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RELATIONSHIP BETWEEN THE OPERATORS OF CAUCHY AND LOGARITHMIC POTENTIAL TYPE

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Abstract. Asymptotic behaviour of singular values, together with the estimate of the remainder term, of Cauchy operator and the operator of logarithmic potential type, as well as the corresponding trace formula are determined.

1. Introduction and notation

Let $\Omega \subset \mathbb{C}$ be bounded domain. Denote by $L^2(\Omega)$ the space of complex valued functions in Ω for which the norm $\|f\|_2 = \left(\int_\Omega |f(\xi)|^2 \, dA(\xi)\right)^{1/2}$ is finite. Here $dA(\xi)$ denotes the area measure in Ω. The operators L and $C\colon L^2(\Omega) \to L^2(\Omega)$ defined by

$$Lf(z) = -\frac{1}{2\pi}\int_\Omega \ln|z-\xi|\, f(\xi)\, dA(\xi)$$

and

$$Cf(z) = -\frac{1}{\pi}\int_\Omega \frac{f(\xi)}{\xi - z}\, dA(\xi)$$

will be called the operator of logarithmic potential type and Cauchy operator respectively. It is well known that C and L are compact operators on $L^2(\Omega)$ and $L > 0$ iff Ω is inside the unit disc; see [1]. The spectral properties of the operators L and C are studied in many papers (see [1], [2], [5], [7]).

In what follows we denote by $\int_\Omega M(z,\xi) \cdot dA(\xi)$ the integral operator acting on $L^2(\Omega)$ whose kernel is $M(z,\xi)$.

For compact operator T we denote by $s_n(T)$ the n-th eigenvalue of the operator $|T| = (T^*T)^{1/2}$, i.e. $s_n(T) = \lambda_n(|T|)$. By c_p we denote the set of compact operators T such that

$$|T|_p = \left(\sum_{k\geq 1} s_k^p(T)\right)^{1/p}.$$

In particular c_1 is the set of nuclear operators.

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2. Results

The operators L and $\frac{C^*C}{4}$ have similar spectral asymptotics. Let $\lambda_n(L)$ and $\lambda_n\left(\frac{C^*C}{4}\right)$ be eigenvalues of the operators L and $\frac{C^*C}{4}$ arranged in the decreasing order, according to their multiplicity.

THEOREM 1. *If $\Omega \subset \mathbb{C}$ is a bounded simply connected domain with analytic boundary, then the following asymptotic formulae hold:*

a)

$$\lambda_n(L) = \frac{|\Omega|}{4\pi n} + O\left(\frac{1}{n^{4/3}}\right)$$

b)

$$\lambda_n\left(\frac{C^*C}{4}\right) = \frac{|\Omega|}{4\pi n} + O\left(\frac{1}{n^{4/3}}\right)$$

Proof. For the eigenvalues λ_n of the problem

$$-\Delta u = \lambda u, \qquad u|_{\partial\Omega} = 0 \tag{$*$}$$

the following Weyl formula holds

$$\frac{1}{\lambda_n} = \frac{|\Omega|}{4\pi n} - \frac{|\partial\Omega|}{\sqrt{4\pi|\Omega|}}\frac{1}{n^{3/2}} + o\left(\frac{1}{n^{3/2}}\right).$$

Here $|\partial\Omega|$ denotes the length of the $|\partial\Omega|$. By the inversion of the boundary problem

$$-\Delta u = \lambda u, \qquad u|_{\partial\Omega} = 0$$

we get

$$Sf(z) = \int_\Omega G(z,\xi) f(\xi)\, dA(\xi)$$

where $G(\cdot,\cdot)$ is the Green function of the problem $(*)$.

It is clear that $s_n(S) = \frac{1}{\lambda_n}$. In terms of conformal mapping $h : \Omega \to D$ (D is the unit disc) we have

$$G(z,\xi) = -\frac{1}{2\pi}\ln\left|\frac{h(z)-h(\xi)}{1-\overline{h(\xi)}h(z)}\right|.$$

(Observe that the function h can be analytically continued to some neighborhood of $\overline{\Omega}$. The same holds for $\varphi(z) = h^{-1}(z)$. Therefore, $\varphi'(z)$ is a bounded function on $\overline{D}$.)

