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Finite Rank $\mathbb{Z}^d$ Actions and the Loosely Bernoulli Property

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Abstract. We define finite rank for $\mathbb{Z}^d$ actions and show that those finite rank actions with a certain tower shape are loosely Bernoulli for $d \geq 1$.

Contents

1. Introduction

In [4], the authors show that rank 1 $\mathbb{Z}^d$ transformations are loosely Bernoulli, extending the $\mathbb{Z}$ result from [7]. D. Ornstein, D. Rudolph, and B. Weiss also show in [7] that all finite rank transformations are loosely Bernoulli. In this paper we give the $\mathbb{Z}^d$ generalization of this result.

Intuitively, a zero entropy loosely Bernoulli (LB) action has one name up to the $\overline{f}$ metric on processes (cf [6, 2]). The proof in [4] rests on the fact that for a rank one action most large enough names are well covered by towers of various sizes. Given two large names we use this fact to identify towers in one name who have a same size tower close by in the second name. We use these pairs of neighbours to show that the names are $\overline{f}$ close.

In the rank r case with $r > 1$, it is still true that large names are well covered by towers of many sizes, but now each size tower is of r different types. Hence,

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while it is still possible to find close by towers, the methods of [4] do not guarantee that nearby towers are of the same type. In this paper we generalize the matching argument of [4] to address this issue.

The next two sections contain the necessary definitions. There is then a section containing a Generalized Matching Lemma. The final section uses this lemma to prove the result.

2. Background

Let $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space. Take T to be an ergodic $\mathbb{Z}^d$ action on $(X, \mathcal{A}, \mu)$. We can think of T as being generated by d commuting measure preserving 1-dimensional transformations on X, $\{T_{\vec{e}_1}, ..., T_{\vec{e}_d}\}$, where $\{\vec{e}_1, \ldots, \vec{e}_d\}$ is the standard basis for $\mathbb{Z}^d$. Then $T_{\vec{v}}(x) = T_{\vec{e}_1}^{v_1} \circ \cdots \circ T_{\vec{e}_d}^{v_d}(x)$, where $\vec{v} = (v_1, ..., v_d)$. We call $(X, \mathcal{A}, \mu), T$ a $\mathbb{Z}^d$-dynamical system. Often we will simply write (X, T).

Let P be a finite label set, or equivalently, a finite measurable partition $P = \{p_1, ..., p_h\}$ on X. (T, P) is then the usual process associated with T and the partition P. Set $||\vec{v}|| = \max\{|v_i| : 1 \leq i \leq d\}$, and for $n \in \mathbb{N}$,

$$B_n = \{\vec{v} = (v_1, \ldots, v_d) \in \mathbb{Z}^d : 0 \leq v_i \leq n\}.$$

For each x we can then define its P_n-name to be $P_n(x) : B_n \to P$ by $P_n(x)(\vec{v}) = i$ if $T_{\vec{v}}(x) \in p_i$. In order to define a loosely Bernoulli process we start with $\pi : B_n \to B_n$, a permutation of the indices in B_n, and define a size for this permutation. This idea is defined and extended in [1] and [5].

Definition 2.1. Let $\pi : B_n \to B_n$ be a permutation of the indices of B_n. We say π is of size ϵ, denoted by $m(\pi) < \epsilon$, if there exists a subset S of B_n satisfying

(i) $|S| > (1 - \epsilon)|B_n|$, where $|S|$ is the cardinality of the set S,

(ii) $||\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})|| < \epsilon||\vec{u} - \vec{v}||$ for every $\vec{u}, \vec{v} \in S$.

Definition 2.2. Given two P_n-names η and ξ, we define the $\overline{f}_n$-distance between them to be

$$\begin{aligned} \overline{f}_n(\eta, \xi) = \inf\{\epsilon > 0 : & \text{ there exists a permutation } \pi \text{ of } B_n \text{ such that} \\ & (i)\ m(\pi) < \epsilon \\ & (ii)\ \overline{d}(\eta \circ \pi, \xi) < \epsilon\}. \end{aligned}$$

Here $\overline{d}(.,.)$ denotes the Hamming metric which simply gives the proportion of locations of B_n on which the two names disagree.

