}^{a_n^1}\hat{S}^{a_n^2}(G'_{\mathbf{r}_n} - H'_{\mathbf{r}_n})dm.$$

Since $\lim_n \frac{1}{r_n^2}T^{a_n^1}S^{a_n^2}(G_{\mathbf{r}_n} - H_{\mathbf{r}_n})$ exists μ-a.e., $\lim_n \frac{1}{r_n^2}\hat{T}^{a_n^1}\hat{S}^{a_n^2}(G'_{\mathbf{r}_n} - H'_{\mathbf{r}_n})$ exists m-a.e. too. Therefore, using Fatou's Lemma,

$$\int \limsup_n \frac{1}{r_n^2}\hat{T}^{a_n^1}\hat{S}^{a_n^2}(G'_{\mathbf{r}_n} - H'_{\mathbf{r}_n})dm \leq \lim_n \int \frac{1}{r_n^2}\hat{T}^{a_n^1}\hat{S}^{a_n^2}(G'_{\mathbf{r}_n} - H'_{\mathbf{r}_n})dm.$$

Consequently,

$$\begin{aligned}0 &\leq \int \limsup_n \frac{1}{r_n^2}\hat{T}^{a_n^1}\hat{S}^{a_n^2}(F'_{\mathbf{r}_n} - H'_{\mathbf{r}_n})dm \\ &\leq \lim_n \frac{1}{r_n^2}\int G'_{\mathbf{r}_n}dm - \lim_n \frac{1}{r_n^2}\int H'_{\mathbf{r}_n}dm \leq \int h^{-1}\phi dm - \lim_n \frac{u^2}{r_n^2}\int \frac{1}{k^2}h^{-1}F_{\mathbf{k}}dm \\ &= \int h^{-1}\phi dm - (\gamma_{F'} - \epsilon),\end{aligned}$$

where $u = [r_n/k]$, integer part of r_n/k. On the other hand, observe that $\int h^{-1}\phi dm = \int h^{p-1}\phi d\mu$ and $h^{p-1} \in L_q(\mu)$. Since $\phi = w-\lim \phi_{v_k}$, and $\hat{T}$ and $\hat{S}$ are Markovian, (using v instead of v_k, for convenience) we have

$$\begin{aligned}\int h^{p-1}\phi d\mu &= \lim_v \int h^{p-1}\phi_v d\mu \\ &= \lim_v \int h^{p-1}[\frac{1}{v^2}\sum_{i,j=0}^{v-1} F_{(i,j)} - TF_{(i-1,j)} - SF_{(i,j-1)}r + TSF_{(i-1,j-1)}]d\mu \\ &= \lim_v \int [\frac{1}{v^2}\sum_{i,j=0}^{v-1} F'_{(i,j)} - \hat{T}F'_{(i-1,j)} - \hat{S}F'_{(i,j-1)} + \hat{T}\hat{S}F'_{(i-1,j-1)}]dm \quad . \\ &= \lim_v \int [\frac{1}{v^2}\sum_{i,j=0}^{v-1} F'_{(i,j)} - F'_{(i-1,j)} - F'_{(i,j-1)} + F'_{(i-1,j-1)}]dm \\ &= \lim_v \frac{1}{v^2}\int F'_{(v-1,v-1)}dm = \gamma_{F'}.\end{aligned}$$

This equality, combined with the inequality above, implies that

$$0 \le \int \limsup_n \frac{1}{r_n^2} \hat{T}^{a_n^1} \hat{S}^{a_n^2} (F'_{\mathbf{r}_n} - H'_{\mathbf{r}_n}) dm < \gamma_{F'} - (\gamma_{F'} - \epsilon) = \epsilon.$$

Hence $\lim_n \frac{1}{r_n^2} \hat{T}^{a_n^1} \hat{S}^{a_n^2} F'_{\mathbf{r}_n}$ exists m-a.e., which in turn, shows that $\lim_n \frac{1}{r_n^2} T^{a_n^1} S^{a_n^2} F_{\mathbf{r}_n}$ exists μ-a.e. □

Remark 6. The assertion of Theorem 3.2 is also valid for positive power bounded Lamperti operators (See [JO$_1$] for details).