Then we get

$$\begin{aligned} S &= -\frac{1}{2\pi}\int_\Omega \ln\left|\frac{h(z)-h(\xi)}{z-\xi}\right|\cdot dA(\xi) + \frac{1}{2\pi}\int_\Omega \ln\left|1-\overline{h(\xi)}h(z)\right|\cdot dA(\xi) + L \\ &= S_1 + S_2 + L. \end{aligned}$$

From the Birman-Solomjak theorem on the growth of singular values of operators with an analytic kernel, [3], we obtain

$$s_n\left(S_1\right) = O\left(e^{-c_0\sqrt{n}}\right) \quad (c_0 \text{ does not depend on } n). \tag{1}$$

Since $s_n\left(\int_D \ln\left|1-\overline{\xi}z\right| \cdot dA\right) = O(1/n^2)$ and

$$s_n\left(S_2\right) = s_n\left(\frac{1}{2\pi}\int_D \varphi'(z)\,\overline{\varphi'(\xi)}\ln\left|1-\overline{\xi}\;z\right| \cdot dA\right)$$

we have $s_n\left(S_2\right) = O(1/n^2)$ and from (1) and properties of singular values of the sum of two operators we get

$$s_n\left(S_1+S_2\right) = O\left(\frac{1}{n^2}\right).$$

Now, we need the following simple generalization of the Ky-Fan theorem (it can be deduced in the same way as the Ky-Fan theorem [9], see also [8]).

Lemma 1. *If A and B are compact operators such that*

$$s_n(A) = \frac{a}{n^\alpha} + O\left(n^{-\beta}\right) \quad \alpha+1>\beta>\alpha>0$$

$$s_n(B) = O\left(n^{-\beta_1}\right) \quad \beta_1 \geq \frac{\beta}{\alpha+1-\beta},$$

then

$$s_n(A+B) = \frac{a}{n^\alpha} + O\left(n^{-\beta}\right)$$

Applying Lemma 1 to the operators S and $-\left(S_1+S_2\right)$ we get the statement of Theorem 1 a).

For the proof of Theorem 1 b) we observe that the operator $R = \frac{C^*C}{4} - L$ has the following kernel

$$\mathcal{X}(z,\xi) = \frac{1}{4\pi^2 i}\int_{\partial\Omega}\frac{\ln|t-z|}{t-\xi}\,dt$$

and hence

$$s_n(R) = O\left(\frac{1}{n^2}\right). \tag{2}$$

(Indeed, the kernel of the operator $\frac{C^*C}{4}$ is equal to the function

$$K(z,\xi) = \frac{1}{4\pi^2}\int_\Omega \frac{dA(t)}{(\xi-t)\left(\overline{z}-\overline{t}\right)},$$

so applying the Cauchy-Green formula we obtain

$$K(z,\xi) = -\frac{1}{2\pi}\ln|z-\xi| + \mathcal{X}(z,\xi)$$

i.e. the kernel of the operator R is equal to $\mathcal{X}(z,\xi)$.)

Let us show (2). It is enough to prove that for the singular values of the operator $R\colon L^2(D) \to L^2(D)$ with the kernel

$$\int_{\partial D} \frac{\ln|\varphi(t) - \varphi(z)|}{\varphi(t) - \varphi(\xi)} \cdot \varphi'(t)\, dt$$

there holds $s_n(R_1) = O\left(1/n^2\right)$. Observe that

$$\begin{aligned}
&\int_{\partial D} \frac{\ln|\varphi(t) - \varphi(z)|}{\varphi(t) - \varphi(\xi)} \cdot \varphi'(t)\, dt \\
&= \int_{\partial D} \frac{(t-\xi)\varphi'(t)}{\varphi(t) - \varphi(\xi)} \cdot \frac{\ln|t-z|}{t-\xi}\, dt + \int_{\partial D} \ln\left|\frac{\varphi(t) - \varphi(z)}{t-z}\right| \cdot \frac{\varphi'(t) - \varphi'(\xi)}{\varphi(t) - \varphi(\xi)}\, dt \\
&\quad + \varphi'(\xi) \int_{\partial D} \ln\left|\frac{\varphi(t) - \varphi(z)}{t-z}\right| \cdot \left(\frac{1}{\varphi(t) - \varphi(\xi)} - \frac{1}{t-\xi}\right) dt \\
&\quad + \varphi'(\xi) \int_{\partial D} \frac{\ln\left|\frac{\varphi(t)-\varphi(z)}{t-z}\right|}{t-\xi}\, dt = K_1 + K_2 + K_3 + K_4.
\end{aligned}$$

