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SIMULTANEOUS APPROXIMATION AND CHEBYSHEV CENTRES IN METRIC SPACES

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Abstract. The problem of best simultaneous approximation is studied in convex metric spaces.

Let A be a subset of a metric space (X,d). For each $x \in X$, the distance from x to A is defined by $d(x,A) = \inf\{\, d(x,a) : a \in A\,\}$. An element $a_0 \in A$ satisfying $d(x,a_0) = d(x,A)$ is called a best approximant to x in A. If a set of elements B is given in X, one might like to approximate all the elements of B simultaneously by a single element of A. This type of problem arises when a function being approximated is not known precisely, but is known to belong to a set. Several mathematicians have studied this problem of simultaneous approximation in normed linear spaces. We study this problem in metric spaces.

Let K be a subset of a metric space (X,d). Given any bounded subset F of X, define

$$\delta(F,K) = \inf_{x\in K} \sup_{y\in F} d(y,x).$$

An element $k^* \in K$ is said to be a best simultaneous approximation (b.s.a.) to F if $\sup_{y\in F} d(y,k^*) = \delta(F,K)$.

C. B. Dunham, J. B. Diaz and H. W. McLaughlin (see [6]) have considered the problem of best simultaneous approximation in the following case: $X = C[a,b]$, K a non-empty subset of X and $F = \{f_1, f_2\}$. Goel, Holland, Nasim and Sahney [3] studied the problem when X is a normed linear space, K any subset of X and $F = \{x_1, x_2\} \subset X$. Using the same procedure as in [3], it is possible to study the problem when $F = \{x_1, x_2, \dots, x_n\}$. Holland, Sahney and Tzimbalario [6] studied the problem when F is a compact subset of a normed linear space. In this paper, we study this problem in convex metric spaces. For this we recall a few definitions.

A bounded subset F of a metric space (X,d) is said to be *remotal* with respect to a subset K of X if for each $k \in K$ there exists a point $f_0 \in F$ farthest from k,

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i.e. $d(k, f_0) \geqslant d(k, f)$ for all $f \in F$. It is easy to see that compact subsets F of X satisfy this property.

Let (X, d) be a metric space and $I = [0, 1]$ be the closed unit interval. The continuous mapping $W\colon X \times X \times I \to X$ is said to be a *convex structure* on X [17] if for all $x, y \in X$, $\lambda \in I$

$$d(u, W(x, y, \lambda)) \leqslant \lambda d(u, x) + (1 - \lambda) d(u, y)$$

for all $u \in X$. The metric space (X, d) together with a convex structure is called a *convex metric spaces* [17].

A subset K of a convex metric space (X, d) is said to be *convex* [17] if $W(x, y, \lambda) \in K$ whenever $x, y \in K$ and $\lambda \in I$. This notion of convexity is closely related to that given by K. Menger (see [7]).

A convex metric spaces (X, d) is said to be *strictly convex* [7] if for every $x, y \in X$ and $r > 0$, $d(u, x) \leqslant r$, $d(u, y) \leqslant r$ imply $d(u, W(x, y, \lambda)) < r$ unless $x = y$, where u is arbitrary but fixed point of X.

We show that the problem of b.s.a. is equivalent to the problem of minimizing certain functional. For this we prove a lemma.

Lemma 1. *Let K be any subset of a metric space (X, d) and F a bounded subset of X. Then the functional $\phi\colon K \to \mathbf{R}$ defined by*

$$\phi(k) = \sup_{f \in F} d(f, k)$$

is continuous.

Proof. Let $\varepsilon > 0$ be given. For any $f \in F$ and $k, k' \in K$ we have $d(f, k) \leqslant d(f, k') + d(k', k)$ and so

$$\sup_{f \in F} d(f, k) \leqslant \sup_{f \in F} d(f, k') + d(k', k)$$

i.e. $\phi(k) - \phi(k') \leqslant d(k', k)$.

Interchanging k and k', we get $\phi(k') - \phi(k) \leqslant d(k, k')$ and so $|\phi(k) - \phi(k')| \leqslant d(k, k')$. Therefore, if $d(k, k') < \varepsilon$ then $|\phi(k) - \phi(k')| < \varepsilon$ and so ϕ is continuous. ■

Note. In normed linear spaces this lemma was proved in [6].