Informally, we will think of π as rearranging the name η to make it $\overline{d}$ close to the name ξ and we will often refer to π as acting on a name instead of the (technically correct) set of indices. If π, η, and ξ satisfy (ii) of the above definition we say π matches a $(1 - \epsilon)$-proportion of η and ξ.

Intuitively, a zero entropy loosely Bernoulli process has one name up to $\overline{f}$. Formally,

Definition 2.3. A zero entropy process (T, P) is loosely Bernoulli (LB) if for any $\epsilon > 0$ there exists an integer N_ϵ such that for any $n \geq N_\epsilon$ and ϵ-a.e. atoms ω and ω' of $\bigvee_{\vec{v} \in B_n} T_{\vec{v}}P$,

$$\overline{f}_n(\omega, \omega') < \epsilon.$$

Definition 2.4. We say $(X, \mathcal{A}, \mu), T$ is LB if for every partition P of X, (T, P) is LB.

3. Finite Rank

Definition 3.1. Let $r \in \mathbb{N}$. We say $(X, \mathcal{A}, \mu), T$ is a $\mathbb{Z}^d$ rank r transformation if there exists a sequence of sets $F_i^j \subset X$, $1 \le j \le r$, and Følner sequences D_i^j, $1 \le j \le r$, of subsets of $\mathbb{Z}^d$, such that for each i, $\{T_{\vec{v}} F_i^j\}$ are pairwise disjoint for $\vec{v} \in D_i^j$ and $1 \le j \le r$, and the partitions

$$P_i = \{T_{\vec{v}} F_i^j : \vec{v} \in D_i^j \ \text{ and } \ 1 \le j \le r, \ \ X - \cup_{j=1}^r \cup_{\vec{v} \in D_i^j} T_{\vec{v}} F_i^j\}$$

converge to $\mathcal{A}$ as $i \to \infty$. We also assume that r is the smallest integer for which the above sets can be found.

For each i, we will have r disjoint towers, $\cup_{\vec{v} \in D_i^j} T_{\vec{v}} F_i^j$, where $1 \le j \le r$, which will be denoted by τ_i^j. As in the one dimensional case, any transformation which is rank r is ergodic and has zero entropy [3, 8].

In this paper we restrict our attention to rank r transformations with a special tower shape.

Definition 3.2. $(X, \mathcal{A}, \mu), T$ is a $\mathbb{Z}^d$ uniform square rank r transformation if it is rank r and there exists $\alpha \ge 1$ such that for all i and j, the sets D_i^j of Definition 3.1 satisfy:

1. D_i^j is a rectangle of dimensions $\{l_1^{j,i}, ..., l_d^{j,i}\}$.
2. If $s^{j,i} = \min_{k=1\cdots d} \{l_k^{j,i}\}$ and $b^{j,i} = \max_{k=1\cdots d} \{l_k^{j,i}\}$, and we set
$$\mathsf{s}_i = \min_{j=1\cdots r} \{s^{j,i}\}, \qquad \mathsf{b}_i = \max_{j=1\cdots r} \{b^{j,i}\}$$
then $\frac{\mathsf{s}_i}{\mathsf{b}_i} \ge \frac{1}{\alpha}$.

Let $v_i^s = (\mathsf{s}_i)^d$, the smallest possible volume of a tower at stage i, and $v_i^b = (\mathsf{b}_i)^d$, the biggest possible volume of a tower at stage i. Note then $\frac{v_i^s}{v_i^b} \ge \frac{1}{\gamma}$, where $\gamma = \alpha^d \ge 1$.

Definition 3.3. Given a rectangle $R \subset \mathbb{Z}^d$ of size $l_1 \times \cdots \times l_d$, the ϵ-interior of R is the collection of indices in R which are at least a distance ϵl_k from the k^{th} edge of R. The ϵ-collar of R is the complement of the ϵ-interior and corresponds to the set of indices within ϵl_k of the k^{th} edge in the boundary of R.

