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CONVERGENCE OF A FINITE DIFFERENCE METHOD FOR THE THIRD BVP FOR POISSON'S EQUATION

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Abstract. In this paper we study the convergence of finite difference schemes to weak solutions of the third boundary value problem for Poisson's equation on the unit square. Using the theory of interpolation of function spaces, we obtain error estimates in a discrete W_2^1 Sobolev norm consistent, or "almost" consistent, with the smoothness of the data.

1. Introduction

For a class of finite difference schemes (FDS) for elliptic boundary value problems (BVP), estimates of the convergence rate consistent with the smoothness of the data:

$$\|u-v\|_{W_p^k(\omega)} \leqslant Ch^{s-k}\|u\|_{W_p^s(\Omega)}, \qquad s \geqslant k, \tag{1}$$

are of major interest (see [13], [7]). Here $u=u(x)$ denotes the solution of the BVP, v denotes the solution of corresponding FDS, h is the discretisation parameter, $W_p^k(\omega)$ denotes the discrete Sobolev space, and C is a positive generic constant, independent of h and u.

A standard technique for the derivation of such estimates (see [7], [10], [18]) is based on the Bramble–Hilbert lemma (see [4], [6]). In particular, for the third BVP estimates of type (1) were derived in [5], [12], [15] and [16]. In this paper we, firstly, obtain better convergence then in [12], [15] or [16] and, secondly, expose an alternative technique, based on the theory of interpolation of Banach spaces (see [3], [8], [9], [11], [19]).

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2. Boundary value problem and its approximation

As a model problem let us consider the third boundary value problem (TBVP) for the Poisson's equation in the domain $\Omega = (0,1)^2$:

$$\begin{aligned} L(u) &\equiv -\sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i^2} = f \quad \text{in } \Omega, \\ l(u) &\equiv \frac{\partial u}{\partial n} + \sigma u = 0 \quad \text{on } \Gamma = \partial\Omega\,, \end{aligned} \tag{2}$$

where $\dfrac{\partial u}{\partial n} = \sum\limits_{i=1}^{2} \dfrac{\partial u}{\partial x_i} \cos(\nu, x_i)$ and ν—the unit outward normal to $\partial\Omega$.

We assume that the generalized solution of the problems (2) belongs to the Sobolev space $W_2^s(\Omega)$, $1 < s \leqslant 3$, and that the right-hand side $f(x) \in W_2^{s-2}(\Omega)$. Then, the coefficient σ belongs to the corresponding space of multipliers [14], i.e. $\sigma \in M\left(W_2^{s-1/2}(\Gamma) \mapsto W_2^{s-3/2}(\Gamma)\right)$. Sufficient conditions for this are: $\sigma \in L_2(\Gamma)$ for $1 < s \leqslant \frac{3}{2}$ and $\sigma \in W_2^{s-\frac{3}{2}}(\Gamma)$ for $s > \frac{3}{2}$. Also, let the condition $\sigma(x) \geq \sigma_0 > 0$ be satisfied. We shall assume that the other additional conditions which are possible and which guarantee the existence of the solution $u \in W_2^s(\Omega)$ are also satisfied (consistency conditions of input data at the vertices of the domain Ω).

Let $\bar\omega$ be a uniform mesh in Ω with the step size $h = \frac{1}{n}$, $\omega = \bar\omega \cap \Omega$, $\gamma = \bar\omega \cap \Gamma$, $\gamma_{i\alpha} = \{x \in \bar\omega \,:\, x_i = \alpha = \text{const},\ 0 < x_{3-i} < 1\}$, $\tilde\gamma_{i\alpha} = \{x \in \bar\omega \,:\, x_i = \alpha = \text{const},\ 0 \leqslant x_{3-1} < 1\}$, $\bar\gamma_{i\alpha} = \{x \in \bar\omega \,:\, x_i = \alpha = \text{const},\ 0 \leqslant x_{3-1} \leqslant 1\}$, $\gamma_{i\alpha}^\star = \bar\gamma_{i\alpha} \cap \gamma_{i\alpha}$, $\omega_i = \omega \cup \gamma_{i0}$, $\gamma^\star = \gamma \setminus \left\{\bigcup_{i,k} \gamma_{ik}\right\}$, $i = 1,2;\ k = 0,1$. The finite difference operators are defined in the following maner: $v_{x_i} = (v^{+i} - v)/h$, $v_{\bar x_i} = (v - v^{-i})/h$, where $v^{\pm i}(x) = v(x \pm hr_i)$, r_i is the unit vector of the axis x_i.