If there exists a $k^* \in K$ such that $\phi(k^*) = \inf_{k \in K} \phi(k)$ then $k^* \in K$ is a b.s.a. to F. So the problem of b.s.a. reduces to the problem of minimizing the functional ϕ on K and so we have

Theorem 1. *Let F be a bounded subset of a metric space (X, d) and K a subset of X such that the continuous functional $\phi\colon K \to \mathbf{R}$ defined by $\phi(k) = \sup_{f \in K} d(f, k)$ attains its infimum at some point of K then there always exists a b.s.a. in K to F.*

Since for compact sets K and for sets K which are approximatively compact with respect to F (i.e. any sequence (k_n) in K satisfying $\sup_{f \in F} d(k_n, f) \to \delta(F, K)$ is compact in K), we can find $k^* \in K$ such that $\phi(k^*) = \inf_{k \in K} \phi(k)$, we have

Corollary 1. *Let K be a compact subset of a metric space (X, d) and F be any bounded subset of X. Then there exists a b.s.a. in K to F.*

Corollary 2. [4] *If F is a bounded subset of a metric space (X, d) and K is approximatively compact with respect to F then there exists a b.s.a. in K to F.*

The following result which generalizes Lemma 3 of [6], deals with the convexity of the set of best simultaneous appoximants.

Lemma 2. *Let K be a convex subset of a convex metric space (X, d) and F a bounded subset of X. If $k_1^*, k_2^* \in K$ are b.s.a. to F then $W(k_1^*, k_2^*, \lambda)$ is also a b.s.a. in K to F for every $\lambda \in I$.*

Proof. Since $k_1^*, k_2^* \in K$ are b.s.a. to F,

$$\sup_{f\in F} d(f, k_1^*) = \delta(F, K) = \sup_{f\in F} d(f, k_2^*).$$

For any $f \in F$, consider $d(f, W(k_1^*, k_2^*, \lambda)) \leqslant \lambda d(f, k_1^*) + (1-\lambda) d(f, k_2^*)$. This implies

$$\begin{aligned}\sup_{f\in F} d(f, W(k_1^*, k_2^*, \lambda)) &\leqslant \lambda \sup_{f\in F} d(f, k_1^*) + (1-\lambda)\sup_{f\in F} d(f, k_2^*)\\ &= \lambda\delta(F,K) + (1-\lambda)\delta(F,K) = \delta(F,K) \leqslant \sup_{f\in F} d(f, W(k_1^*, k_2^*, \lambda))\end{aligned}$$

as $W(k_1^*, k_2^*, \lambda) \in K$ by the convexity of K. Therefore, $\sup_{f\in F} d(f, W(k_1^*, k_2^*, \lambda)) = \delta(F, K)$, proving thereby that $W(k_1^*, k_2^*, \lambda)$ is a b.s.a. in K to F for every λ. ∎

The following result deals with the uniqueness of b.s.a.

Theorem 2. *Let K be a convex subset of a strictly convex metric space (X, d) and F be a subset of X which is remotal w.r.t. K. Then there exists at most one b.s.a. in K to F.*

Proof. Suppose k_1^*, k_2^*, $k_1^* \neq k_2^*$ are two b.s.a. in K to the set F, i.e. $\sup_{f\in F} d(f, k_1^*) = \delta(F, K) = \sup_{f\in F} d(f, k_2^*)$. By Lemma 2, $W(k_1^*, k_2^*, \lambda) \in K$ is also a b.s.a. to F, i.e. $d(f, W(k_1^*, k_2^*, \lambda)) = \delta(F, K)$ for every $\lambda \in I$. Let $\lambda \in I$ be arbitrary. Keep it fixed. Since F is remotal w.r.t. K, there exists an element $f^* \in F$ such that

$$d(f^*, W(k_1^*, k_2^*, \lambda)) = \delta(F, K). \tag{1}$$

Now $d(f^*, k_1^*) \leqslant \delta(F, K)$ and $d(f^*, k_2^*) \leqslant \delta(F, K)$ and since the space is strictly convex, we have $d(f^*, W(k_1^*, k_2^*, \lambda)) < \delta(F, K)$ unless $k_1^* = k_2^*$. This contradicts (1) and hence the uniqueness. ∎

Combining Theorems 1 and 2 we have

Theorem 3. *Let K be a convex subset of a strictly convex metric space (X, d) and F a subset of X which is remotal w.r.t. K. If the continuous functional $\phi\colon K \to$ $\mathbf{R}$ defined by $\phi(k) = \sup_{f\in F} d(f, k)$ attains its infimum on K then there exists a unique b.s.a. in K to F.*

Since the functional ϕ attains its infimum when K is compact subset of a metric space (see [10]) or K is approximatively compact w.r.t. a bounded subset F of a metric space (see [4]) or K is weakly compact subset of a normed linear space (see [13]) or K is boundedly sequentially weakly compact subset of a normed linear space (see [15]) or K is a reflexive subspace of a normed linear space (see [1]), we have

Corollary 1. *Let K be compact convex subset of a strictly convex metric space (X, d) and F a subset of X which is remotal w.r.t. K then there exists a unique b.s.a. in K to F.*