Denote the volume of D_i^j by $v_i^j = l_1^{j,i} \times \cdots \times l_d^{j,i}$ and notice that the volume of the ϵ-interior of D_i^j is $(1-2\epsilon) l_1^{j,i} \times \cdots \times (1-2\epsilon) l_d^{j,i} = (1-2\epsilon)^d v_i^j$. Notice also that by Definition 3.2 we have that for $1 \le j, k \le r$

$$\frac{1}{\gamma} \le \frac{v_i^s}{v_i^b} \le \frac{v_i^j}{v_i^k} \le \frac{v_i^b}{v_i^s} \le \gamma.$$

4. The Matching Lemma

The following result is a generalization of the Matching Lemma in [4]. We have already discussed the difference between the rank 1 case and the case with rank r with $r > 1$. To deal with this difference the proof of this result uses two applications of the ergodic theorem. We pick a tower stage k such that there is some k-tower which has measure approximately $\frac{1}{r}$. By the ergodic theorem we know that most large enough names will visit this tower approximately $\frac{1}{r}$ of the time. We match even larger names which visit the previous large names frequently enough. This second use of the ergodic theorem guarantees that there is "randomness" in the location of the different types of towers.

Lemma 4.1 (The Generalized Matching Lemma). *Let ϵ and $c < \frac{1}{r}$ be fixed. Let $a = \frac{1}{\gamma^2}(1-2\epsilon)^d(1-\epsilon)\epsilon^d c^2$.*

There is an integer $K(\epsilon, c) > 0$ such that for all $k \geq K(\epsilon, c)$, if P_k is the partition associated with the kth towers, then there exist integers $N(k)$ and $m(k)$ such that for all $n \geq N(k)$, we can find a set W with $\mu(W) > 1-\epsilon$ and for ω, ω', $(P_k)_n$-names of two points $x, x' \in W$, there exists a permutation $\pi : B_n \to B_n$ such that

$$d(\omega \circ \pi, \omega') < 1 - a,$$

and the action of π can be described as follows:

(a) *π translates all the indices of B_n by a vector $\vec{v}$, with $\|\vec{v}\| < m(k)$, except for those indices i for which $i + \vec{v} \notin B_n$. On these, $\pi(i)$ is defined to be one of the indices vacated by the translation.*

(b) *For a subset of the τ_k^j's occurring in ω, π moves the ϵ-interiors of these towers by an additional amount which can vary for each tower but is always less in magnitude than $\epsilon \mathsf{s}_k$. The resulting location of the ϵ-interiors of these towers matches perfectly with the corresponding interior of a τ_k^j in ω'.*

Proof. Let ϵ and c be given. Pick ϵ_1 satisfying

$$0 < \epsilon_1 < \min\left\{\frac{\epsilon}{4}, \frac{1}{3}(\frac{1}{r} - c)\right\}, \tag{1}$$

ϵ_2 satisfying

$$0 < \epsilon_2 < \frac{1}{2\gamma^2}\epsilon_1 \epsilon^d (1 - 3\epsilon_1), \tag{2}$$

and choose K such that for all $k \geq K$ $\mu\big(\cup_{j=1}^r \tau_k^j\big) > 1 - \frac{\epsilon_1}{2}$. Fix some $k \geq K$.

For $1 \leq j \leq r$ set $m_j = \mu\tau_k^j$, and notice that for some j we must have $m_j \geq \frac{1}{r} - \frac{\epsilon_1}{2}$. Say this is true for $j = 1$.