We define the following scalar products and discrete norms:

$$[u,v] = h^2 \sum_{x\in\omega} u(x)v(x) + \frac{h^2}{2} \sum_{x\in\gamma\setminus\gamma^\star} u(x)v(x) + \frac{h^2}{4} \sum_{x\in\gamma^\star} u(x)v(x)$$

$$[u,v]_i = h^2 \sum_{x\in\omega_i} u(x)v(x) + \frac{h^2}{2} \sum_{x\in\gamma\setminus(\gamma_{i0}\cup\bar\gamma_{i1})} u(x)v(x);$$

$$[u,v]_{\bar\gamma_{i\alpha}} = h \sum_{x\in\gamma_{i\alpha}} u(x)v(x) + \frac{h}{2} \sum_{x\in\gamma_{i\alpha}^\star} u(x)v(x); \qquad |[v]|_{\bar\gamma_{i\alpha}} = [v,v]_{\bar\gamma_{i\alpha}}^{1/2};$$

$$[u,v]_{\tilde\gamma_{i\alpha}} = h \sum_{x\in\tilde\gamma_{i\alpha}} u(x)v(x); \qquad |[v]|_{\tilde\gamma_{i\alpha}} = [v,v]_{\tilde\gamma_{i\alpha}}^{1/2};$$

$$|[v]| = |[v]|_{L_2(\bar\omega)} = |[v]|_{W_2^0(\bar\omega)} = [u,v]^{1/2}; \qquad |[v]|_i = [v,v]_i^{1/2};$$

$$|[v]|^2_{W_2^1(\bar\omega)} = |[v]|^2 + |[v_{x_1}]|_1^2 + |[v_{x_2}]|_2^2; \qquad |[v]|_{C(\bar\omega)} = \max_{x\in\bar\omega} |v(x)|.$$

We also define the Steklov smoothing operators with the step size h:

$T_i^+ f(x) = \int_0^1 f(x + htr_i)\,dt = T_i^- f(x + hr_i) = T_i f(x + \frac{h}{2} r_i)$,

$T_i^2 f(x) = \int_{-1}^{1}(1-|t|)f(x+htr_i)\,dt$ and

$T_{i\pm}^2 f(x) = 2\int_0^1 (1-t)f(x\pm htr_i)\,dt,$

where $i=1,2$. These operators commute and transform derivatives to differences:

$$T_i^+\left(\frac{\partial u}{\partial x_i}\right) = u_{x_i}\,,\ T_i^-\left(\frac{\partial u}{\partial x_i}\right) = u_{\bar{x}_i}\,,\ T_i^2\left(\frac{\partial^2 u}{\partial x_i^2}\right) = u_{\bar{x}_i x_i}\,;\ i=1,2\,.$$

We approximate TBVP (2) with the FDS:

$$L_h v = g(x), \tag{3}$$

where $L_h(v) = L_{h,1}v + L_{h,2}v$,

$$L_{h,i}v = \begin{cases} -\frac{2}{h}\left(v_{x_i} - \bar{\sigma}v\right), & x\in\bar{\gamma}_{i0}, \\ -v_{x_i\bar{x}_i}, & x\in\omega\cup\gamma_{3-i,0}\cup\gamma_{3-i,1}, \\ \frac{2}{h}\left(v_{\bar{x}_i} + \bar{\sigma}v\right), & x\in\bar{\gamma}_{i1}, \end{cases} \tag{4}$$

$g(x)$ is equal to:

$$T_1^2T_2^2 f\,,\ x\in\omega\,;\ T_{i+}^2T_{3-i}^2 f\,,\ x\in\gamma_{i0};\ T_{i-}^2T_{3-i}^2 f\,,\ x\in\gamma_{i1};$$
$$T_{1\pm}^2T_{2\pm}^2 f,\ x=(0.5\mp 0.5,\,0.5\mp 0.5)\in\gamma^\star$$

and

$$\begin{aligned} &\bar{\sigma} = T_i^2\sigma\,, \quad x\in\gamma_{3-i,0}\cup\gamma_{3-i,1}\,, \\ &\bar{\sigma}_{00} = T_{i+}^2\sigma(0,0), \quad \bar{\sigma}_{11} = T_{i-}^2\sigma(1,1) \quad (\text{in } L_{h,i}v)\ i=1,2\,, \\ &\bar{\sigma}_{01} = T_{1+}^2\sigma(0,1) \quad (\text{in } L_{h,2}v), \quad \bar{\sigma}_{01} = T_{2-}^2\sigma(0,1) \quad (\text{in } L_{h,1}v)\,, \\ &\bar{\sigma}_{10} = T_{1-}^2\sigma(1,0) \quad (\text{in } L_{h,2}v), \quad \bar{\sigma}_{10} = T_{2+}^2\sigma(1,0) \quad (\text{in } L_{h,1}v)\,. \end{aligned}$$