4. Concluding Discussions

In two parameter case, for a MAS $\mathbf{w} = \{(\mathbf{a}_n, \mathbf{r}_n)\}$ and commuting MPTs T and S if we define

$$\nu_n f(x) = \frac{1}{|\mathbf{r}_n|} \sum_{i=0}^{r_n^1 - 1} \sum_{j=0}^{r_n^2 - 1} T^{a_n^1 + i} S^{a_n^2 + j} f(x),$$

then $\lim_n \hat{\nu}_n(\gamma, \delta) = 0$ for all $(\gamma, \delta) \neq (1,1)$, $|\gamma| = 1$, $|\delta| = 1$. Hence, this is the same conclusion of Corollary 2.2 when T and S are MPTs. One way of extending this result to L_p-contractions, $1 < p < \infty$, or to Dunford-Schwartz operators on L_1, would be via multidimensional version of Theorem A. However, multidimensional Theorem A is not available yet, due to the fact that Theorem A uses Akcoglu-Sucheston Dilation Theorem [AS$_2$], which is known in one parameter case only.

In a related matter, it is not known whether the condition $(*)$ is the only assumption that guarantees the existence of a dominant for a strongly superadditive process in L_p-spaces, $1 < p < \infty$. (See also Remark 10 below.) It seems that, in order to obtain a dominant for multiparameter processes in an alternative way, either one has to derive a method of "reduction of order" for B-sequences, as in [Ç$_1$, EH], or one has to employ a dilation argument to reduce the problem to the case of processes with respect to commuting invertible isometries of L_p-spaces.

If a two-parameter extension of Akcoglu-Sucheston Dilation Theorem were available, it would yield some nice results mentioned above as well as a dominated estimate for moving averages of two-parameter superadditive processes over B-cubes, as in [Ç$_2$]. (Such a dominated estimate would yield to mean convergence, provided that one has the right (exact) dominant.) The only multiparameter dilation result known in the literature is due to Ando, and is valid only for the two-parameter case in the Hilbert space setting:

Theorem B. [SF, Theorem I.6.4] *Let T and S be commuting contractions of a Hilbert space $\mathbb{H}$. Then there exists a larger Hilbert space $\mathbb{H}'$ and commuting unitary operators Q and R on $\mathbb{H}'$ such that, for all $n \ge 1$, $T^n = PQ^nI$ and $S^n = PR^nI$, where I is the natural imbedding of $\mathbb{H}$ into $\mathbb{H}'$ and $P : \mathbb{H}' \to \mathbb{H}'$ is the orthogonal projection onto the subspace $\mathbb{H}$.*

Using this theorem, one easily obtains a dominated estimate for moving averages of two-parameter superadditive processes over B-cubes via the dominated estimate of Jones and Olsen [JO$_1$] for the moving averages of multiparameter additive processes relative to positive L_p-contractions. Since the proof is very similar to the analogous statement in [Ç$_2$], it will be omitted.

Theorem 4.1. *Let T and S be positive L_2-contractions, $F \subset L_2$ be a positive (T,S)-superadditive process with a dominant ϕ, and let $\{(\mathbf{a}_n, \mathbf{r}_n)\}$ be a B-sequence. Then*

$$\| \frac{1}{|\mathbf{r}_n|} T^{a_n^1} S^{a_n^2} F_{\mathbf{r}_n} \|_2 \leq C \|\phi\|_2,$$

where C is a constant independent of the sequence and the process.

Remark 7. It has been shown in [SF] that Theorem B cannot be extended to n-parameters, when $n > 2$.

Remark 8. Obtaining a "reduction of order" procedure for the moving averages of multiparameter (super)additive processes is an open problem.

Remark 9. The proofs of Theorems 2.5 and 3.2 depend upon the assumption of strong superadditivity, which is defined in two parameter case only. Strong superadditivity is needed for the existence of an exact dominant.

Remark 10. Akcoglu and Krengel gave an example of two-parameter bounded superadditive process (in L_1), which is not strongly superadditive [AK]. (Since, for instance $F_{(11,10)} = 18$, while $F_{(6,5)} + F_{(5,10)} + F_{(11,5)} - F_{(5,5)} = 19$.) On the other hand, that process is dominated by the additive process $\{G_I\}$, where $G_I = \frac{1}{6}|I|$. Actually, this additive process is generated by $g = \frac{1}{6}I_0$, with $I_0 =$[0,1]×[0,1]. Since $\gamma_F = \frac{1}{6}$, g is an *exact* dominant for F.

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