Choose $n_1 \in \mathbb{N}$ such that

$$\frac{\mathsf{b}_k}{n_1} < \frac{\epsilon_1}{2d} \tag{3}$$

and there is a set U with $\mu U > 1 - \epsilon_2$ such that for all $x \in U$,

$$\frac{|\{\vec{v} \in B_{n_1} : T_{\vec{v}}(x) \in \tau_k^1\}|}{n_1^d} > \frac{1}{r} - \epsilon_1. \tag{4}$$

Next, choose $N > n_1$ large enough so that for every $n > N$

$$\frac{n_1}{n} < \frac{\epsilon_1}{2d}, \tag{5}$$

and there is a set W with $\mu W > 1-\epsilon$ so that for all $x \in W$

$$\frac{|\{\vec{v} \in B_n : T_{\vec{v}}(x) \in U\}|}{n^d} > 1 - 2\epsilon_2 \tag{6}$$

$$\frac{|\{\vec{v} \in B_n : T_{\vec{v}}(x) \in \cup_{j=1}^{r} \tau_k^j\}|}{n^d} > 1 - 2\epsilon_1 \tag{7}$$

$$\frac{|\{\vec{v} \in B_n : T_{\vec{v}}(x) \in \tau_k^1\}|}{n^d} > \frac{1}{r} - \epsilon_1. \tag{8}$$

Fix such an integer n and a set W. Thus points in W have n-names which are all but $2\epsilon_1$ full of k-towers, are all but $2\epsilon_2$ full of U, and see τ_k^1 with frequency at least $\frac{1}{r} - \epsilon_1$.

Take two points in W and let ω and ω' be their n-names. We will define a permutation between them satisfying the statement of the lemma. Roughly speaking, to do this we will first identify the towers in ω which have a tower of the same type occurring close by in ω'. We will be able to match a subset of these towers.

Consider all the towers which occur in ω. Notice that by conditions (5) and (7) a proportion

$$> (1 - 3\epsilon_1) \tag{9}$$

of ω lies in a complete tower which is a distance at least n_1 away from the boundary of ω.

Enumerate these complete towers in ω. Denote the position of the base point of tower t by $\vec{z}_t \in B_n$. Consider $B_{\epsilon \mathsf{s}_k}$, the box of size $\epsilon \mathsf{s}_k \times \cdots \times \epsilon \mathsf{s}_k$, centered at $\vec{z}_t$ in ω'. We will say tower t is a good tower if the corresponding $B_{\epsilon \mathsf{s}_k}$ box in ω' contains the base point of an n_1-name from U.

Create an array whose rows correspond to complete towers in ω that lie at least n_1 away from the boundary of ω, and whose columns are the elements of B_{n_1} listed in some order. A row in this array corresponding to a good tower will be called a good row.

Fix a good row of the array, say row t. So tower t in ω is a good tower, namely there is an occurrence of a base point of a name from U in the $B_{\epsilon \mathsf{s}_k}$ box at position $\vec{z}_t$ in ω'. Now shift this box in ω' by each $\vec{v} \in B_{n_1}$. In entry $(t, \vec{v})$ of the array, place a $*$ if the box shifted by $\vec{v}$ contains the base point of a tower of the same type as tower t in ω.

Suppose the array is $c\%$ full of $*$ marks, for some $c > 0$. Then there will be a column, say column $\vec{v} \in B_{n_1}$ such that at least $c\%$ of the column will be full of $*$ marks.

The towers in the $*$ marked rows of column $\vec{v}$ will be the ones we will match. We will define the permutation π from ω to ω' as follows: π will first translate all indices in B_n by the vector $\vec{v}$, except for those in the n_1-collar of B_n which are dislocated by the translation. These indices can be mapped into the indices in the n_1-collar which are vacated by the translation. Note that the towers represented in the array are still intact after this translation. Further, the towers whose entry in column $\vec{v}$ has a $*$ are now within $\epsilon \mathsf{s}_k$ of a tower of the same type in ω'. π will shift the ϵ-interiors of these towers by the amount needed to give an exact match to that tower in ω'.

This construction will give $m(k) \leq n_1$, and the permutation π will have matched a proportion at least

$$\text{(percentage of } \omega \text{ covered by complete towers within } n_1 \text{ of the boundary)}$$
$$\times\text{(the percentage of these towers moved close to a like tower in } \omega' \text{ by } \vec{v})$$
$$\times\text{(the percentage of a tower inside its } \epsilon\text{-interior)}.$$

By equation (9) and the definition of an ϵ-interior, this is at least

$$(1 - 3\epsilon_1) \times \text{(percentage of array which is full of } *) \times (1 - 2\epsilon)^d. \tag{10}$$

Letting c be as above, we will find a lower bound for c by computing the

$$\text{(percentage of rows of the array which correspond to good towers of type 1)}$$
$$\times\text{(the proportion of such a row which is filled with * marks)}.$$

To find the first quantity, first note that by (2), (6), and (9) less than

$$\frac{\epsilon_1}{\gamma} \tag{11}$$

of the rows in the array are bad rows.