Let u be the solution of the TBVP (2), and v the solution of the FDS (3). For $s>1$, the error of the FDS (3) $z=u-v$ is defined and satisfies the condition:

$$L_h z = \psi, \tag{5}$$

where

$$\psi = \begin{cases} \sum\limits_{i=1}^{2}\eta_{i,x_i\bar{x}_i}\,, & x\in\omega, \\ \bar{\eta}_{3-i,x_{3-i}\bar{x}_{3-i}} + \frac{2}{h}(\eta_{i,x_i}+\zeta_i), & x\in\gamma_{i0},\ i=1,2, \\ \frac{2}{h}\sum\limits_{i=1}^{2}(\bar{\eta}_{i,x_i}+\bar{\zeta}_i), & x=(0,0), \end{cases} \tag{6}$$

and analogously in the other boundary nodes. Here

$$\begin{aligned} &\eta_i = T_{3-i}^2 u - u\,, \quad x\in\omega, \\ &\bar{\eta}_i = T_{(3-i)\pm}^2 u - u \quad x\in\gamma_{3-i,0.5\mp0.5}, \\ &\zeta_i = \left(T_{3-i}^2\sigma\right)u - T_{3-i}^2(\sigma u), \quad x\in\gamma_{ik}, \\ &\bar{\zeta}_i = \left(T_{(3-i)+}^2\sigma\right)u - T_{(3-i)+}^2(\sigma u), \quad x=(0,0), \end{aligned} \tag{7}$$

and analogously in the other nodes $x\in\gamma^\star$.

3. Stability of the finite difference scheme

Stability of the scheme (3) for $s \in (1,3]$ is proved in [12] and [15]. However, we need a new a priori estimate specially for the case $s=3$. Note that in this case $\sigma(x) \in W_2^{3/2}(\Gamma)$ and $C(\Gamma) \subset W_2^{3/2}(\Gamma)$ (see [1]). It is not so hard to prove the following lemma:

LEMMA 1. *Let v denote a mesh-function on $\bar{\omega}$, operator L_h defined by* (4) *and $0 \leqslant \sigma_0 \leqslant \sigma(x) \leqslant \sigma_1$. Then there exist positive constants C_1 and C_2 and following inequalities hold:*

$$C_1|[v]|^2 \leqslant [L_h v, v] \leqslant C_2|[v]|^2.$$

Using (6) and (7), we can transform $[L_h z, z]$ into

$$\begin{aligned}
[L_h z, z] = &-h^2 \sum_{x\in\omega_1} \eta_{1,x_1}\, z_{x_1} - h^2 \sum_{x\in\omega_2} \eta_{2,x_2}\, z_{x_2} - \frac{h^2}{2} \sum_{k=0}^{1} \sum_{i=1}^{2} \sum_{x\in\bar{\gamma}_{3-i,k}} \bar{\eta}_{i,3-i,x_i}\, z_{x_i} \\
&+ I_{10}(\bar{\eta}_{22}) + I_{11}(\bar{\eta}_{22}) + I_{20}(\bar{\eta}_{11}) + I_{21}(\bar{\eta}_{11}) \\
&+ h \sum_{k=0}^{1} \sum_{i=1}^{2} \sum_{x\in\gamma_{ik}} \zeta_i\, z + \frac{h}{2} \sum_{x\in\gamma^\star} (\bar{\zeta}_1 + \bar{\zeta}_2)\, z,
\end{aligned} \tag{8}$$

where $\bar{\eta}_k = \bar{\eta}_{k1} + \bar{\eta}_{k2}$, $k=1,2$ and, for example in the node $(0,x_2)$,

$$\begin{aligned}
\bar{\eta}_{21} &= \frac{2}{h} \int_0^h \int_0^{x_1'} \int_0^{x_1''} \left(1 - \frac{x_1'}{h}\right) \frac{\partial^2 u(x_1''', x_2)}{\partial x_1^2}\, dx_1'''\, dx_1''\, dx_1' \\
\bar{\eta}_{22} &= \frac{h}{3}\, \sigma(x_2)\, u(0,x_2)
\end{aligned} \tag{9}$$

and

$$I_{10}(\bar{\eta}_{22}) = h \sum_{x\in\gamma_{10}} \bar{\eta}_{22,x_2\bar{x}_2} z + \bar{\eta}_{22,x_2}(0,0)\, z(0,0) - \bar{\eta}_{22,\bar{x}_2}(0,1)\, z(0,1) \tag{10}$$

Analogously we define the other I_{ik}.