The functions K_2, K_3, K_4 are real analytic on a neighborhood of $\overline{D} \times \overline{D}$ and thus the corresponding integral operators have exponential decrease of singular values. (Theorem of Birman-Solomjak). The function $(t,\xi) \longmapsto (t-\xi)\,\frac{\varphi'(t)}{\varphi(t)-\varphi(\xi)}$ is analytic on a neighborhood of $\overline{D} \times \overline{D}$ and the rate of growth of the singular values of the operator with the kernel K_1 is equal to the rate of growth of the singular values of the operator with the kernel

$$\int_{\partial D} \frac{\ln|t-z|}{t-\xi}\, dt \quad (= \pi i \ln(1 - \xi\,\overline{z})).$$

Since $s_n\left(\int_D \ln(1-\xi\,\overline{z}) \cdot dA(\xi)\right) = O(1/n^2)$ we obtain (2).

It follows that $s_n(R - S_1 - S_2) = O\left(\frac{1}{n^2}\right)$ and applying Lemma 1 to the operators S and $R - S_1 - S_2$ we obtain the statement of Theorem 1 b). ∎

From Theorem 1 it follows that the function

$$\Psi(z) = \sum_{n=1}^{\infty} \left(\lambda_n^z\left(\frac{C^*C}{4}\right) - \lambda_n^z(L)\right)$$

$\left(\lambda^z = e^{z\ln\lambda}\ ;\ \ln\lambda = \ln|\lambda| + i\arg\lambda,\ 0 \le \arg\lambda < 2\pi\right)$ is analytic for $\operatorname{Re} z > 2/3$.

Theorem 2. *If $\Omega \subset \mathbb{C}$ is a bounded simply connected domain with analytic boundary, then the following trace formula*

$$\sum_{n=1}^{\infty} \left(\lambda_n\left(\frac{C^*C}{4}\right) - \lambda_n(L)\right) = -\frac{1}{8\pi^2} \int_{\partial\Omega} \int_{\partial\Omega} \ln^2|z-\xi|\, dz d\overline{\xi}$$

holds.

In the proof of this theorem we need the following

LEMMA 2. [6] *If C and D are positive compact operators on some complex Hilbert space $\mathcal{H}$ such that $|C-D|^{\operatorname{Re} z} \in c_1$ $(0<\operatorname{Re} z<1)$, then $C^z - D^z \in c_1$ and*

$$|C^z - D^z|_1 \leq \frac{|\sin \pi z|}{\sin \pi (\operatorname{Re} z)} \left| |C-D|^{\operatorname{Re} z} \right|_1 .$$

Proof of Theorem 2. From (2) it follows that for every $k \in N$ holds

$$s_n\left(\left(\frac{C^*C}{4}\right)^k - L^k\right) = O\left(\frac{1}{n^{k+1}}\right)$$

and by Lemma 2, the function

$$z \longmapsto \operatorname{tr}\left(\left(\frac{C^*C}{4}\right)^z - L^z\right) = \operatorname{tr}\left[\left(\left(\frac{C^*C}{4}\right)^k\right)^{z/k} - \left(L^k\right)^{z/k}\right]$$

is analytic for $\frac{1}{k+1} < \operatorname{Re}\left(\frac{z}{k}\right) < 1$ i.e. $\frac{k}{k+1} < \operatorname{Re} z < k$. So, the function

$$z \longmapsto \operatorname{tr}\left(\left(\frac{C^*C}{4}\right)^z - L^z\right)$$

is analytic for $\operatorname{Re} z > 1/2$.