Next, note that by conditions (5) and (8) we have a proportion

$$\geq \frac{1}{\gamma}(\frac{1}{r} - 2\epsilon_1)$$

of the rows of the array corresponding to towers of type 1 in ω. Call these type 1 rows.

Hence the rows which are both good and of type 1 take up at least a proportion

$$\frac{1}{\gamma}(\frac{1}{r} - 2\epsilon_1) - \frac{\epsilon_1}{\gamma} = \frac{1}{\gamma}(\frac{1}{r} - 3\epsilon_1)$$

of the array. To find a bound for c all we are missing is the proportion of a type 1 row which is guaranteed to be filled by $*$ marks.

For this computation we note that by conditions (3) and (4) we have that a name from U sees at least

$$\frac{(\frac{1}{r} - 2\epsilon_1)n_1^d}{v_k^1}$$

base points of towers of type 1. Hence a good row of type 1 is a proportion at least

$$\frac{(\frac{1}{r} - 2\epsilon_1)}{v_k^1}\epsilon^d v_k^s \geq \frac{1}{\gamma}(\frac{1}{r} - 2\epsilon_1)\epsilon^d$$

full of $*$ marks.

So the entire array is

$$> \frac{1}{\gamma^2}(\frac{1}{r} - 3\epsilon_1)(\frac{1}{r} - 2\epsilon_1)\epsilon^d > \frac{1}{\gamma^2}(\frac{1}{r} - 3\epsilon_1)^2\epsilon^d \tag{12}$$

full of $*$ marks.

Finally, we plug (12) in for c in (10) to conclude that we have matched a proportion

$$> (1 - 3\epsilon_1)\frac{1}{\gamma^2}(\frac{1}{r} - 3\epsilon_1)^2\epsilon^d(1 - 2\epsilon)^d$$

of ω. By our choice of ϵ_1 in (1) this is

$$> (1-\epsilon)\frac{1}{\gamma^2}c^2\epsilon^d(1-2\epsilon)^d = a.$$

□

5. The Theorem for Finite Rank

Theorem 5.1. *Let $r \in \mathbb{N}$. If T is a uniform square rank r $\mathbb{Z}^d$ action on a Lebesgue probability space $(X, \mathcal{M}, \mu)$ then T is LB.*

Proof. Let Q be an arbitrary partition on X and consider the process (T, Q). Let $\epsilon > 0$ be fixed.

We will show that there is an integer N such that for all $n \geq N$ we can find a set W with $\mu W > 1-\epsilon$ such that for all $x, x' \in W$ if ω, ω' are the Q_n-names of x and x' then

$$\overline{f}(\omega, \omega') < \epsilon. \tag{13}$$

Let $\overline{\epsilon} = \frac{\epsilon^2}{16d^2\alpha^2}$. Pick $c < \frac{1}{r}$ and let $a = \frac{1}{\gamma^2}(1-2\epsilon)^d(1-\epsilon)\epsilon^d c^2$.

For ease of exposition we assume $(1-a)^2 < \frac{\epsilon}{2}$. The proof of the general case can be constructed directly from our argument. The idea will be to apply the Generalized Matching Lemma (GML) to obtain an integer $K(\overline{\epsilon}, c)$ and to find two tower sizes $K(\overline{\epsilon}, c) < k_1 < k_2$ so that the partition Q is well approximated by $\bigvee_{i=1}^t P_i$, for some $t \geq k_2$. If W is the set from the GML associated to k_2, and ω, ω' are two $(\bigvee_{i=1}^t P_i)_n$-names from W, then for large enough n the GML guarantees the existence of a permutation π_2 such that