Let us define the operator Λ_i on $\bar{\gamma}_{3-i,\alpha}$ as

$$\Lambda_i v = \begin{cases} -\frac{2}{h} v_{x_i}, & x_i = 0, \\ -v_{x_i\bar{x}_i}, & 0 < x_i < 1, \\ \frac{2}{h} v_{\bar{x}_i}, & x_i = 1. \end{cases} \tag{11}$$

The operator Λ_i is self-adjoint with respect to the inner product $[u,v]_{\bar{\gamma}_{3-i,\alpha}}$. The eigenvalues of the operator Λ_i are (see [7])

$$\lambda_k = \frac{4}{h^2} \sin^2 \frac{k\pi h}{2}\,, \quad k = 0,1,\ldots,n,$$

and its corresponding eigenfunction are $\psi_k = \psi_k(x_i) = \cos\, k\pi x_i$. Since $\lambda_0 = 0$, the operator Λ_i is non-negative. Using (9), (10) and (11) we obtain

$$I_{10}(\bar{\eta}_{22}) = [\Lambda_2 \bar{\eta}_{22}, z]_{\bar{\gamma}_{10}} = [\Lambda_2^{3/4} \bar{\eta}_{22}, \Lambda_2^{1/4} z]_{\bar{\gamma}_{10}}\ |[\Lambda_2^{3/4} \bar{\eta}_{22}]|_{\bar{\gamma}_{10}} |[\Lambda_2^{1/4} z]|_{\bar{\gamma}_{10}}$$

and analogously for the other I_{ik} at $\bar{\gamma}_{ik}$.

Let us prove the following assertion:

Lemma 2. *Let v be a mesh-function on $\bar\omega$ and Λ_i defined by* (11). *Then*

$$|[\Lambda_i^{1/4} v]|_{\bar\gamma_{3-i,\alpha}} \leqslant C\ |[v]|_{W_2^1(\bar\omega)}.$$

Proof. If we take $i = 2$ (proof in the other case is the same), the function v can be presented in the form

$$v(x_1, x_2) = \sum_{l=0}^{n}\sum_{k=0}^{n} a_{kl}\psi_k(x_1)\psi_l(x_2) = \sum_{l=0}^{n} C_l(x_1)\psi_l(x_2)$$

where $\psi_k(x_1) = \cos k\pi x_1$, $\psi_l(x_2) = \cos l\pi x_2$ and then (see [7])

$$\begin{aligned}
|[v]|^2_{\bar\gamma_{1,x_1}} &= \frac{1}{2}\sum_{l=0}^{n}{}' C_l^2(x_1) \equiv \frac{1}{2}\sum_{l=1}^{n-1} C_l^2(x_1) + \frac{1}{4}\left(C_0^2(x_1) + C_n^2(x_1)\right)\\
|[\Lambda_2^{1/2} v]|^2_{\bar\gamma_{1,x_1}} &= \frac{1}{2}\sum_{l=1}^{n}{}' \lambda_l\, C_l^2(x_1) \equiv \frac{1}{2}\sum_{l=1}^{n-1} \lambda_l\, C_l^2(x_1) + \frac{\lambda_n}{4}\, C_n^2(x_1).\\
|[\Lambda_2^{1/4} v]|^2_{\bar\gamma_{1,x_1}} &= \frac{1}{2}\sum_{l=1}^{n}{}' \sqrt{\lambda_l}\, C_l^2(x_1) \equiv \frac{1}{2}\sum_{l=1}^{n-1} \sqrt{\lambda_l}\, C_l^2(x_1) + \frac{\sqrt{\lambda_n}}{4}\, C_n^2(x_1).
\end{aligned} \tag{12}$$

Using the inequality (see [2]):

$$\max_{x\in\bar\gamma_{2,x_2}} |v(x)|^2 \leqslant \varepsilon\, |[v_{x_1}]|^2_{\bar\gamma_{2,x_2}} + \left(\frac{1}{\varepsilon} + 1\right) |[v]|^2_{\bar\gamma_{2,x_2}}, \quad \varepsilon > 0,$$

from (12) we obtain

$$|[\Lambda_2^{1/4} v]|^2_{\bar\gamma_{1,x_1}} \leqslant \frac{1}{2}\sum_{l=1}^{n}{}' \sqrt{\lambda_l}\left\{\varepsilon_l h \sum_{m=0}^{n-1} C^2_{l,x_1}(mh) + \left(\frac{1}{\varepsilon_l} + 1\right) h \sum_{m=0}^{n}{}' C_l^2(mh)\right\}$$

If we choose $\varepsilon_l = \dfrac{1 + \sqrt{1 + 4\lambda_l}}{2\lambda_l}$, then we have

$$|[\Lambda_2^{1/4} v]|^2_{\bar\gamma_{1,x_1}} \leqslant \max_{1\leqslant l\leqslant n} \frac{1 + \sqrt{1 + 4\lambda_l}}{2\sqrt{\lambda_l}} \cdot \frac{1}{2}\sum_{l=1}^{n}{}'\left\{h \sum_{m=0}^{n-1} C^2_{l,x_1}(mh) + \lambda_l h \sum_{m=0}^{n}{}' C_l^2(mh)\right\}.$$