If $\operatorname{Re} z > 1$, then both operators $\left(\frac{C^*C}{4}\right)^z$ and L^z are nuclear and hence

$$\operatorname{tr}\left(\left(\frac{C^*C}{4}\right)^z - L^z\right) = \operatorname{tr}\left(\frac{C^*C}{4}\right)^z - \operatorname{tr} L^z = \Psi(z).$$

Hence

$$\Psi(z) \equiv \operatorname{tr}\left(\left(\frac{C^*C}{4}\right)^z - L^z\right)$$

for $\operatorname{Re} z > \frac{2}{3}$. Particularly, for $z=1$ we obtain

$$\sum_{n=1}^{\infty}\left(\lambda_n\left(\frac{C^*C}{4}\right) - \lambda_n(L)\right) = \operatorname{tr}\left(\frac{C^*C}{4} - L\right) = \operatorname{tr} R.$$

Having in mind that $R \in c_1$, by [4], Theorem 3.1 it follows

$$\operatorname{tr} R = \frac{1}{4\pi^2 i}\int_\Omega dA(z)\int_{\partial\Omega}\frac{\ln|z-t|}{z-t}\,dt = \frac{1}{4\pi^2 i}\int_{\partial\Omega} dt\int_\Omega \frac{\ln|z-t|}{z-t} dA(z)$$

Let $\Omega_\varepsilon = \Omega \backslash \{z : |z-t| < \varepsilon\}$, $t \in \partial\Omega$. Since

$$\int_\Omega \frac{\ln|z-t|}{z-t}\,dA(z) = -\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\frac{\ln|z-t|}{z-t} dA(z) = -\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\frac{\partial}{\partial z}\left(\ln^2|z-t|\right)\,dA(z)$$

according to Green's formula, [10], we get

$$\int_\Omega \frac{\ln|z-t|}{z-t}\,dA(z) = -\lim_{\varepsilon\to 0}\left(-\frac{1}{2i}\int_{\partial\Omega_\varepsilon}\ln^2|z-t|\,d\overline{z}\right) = \frac{1}{2i}\int_{\partial\Omega}\ln^2|z-t|\,d\overline{z}$$

So,

$$\operatorname{tr} R = \frac{1}{4\pi^2 i}\int_{\partial\Omega} dt\, \frac{1}{2i}\int_{\partial\Omega} \ln^2|z-t|\, d\overline{z} = -\frac{1}{8\pi^2}\int_{\partial\Omega}\int_{\partial\Omega} \ln^2|z-t|\, dt\, d\overline{z}. \quad \blacksquare$$

REMARK. In the case $\Omega = D$ (D is the unit disc) it was proved in [1] that the spectra of the operators $\frac{C^*C}{4}$ and L are equal and we may expect that

$$\sum_{n=1}^{\infty}\left(\lambda_n\left(\frac{C^*C}{4}\right) - \lambda_n(L)\right) = 0.$$

But direct calculation shows that

$$\int_{\partial D}\int_{\partial D} \ln^2|z-\xi|\, dz\, d\overline{\xi} = 2\pi^2 .$$

The explanation lies in the fact that the operators $\frac{C^*C}{4}$ and L have the same spectra (in the case $\Omega = D$) but their multiplicities differ.

COROLLARY. *Assume the boundary of a simply connected bounded domain G_ε is given by $z(\theta) = (1+\varepsilon\varphi(\theta))\,e^{i\theta}$. Then applying Theorem 2 to the operators $\frac{C^*C}{4}$ and L (acting on $L^2(G_\varepsilon)$) we have*

$$\sum_{n=1}^{\infty}\left(\lambda_n\left(\frac{C^*C}{4}\right) - \lambda_n(L)\right) = -\frac{1}{8\pi^2}\int_{\partial D}\int_{\partial D} \ln^2|z-t|\, dz\, d\overline{t} + \varepsilon\cdot A + O\left(\varepsilon^2\right), \varepsilon \to 0$$

where A depends only on φ and φ'.

So the operators $\frac{C^*C}{4}$ and L on domains which are close to the unit disc, either have different spectra or if they have the same spectrum the multiplicities of eigenvalues are not the same.

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