$$\overline{d}(\omega \circ \pi_2, \omega') < 1-a.$$

The trick is to choose k_1 and k_2 such that if $\tau_{k_1}^1$ satisfies $\mu(\tau_{k_1}^1) \sim \frac{1}{r}$, and G^c is the set of unmatched indices of $\omega \circ \pi_2$, then for some n_1, G^c sees n_1-names which visit $\tau_{k_1}^1$ approximately $\frac{1}{r}$ of the time. This is exactly the setup of the GML, and we can construct a permutation π_1 of G^c as in that lemma which will match a percent of G^c.

We now choose our parameters and give the details of the argument.

Apply the GML with $\overline{\epsilon}$ and c to obtain an integer $K(\overline{\epsilon}, c)$ as in the statement of that lemma. Pick ϵ_1 satisfying

$$0 < \epsilon_1 < \min\left\{\frac{\overline{\epsilon}}{4}, \frac{1}{3}(\frac{1}{r} - c)\right\} \tag{1'}$$

and ϵ_2 satisfying

$$0 < \epsilon_2 < \frac{1}{2\gamma^2}\epsilon_1\overline{\epsilon}^d(1-3\epsilon_1). \tag{2'}$$

Pick $k_1 > K$ such that

$$\mu(\cup_{j=1}^r \tau_{k_1}^j) > 1 - \frac{\epsilon_1^2}{16}.$$

If $m_j = \mu(\tau_{k_1}^j)$, then for some j we must have

$$m_j > \frac{1}{r} - \frac{\epsilon_1^2}{16} > \frac{1}{r} - \frac{\epsilon_1}{4}.$$

Say this happens for $j = 1$.

Choose $n_1 \in \mathbb{N}$ such that

$$\frac{\mathsf{b}_{k_1}}{n_1} < \frac{\epsilon_1}{2d} \tag{3'}$$

and such that there is a set U with $\mu U > 1 - \frac{\epsilon_2^2}{2}$ such that for all $x \in U$

$$\frac{|\{\vec{v} \in B_{n_1} : T_{\vec{v}}(x) \in \tau_{k_1}^1\}|}{n_1^d} > \frac{1}{r} - \frac{\epsilon_1}{2}. \tag{4'}$$

Next choose $k_2 > k_1$ such that $m(k_2) > n_1$ and

$$\frac{n_1}{\mathsf{s}_{k_2}} < \frac{\epsilon_2^2}{16d\gamma}. \tag{5'}$$

Choose $t \in \mathbb{N}$ such that $t > k_2$ and there is a partition $Q_t \subset \bigvee_{i=1}^t P_i$ such that $d(Q, Q_t) < \frac{\epsilon}{8}$. Then pick $n > N(k_2)$ large enough so that we can find a set W which not only satisfies the statement of the GML, but in addition, for all $x \in W$ we have

$$\frac{|\{\vec{v} \in B_n : T_{\vec{v}}(x) \in U\}|}{n^d} > 1 - \epsilon_2^2 \tag{14}$$

$$\frac{|\{\vec{v} \in B_n : T_{\vec{v}}(x) \in \cup_{j=1}^r \tau_{k_2}^j\}|}{n^d} > 1 - \frac{3}{2}\epsilon_1^2 \tag{15}$$

$$\frac{m(k_2) + n_1}{n} < \frac{\epsilon_2^2}{16d} \tag{16}$$

and the $(Q_t)_n$-name of x and the (Q_n) name of x differ less than $\frac{\epsilon}{4}$ of the time. Consider ω, ω' two $(\bigvee_{i=1}^t P_i)_n$-names of points in W. We will define a permutation $\pi : B_n \to B_n$ such that

$$\overline{d}(\omega \circ \pi, \omega') < \frac{\epsilon}{2} \qquad \text{and} \qquad m(\pi) < \epsilon.$$

We can then use this same π on Q names, and by our choice of t, we will have obtained (13).