However, we know that $\lambda_l \geqslant \lambda_1 \geqslant 8$, and then

$$\begin{aligned}
|[\Lambda_2^{1/4} v]|^2_{\bar\gamma_{1,x_1}} &\leqslant \frac{5}{4}\, h \sum_{m=0}^{n-1}\left(\frac{1}{2}\sum_{l=1}^{n}{}' C^2_{l,x_1}(mh)\right) + \frac{5}{4}\, h \sum_{m=0}^{n}{}'\left(\frac{1}{2}\sum_{l=1}^{n}{}' \lambda_l\, C_l^2(mh)\right)\\
&\leqslant \frac{5}{4}\, h \sum_{m=0}^{n-1} |[v_{x_1}]|^2_{\bar\gamma_{1,mh}} + \frac{5}{4}\, h \sum_{m=0}^{n}{}' |[v^2_{x_2}]|^2_{\bar\gamma_{1,mh}}\\
&\leqslant \frac{5}{4}\left(|[v_{x_1}]|_1^2 + |[v_{x_2}]|_2^2\right) \leqslant C\ |[v]|^2_{W_2^1(\bar\omega)}.
\end{aligned}$$

In the similar way, we can prove following two lemmas (see [2], [7]):

LEMMA 3. *Let v be a mesh-function on $\bar\omega$ and Λ_{3-i} defined by* (11). *Then*

$$|[\Lambda_{3-i}^{3/4} v]|^2_{\tilde\gamma_{1,x_i}} \leqslant C\ h^2 \sum_{x,x'\in\tilde\gamma_{1,x_i},\ x\neq x'} \frac{\left[v_{x_{3-i}}(x) - v_{x_{3-i}}(x')\right]^2}{(x_{3-i} - x'_{3-i})^2}.$$

LEMMA 4. *Let v be a mesh-function on $\bar\omega$, then*

$$|[v]|_{C(\bar\omega)} \leqslant C \ln\frac{1}{h}\, |[v]|_{W_2^1(\bar\omega)}.$$

From (8), using lemmas 1, 2 and 4, the following assertion holds true:

LEMMA 5. *The finite difference scheme* (5) *is stable in the sense of the a priori estimate:*

$$\begin{aligned} |[z]|_{W_2^1(\bar\omega)} \leqslant C\cdot \Bigg\{ & h^2 \sum_{x\in\omega_1} \eta^2_{1,x_1} + h^2 \sum_{x\in\omega_2} \eta^2_{2,x_2} + h^2 \sum_{k=0}^{1} \Bigg(\sum_{x\in\tilde\gamma_{2k}} \bar\eta^2_{12,x_1} + \sum_{x\in\tilde\gamma_{1k}} \bar\eta^2_{21,x_2} \Bigg) \\ & + h^2 \Big(|[\Lambda_2^{3/4}\bar\eta_{22}]|^2_{\tilde\gamma_{10}} + |[\Lambda_2^{3/4}\bar\eta_{22}]|^2_{\tilde\gamma_{11}} + |[\Lambda_1^{3/4}\bar\eta_{11}]|^2_{\tilde\gamma_{20}} + |[\Lambda_1^{3/4}\bar\eta_{11}]|^2_{\tilde\gamma_{21}} \Big) + \\ & + h \sum_{i=1}^{2}\sum_{k=0}^{1}\sum_{x\in\gamma_{ik}} \zeta_i^2 + h^2 \ln^2\frac{1}{h} \sum_{x\in\gamma^*} \left(\bar\zeta_1^2 + \bar\zeta_2^2\right) \Bigg\}^{1/2}. \end{aligned}$$

4. Convergence of the finite difference scheme

The problem of deriving the convergence rate estimate for the FDS (3) is reduced to estimating the right-hand side of the inequality in Lemma 5.

Note that in [16] the following estimates are proved:

$$\begin{aligned} &|[u-v]|_{W_2^1(\bar\omega)} \leqslant Ch^{s-1}\|\sigma\|_{L_2(\Gamma)}\|u\|_{W_2^{s+\varepsilon(2-s)}(\Omega)}, \quad 1\leqslant s<3/2, \\ &|[u-v]|_{W_2^1(\bar\omega)} \leqslant Ch^{s-1}\|\sigma\|_{W_2^{s-3/2}(\Gamma)}\|u\|_{W_2^{s+\varepsilon(2-s)}(\Omega)}, \quad 3/2<s\leqslant 2. \end{aligned} \tag{13}$$

(i) The estimate for η_i in ω_i, $i=1,2$ was obtained in [9]:

$$\Bigg\{ h^2 \sum_{x\in\omega_i} \eta^2_{i,x_i} \Bigg\}^{1/2} \leqslant Ch^{s-1}\, \|u\|_{W_2^s(\Omega)}, \quad 2<s\leqslant 3. \tag{14}$$

(ii) The value $\bar\eta_{21,x_2}$ in the nodes $x\in\tilde\gamma_{10}$ may be represented in the form:

$$\bar\eta_{21,x_2} = \frac{2}{h^2}\int_0^h\int_0^{x_1'}\int_0^{x_1''}\int_{x_2}^{x_2+h} \left(1-\frac{x_1'}{h}\right) \frac{\partial^3 u(x_1''', x_2')}{\partial x_1^2 \partial x_2}\, dx_2'\ dx_1'''\ dx_1''\ dx_1'.$$

Now, from above it follows that:

$$|\bar\eta_{21,x_2}| \leqslant Ch \left\| \frac{\partial^3 u}{\partial x_1^2\partial x_2} \right\|_{L_2(e)}, \quad e=(0,h)\times(x_2,x_2+h),$$

and summing over the nodes $x \in \tilde{\gamma}_{10}$ we obtain:

$$\left\{ h^2 \sum_{x \in \tilde{\gamma}_{10}} \bar{\eta}^2_{21,x_2} \right\}^{1/2} \leqslant C h^2 \|u\|_{W_2^3(\Omega)}. \tag{15}$$

Analogous estimates hold true for the other sum with $\bar{\eta}_{21,x_2}$ and $\bar{\eta}_{12,x_1}$.

(iii) Let us estimate, for example, $h^2 |[\Lambda_2^{3/4} \bar{\eta}_{22}|]^2_{\gamma_{10}}$. Firstly, setting

$$S(x_2) = \sigma(x_2)\, u(0,x_2) = \frac{\partial u(0,x_2)}{\partial x_1}$$

from (9) it follows that

$$\bar{\eta}_{22,x_2} = \frac{h}{3}\, S_{x_2} = \frac{1}{3} \int_{x_2}^{x_2+h} S'(y_2)\, dy_2.$$

Now, let us derive an estimate for the following expression:

$$\begin{aligned} \bar{\eta}_{22,x_2}(0,x_2) - \bar{\eta}_{22,x_2}(0,x'_2) &= \frac{1}{3h} \int_{x_2}^{x_2+h} \int_{x'_2}^{x'_2+h} \frac{S'(y_2) - S'(y'_2)}{y_2 - y'_2} (y_2 - y'_2)\ dy_2\ dy'_2 \\ &\leqslant \frac{1}{3} \left\{ \frac{7}{6} (x_2 - x'_2)^2 \right\}^{1/2} \left\{ \int_{x_2}^{x_2+h} \int_{x'_2}^{x'_2+h} \frac{[S'(y_2) - S'(y'_2)]^2}{|y_2 - y'_2|^2}\, dy_2 dy'_2 \right\}^{1/2}. \end{aligned} \tag{16}$$

Now, using (16) and Lemma 3, we obtain

$$\begin{aligned} h^2 |[\Lambda_2^{3/4} \bar{\eta}_{22}|]^2_{\gamma_{10}} &\leqslant h^2 \cdot C\ h^2 \sum_{x,x' \in \tilde{\gamma}_{10},\ x \neq x'} \frac{[\bar{\eta}_{22,x_2}(x) - \bar{\eta}_{22,x_2}(x')]^2}{(x_2 - x'_2)^2} \\ &\leqslant C\ h^4 \sum_{x,x' \in \tilde{\gamma}_{10},\ x \neq x'} \int_{x_2}^{x_2+h} \int_{x'_2}^{x'_2+h} \frac{[S'(y_2) - S'(y'_2)]^2}{|y_2 - y'_2|^2}\, dy_2 dy'_2 \\ &\leqslant C\ h^4 \int_0^1 \int_0^1 \frac{[S'(y_2) - S'(y'_2)]^2}{|y_2 - y'_2|^2}\, dy_2 dy'_2 = C\ h^4\ |S|^2_{W_2^{3/2}(0,1)} \\ &=\ C\ h^4 \left| \frac{\partial u(0,\cdot)}{\partial x_1} \right|^2_{W_2^{3/2}(0,1)} \leqslant C\ h^4\ \|u\|^2_{W_2^3(\Omega)}. \end{aligned} \tag{17}$$

(iv) Let us derive the estimate for $h \sum\limits_{i=1}^{2} \sum\limits_{k=0}^{1} \sum\limits_{x \in \gamma_{ik}} \zeta_i^2$.