Let ω, ω' be as above and apply the GML to ω and ω' with tower τ_{k_2} and $\overline{\epsilon}$ to obtain a permutation $\pi_2 : B_n \to B_n$ such that (a) and (b) of that lemma hold. Let G be the set of matched indices of $\omega \circ \pi_2$.

If $\frac{|G^c|}{n^d} < \frac{\epsilon}{2}$ then to complete the proof we need only show that $m(\pi_2) < \epsilon$. Suppose that

$$\frac{|G^c|}{n^d} \geq \frac{\epsilon}{2}. \tag{17}$$

We have already chosen ϵ_1 and ϵ_2 appropriately (in $(1')$ and $(2')$ respectively) and have chosen our parameters to guarantee $(3')$,$(4')$ and $(5')$. So to construct a permutation π_1 as in the GML it remains to show that G^c is all but $2\epsilon_2$ full of occurrences of U, is all but $2\epsilon_1$ full of occurrences of k_1 towers and visits $\tau_{k_1}^1$ all but $\frac{1}{r} - \epsilon_1$ of the time. These estimates will be the analogs of (6),(7) and (8).

For all these estimates we will need to know what proportion of the indices in G^c lie in an $(m(k_2) + 2n_1)$-collar around the boundary of G^c. The boundary of G^c consists of the boundary of B_n, and the disjoint union of boundaries of boxes of dimensions at most $(1-2\epsilon)\mathsf{b}_{k_2} \times \cdots \times (1-2\epsilon)\mathsf{b}_{k_2}$. Using $(3')$, (16), (17), and the fact that the largest number of k_2 towers matched is $< \frac{n^d}{v^s_{k_2}}$, we see that the number of indices within $m(k_2) + 2n_1$ of the boundary of G^c is at most

$$\frac{3\epsilon_2}{4}|G^c|. \tag{18}$$

Using ($2'$), (14), (16), (17), and (18) we have that for both $\omega \circ \pi_2$ and ω'

$$\frac{|\{\vec{v} \in G^c : \vec{v} \text{ is the base point of an } n_1\text{-name from } U\}|}{|G^c|} > 1 - 2\epsilon_2. \tag{6'}$$

Further, by (3), (15), (16), and (18),

$$\frac{|\{\vec{v} \in G^c : \vec{v} \text{ is in a } k_1 \text{ tower } \}|}{|G^c|} > 1 - 2\epsilon_1. \tag{7'}$$

We also claim that a proportion

$$> 1 - 3\epsilon_1 \tag{9'}$$

of $\omega \circ \pi_2$ is covered by complete towers who are further than n_1 to the boundary of $\omega \circ \pi_2$. To see this note that the suspect indices are exactly those who are in a $2n_1$ collar of B_n. By (18) this is a set of indices of proportion less than $\frac{3\epsilon_2}{4}$.

Finally, we claim that by ($4'$), ($6'$), and (18)

$$\frac{|\{\vec{v} \in G^c : \vec{v} \text{ lies in a } \tau^1_{k_1} \text{ tower}\}|}{|G^c|} > \frac{1}{r} - \epsilon_1. \tag{8'}$$

Matching up conditions ($3'$),($4'$), ($5'$),($6'$), ($7'$),($8'$), and ($9'$) with their analogs in the proof of the GML, we see that the conditions for that argument are satisfied and we proceed as in the proof of that lemma to obtain a permutation π_1 which is the identity on G, satisfies (a) and (b) of the GML on G^c with $m(k_1) = n_1$, and matches $a\%$ of G^c with the corresponding indices in ω'.

Set $\pi = \pi_1 \circ \pi_2 : B_n \to B_n$. It follows from our construction that $\overline{d}(\omega \circ \pi, \omega') < (1-a)^2 < \frac{\epsilon}{2}$. To finish the proof, it remains to show that $m(\pi) < \epsilon$. We need, then, to show that there is a set $C \subset B_n$ such that $|C| > (1-\epsilon)n^d$, and for all $\vec{u}, \vec{v} \in C$, $\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| < \epsilon\|\vec{u} - \vec{v}\|$.