For example, consider $h \sum_{x \in \gamma_{10}} \zeta_1^2$. The functional $\zeta_1 = (T_2^2 \sigma) u - T_2^2(\sigma u)$ may be decomposed in the following manner:

$$\zeta_1 = \zeta_{11} + \zeta_{12}, \quad \text{where}$$

$$\zeta_{11} = \frac{1}{h} \int_{x_2-h}^{x_2+h} \left(1 - \frac{|x'_2 - x_2|}{h}\right) [\sigma(x'_2) - \sigma(x_2)][u(0,x_2) - u(0,x'_2)]\ dx'_2,$$

$$\zeta_{12} = \sigma(x_2) \frac{1}{h} \int_{x_2-h}^{x_2+h} \left(1 - \frac{|x'_2 - x_2|}{h}\right) [u(0,x_2) - u(0,x'_2)]\ dx'_2.$$

Transforming ζ_{11} and ζ_{12} to the following forms

$$\zeta_{11} = \frac{1}{h}\int_{x_2-h}^{x_2+h}\left(1-\frac{|x_2'-x_2|}{h}\right)\left[\int_{x_2}^{x_2''}\sigma'(x_2''')dx_2'''\int_{x_2''}^{x_2}\frac{\partial u(0,x_2''')}{\partial x_2}dx_2'''\right]dx_2'',$$

$$\zeta_{12} = \frac{\sigma(x_2)}{h}\int_{x_2-h}^{x_2+h}\left(1-\frac{|x_2'-x_2|}{h}\right)\int_{x_2'}^{x_2}\int_{x_2}^{x_2''}\frac{\partial^2 u(0,x_2''')}{\partial x_2^2}\,dx_2'''dx_2''dx_2',$$

we simply obtain

$$|\zeta_{11}| \leqslant 4\sqrt{2}\,h^{3/2}\left\|\frac{\partial u}{\partial x_2}\right\|_{C(\bar\Omega)}\left(\int_{x_2-h}^{x_2+h}|\sigma'(x_2')|^2dx_2'\right)^{1/2},$$

$$|\zeta_{12}| \leqslant 4\sqrt{2}\,h^{3/2}\,\|\sigma\|_{C[0,1]}\left(\int_{x_2-h}^{x_2+h}\left|\frac{\partial^2 u(0,x_2')}{\partial x_2^2}\right|^2dx_2'\right)^{1/2}.$$

Summing over the nodes $x\in\gamma_{10}$ we obtain

$$h\sum_{x\in\gamma_{10}}\zeta_1^2 \leqslant Ch^4\,\|\sigma\|^2_{W_2^{3/2}(0,1)}\|u\|^2_{W_2^3(\Omega)}$$

and thus we can prove the estimate

$$h\sum_{i=1}^{2}\sum_{k=0}^{1}\sum_{x\in\gamma_{ik}}\zeta_i^2 \leqslant Ch^4\,\|\sigma\|^2_{W_2^{3/2}(0,1)}\|u\|^2_{W_2^3(\Omega)}. \tag{18}$$

(v) Let us derive the estimate for $\bar\zeta_1(0,0)$ as an example of estimates for the functionals $\bar\zeta_i$ in the nodes $x\in\gamma^\star$. The functional $\bar\zeta_1(0,0)$ can be represented in the form

$$\bar\zeta_1(0,0) = \frac{2}{h}\int_0^h\left(1-\frac{x_2'}{h}\right)\sigma(x_2')\int_0^{x_2'}\frac{\partial u(0,x_2'')}{\partial x_2}\,dx_2''\,dx_2'.$$

From this representation we simply obtain:

$$\begin{aligned}&|\bar\zeta_1(0,0)| \leqslant h\|\sigma\|_{C[0,1]}\left\|\frac{\partial u}{\partial x_2}\right\|_{C(\bar\Omega)} \quad \text{and}\\ &\left|h^2\ln^2\frac{1}{h}\,\bar\zeta_1(0,0)^2\right| \leqslant Ch^4\ln^2\frac{1}{h}\,\|\sigma\|^2_{W_2^{3/2}(0,1)}\|u\|^2_{W_2^3(\Omega)}.\end{aligned} \tag{19}$$

Finally, from lemma 5 and estimates (14)–(19) we obtain

$$|[z]|_{W_2^1(\bar\omega)} \leqslant Ch^2\ln\frac{1}{h}\,\|\sigma\|_{W_2^{3/2}(\Gamma)}\|u\|_{W_2^3(\Omega)}.$$

Now, applying interpolation theory (see [3], [19]) and using (13) we can prove the main result of this paper:

Theorem. *The finite difference scheme* (3) *converges in the norm* $W_2^1(\bar\omega)$ *and the following estimates hold true:*

$$|[u-v]|_{W_2^1(\bar\omega)} \leqslant Ch^{s-1}\|\sigma\|_{L_2(\Gamma)}\|u\|_{W_2^{s+\varepsilon(2-s)}(\Omega)},\quad 1\leqslant s<3/2,$$

$$|[u-v]|_{W_2^1(\bar\omega)} \leqslant Ch^{s-1}\|\sigma\|_{W_2^{s-3/2}(\Gamma)}\|u\|_{W_2^{s+\varepsilon(2-s)}(\Omega)},\quad 3/2<s\leqslant 2,$$

$$|[u-v]|_{W_2^1(\bar\omega)} \leqslant Ch^{s-1}\left(\ln\frac{1}{h}\right)^{s-2}\|\sigma\|_{W_2^{s-3/2}(\Gamma)}\|u\|_{W_2^s(\Omega)},\quad 2<s\leqslant 3.$$

Note that the error estimates obtained are "almost" consistent with the smoothness of data.

REFERENCES

[1] R. A. Adams, *Sobolev Spaces.* Academic Press, New York 1975.

[2] V. B. Andreev, *On the convergence of difference schemes for the approximation of second and third boundary-value problems for elliptic equations,* Zh. Vychisl. Mat. Mat. Fiz. **6** (1968), 1218–1231 (Russian).

[3] J. Bergh, J. Löfström, *Interpolation Spaces,* Springer-Verlag, Berlin–Heidelberg–New York 1976.

[4] J. H. Bramble, S. R. Hilbert, *Bounds for a class of linear functionals with application to Hermite interpolation,* Numer. Math. **16** (1971), 362–369.

[5] M. D. Chkhartishvili, G. K. Berikelashvili, *On the convergence in W_2^1 of difference solution of elliptic equation with mixed boundary conditions.* Bull. Acad. Sci. of Georgia **148**, 2 (1993), 180–184 (Russian)

[6] T. Dupont, R. Scott, *Polynomial approximation of functions in Sobolev spaces,* Math. Comput. **34** (1980), 441–463.

[7] B. S. Jovanović, *The finite difference method for boundary value problems with weak solutions,* Posebna izdanja Mat. Instituta, No. **16**, Beograd 1993.

[8] B. S. Jovanović, *Interpolation of function spaces and the convergence rate estimates for the finite difference schemes,* Second International Colloquium on Numerical Analysis, Plovdiv 1993, (D. Bainov and V. Covachev, eds.), VPS, Utrecht 1994, 103–112.

[9] B. S. Jovanović, *Interpolation technique and convergence rate estimates for finite difference method,* Lect. Notes Comp. Sci. **1196** (1997), 200–211.

[10] B. S. Jovanović, L. D. Ivanović, E. E. Süli, *Convergence of finite-difference schemes for elliptic equations with variable coeffients,* IMA J. Numer. Anal. **7** (1987), 301–305.

[11] B. S. Jovanović, B. Z. Popović, *Some convergence rate estimates for finite difference schemes,* Mat. Vesnik **49** (1997), 249–256.

[12] Lj. R. Lalović, *Approximation of generalized solutions of the third boundary value problem for elliptic equation with variable coefficients,* Zbor. rad. PMF Kragujevac **19** (1997), 53–65.

[13] R. D. Lazarov, V. L. Makarov, A. A. Samarskiĭ: *Application of exact difference schemes for the construction and investigation of difference schemes for generalized solutions,* Mat. Sbornik **117** (1982), 469–480 (Russian).

[14] V. G. Maz'ya, T. O. Shaposhnikova, *Theory of multipliers in spaces of diferentiable functions,* Monographs and Studies in Mathematics **23**, Pitman, Boston, Mass. 1985.

[15] B. Z. Popović, *Convergence of finite-difference schemes for Poisson's equation with boundary condition of the third kind,* Mat. Vesnik **47** (1995), 23–30.

[16] B. Z. Popović, D. Bojović, B. S. Jovanović, *Convergence of finite difference method for the third BVP – Interpolation technique,* XIII Conference on Applied Mathematics PRIM'98, Igalo 1998. (to appear)

[17] A. A. Samarskiĭ: *Theory of Difference Schemes,* Nauka, Moscow 1983. (Russian)

[18] E. E. Süli, B. S. Jovanović, L. D. Ivanović, *Finite difference approximations of generalized solutions,* Math. Comput. **45** (1985), 319–327.

[19] H. Triebel: *Interpolation Theory, Function Spaces, Differential Operators,* Deutscher Verlag der Wissenschaften, Berlin 1978.

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