Recall that π_2 matches the $\overline{\epsilon}$-interiors of some k_2-towers, and π_1 matches the $\overline{\epsilon}$-interiors of some k_1 towers. Let C_2 be the indices in the $\sqrt{\overline{\epsilon}}$-interiors of the boxes matched by π_2, and define C_1 similarly. As in the rank one argument we set $C = C_2 \cup C_1$. The rest of the argument is identical to the rank one case; we include it below for completeness' sake.

To compute $\frac{|C|}{|B_n|}$ we count the indices which were matched by π, but are not in C. The $\sqrt{\overline{\epsilon}}$-collar we removed from the $\overline{\epsilon}$-interiors is a proportion

$$\sum_{i=1}^{2} 2d\sqrt{\overline{\epsilon}}(\text{the proportion of the } n\text{-name matched by } \pi_i) < 2d\sqrt{\overline{\epsilon}}.$$

So by our choice of $\overline{\epsilon}$, $\frac{|C|}{|B_n|} > 1 - \frac{\epsilon}{2} - 2d\sqrt{\overline{\epsilon}} > 1 - \epsilon$.

Now pick $\vec{u}, \vec{v} \in C$. Suppose first that $\vec{u}$ and $\vec{v}$ were both matched by π_2. Then, if they are in the same k_2 tower we have

$$\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| = 0.$$

Otherwise, they lie in different k_2 towers, so $\|\vec{u}-\vec{v}\| > 2\sqrt{\bar{\epsilon}}\mathsf{s}_{k_2}$. On the other hand,

$$\begin{aligned}\|\pi\vec{u}-\pi\vec{v}-(\vec{u}-\vec{v})\| &< 2\bar{\epsilon}\mathsf{b}_{k_2}\\ &= \frac{2\bar{\epsilon}\mathsf{b}_{k_2}}{\|\vec{u}-\vec{v}\|}\|\vec{u}-\vec{v}\| < \frac{2\bar{\epsilon}\mathsf{b}_{k_2}}{2\sqrt{\bar{\epsilon}}\mathsf{s}_{k_2}}\|\vec{u}-\vec{v}\|\\ &< \epsilon\|\vec{u}-\vec{v}\|.\end{aligned}$$

The argument for the case where $\vec{u}$ and $\vec{v}$ are both matched by π_1 is similar.

Now suppose $\vec{u}$ is matched by π_2 and $\vec{v}$ is matched by π_1. Then $\|\vec{u}-\vec{v}\| > \sqrt{\bar{\epsilon}}\mathsf{s}_{k_2}$, and

$$\begin{aligned}\|\pi\vec{u}-\pi\vec{v}-(\vec{u}-\vec{v})\| &< \bar{\epsilon}\mathsf{b}_{k_2} + m(k_1) + \bar{\epsilon}\mathsf{b}_{k_1}\\ &\leq \bar{\epsilon}\mathsf{b}_{k_2} + n_1 + \bar{\epsilon}\mathsf{b}_{k_1}.\end{aligned}$$

But $\mathsf{b}_{k_1} < \frac{\epsilon_1}{2d}n_1$ which by $(3')$ is

$$< \frac{\epsilon_1}{2d}\frac{\epsilon_2^2}{16d\gamma}\mathsf{s}_{k_2} < \frac{\epsilon_1}{2d}\frac{\epsilon_2^2}{16d\gamma}\mathsf{b}_{k_2}.$$

So we have that

$$\begin{aligned}\|\pi\vec{u}-\pi\vec{v}-(\vec{u}-\vec{v})\| &< 2\bar{\epsilon}\mathsf{b}_{k_2}\\ &= \frac{2\bar{\epsilon}\mathsf{b}_{k_2}}{\|\vec{u}-\vec{v}\|}\|\vec{u}-\vec{v}\| < \frac{2\bar{\epsilon}\mathsf{b}_{k_2}}{\sqrt{\bar{\epsilon}}\mathsf{s}_{k_2}}\|\vec{u}-\vec{v}\|\\ &< \epsilon\|\vec{u}-\vec{v}\|.\end{aligned}$$

□

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