Corollary 2. *Let F be a bounded subset of a strictly convex metric space (X, d), K a convex subset of X which is appoximatively compact w.r.t. F and F is remotal w.r.t. K then there exists a unique b.s.a. in K to F.*

Corollary 3. *Let K be a weakly compact convex subset of a strictly convex normed linear space X and F a subset of X which is remotal w.r.t. K then there exists a unique b.s.a. in K to F.*

Corollary 4. *If X is a strictly convex normed linear space, K is boundedly weakly sequentially compact convex subset of X and F a subset of X which is remotal w.r.t. K then there exists a unique b.s.a. in K to F.*

Corollary 5. *Let K be a reflexive subspace of a strictly convex normed linear space X and F a subset of X which is remotal w.r.t. K then there exists a unique b.s.a. in K to F.*

Corollary 1 generalizes the best simultaneous approximation theorem of [10], Corollary 2 generalizes Theorem 3 of [5], Corollary 3 generalizes Theorem 1 of [13], Corollary 4 contains Theorem 1 of [15] and Corollary 5 generalizes Theorem 1 of [1].

Since a uniformly convex Banach space is strictly convex and reflexive and a bounded closed convex subset of a reflexive space is weakly compact, Corollary 3 gives the following

Corollary 6. *Let K be a bounded closed convex subset of a uniformly convex Banach space X and F a subset of X which is remotal w.r.t. K then there exists a unique b.s.a. to F from the elements of K.*

Since a bounded closed, convex subset of a reflexive Banach space is weakly compact, Corollary 3 gives the following

Corollary 7. *If X is a strictly convex and reflexive Banach space, K is bounded, closed and convex, and F is a subset of X which is remotal w.r.t. K then there exists a unique b.s.a. in K to F.*

Since a finite dimensional normed linear space is boundedly weakly sequentially compact, Corollary 4 gives the following generalization of Theorem 1 of [6].

Corollary 8. *Let K be a finite-dimnesional subspace of a strictly convex normed linear space X and F a subset of X which is remotal w.r.t. K then there exists a unique b.s.a. in K to F.*

Analogous to the problem of best simultaneous approximation, the following problem of simultaneous furthest points was considered in [8].

Given any bounded subset F and a subset K of a metric space (X, d), define

$$D(F, K) = \sup_{f \in F} \inf_{k \in K} d(f, k).$$

An element $f^* \in F$ is said to be a *simultaneous furthest point* to the set K if $D(F, K) = \inf_{k \in K} d(f^*, k)$ i.e. f^* is farthest from K among the elements of F.

The problem is the same as that of deviation of F from K which in normed linear spaces was considered by V. M. Tihomirov (cf. [16], p. 160). Such an $f^* \in F$ is also called *extremal* with respect to K.

In particular case when $K = \{k\}$, the extremal elements $f^* \in F$ are nothing else but the elements of F which are farthest from k among the elements of the set F, i.e. $d(f^*, k) = \sup_{f \in F} d(f, k)$.

The following solution to the simultaneous furthest point problem was given in [8].

Theorem 4. *Let F be a compact subset of a metric space (X, d). Then given any subset K of X, there exists a simultaneous furthest point in F to the set K.*

This theorem can be formulated sowewhat stronger, i.e. under slighlty weaker assumptions. Let F be a bounded subset of a metric space (X, d). Then given any subset K of X, there exists a simultaneous furthest point in F to the set K if the function $\psi\colon F \to \mathbf{R}$ defined by $\psi(f) = d(f, K)$ attains its supremum at some $f^* \in F$. Its proof runs on the same lines as in [8].

Since the function $\psi\colon F \to \mathbf{R}$ defined by $\psi(f) = d(f, K)$ attains its supremum at a point of F if F is nearly compact with respect to K (i.e. any sequence (f_n) in F such that $d(f_n, K) \to D(F, K)$ is compact in F), we get

Theorem 5. [5] *If K is a subset of a metric space (X, d) and F is a subset of X and is nearly compact with respect to K then there exists a simultaneous furthest point in F to the set K.*

Many authors have studied relative Chebyshev centres in normed linear spaces (see [14]). This concept is closely related to the b.s.a. and was first introduced in 1962 by A. L. Garkavi [2]. We take up this study in metric spaces.

Let M be a subset of a metric space (X, d) and B a bounded subset of X. The Chebyshev radius of B with respect to M is defined as

$$\mathrm{rad}_M(B) = \inf_{m \in M} \sup_{f \in B} d(f, m).$$

The set of all elements $x \in M$ such that

$$\sup_{f \in M} d(f, x) = \mathrm{rad}_M(B)$$

is denoted by $\mathrm{Cent}_M(B)$ and its members, if they exist, are called *relative Chebyshev centres* of B w.r.t. M. They are the solutions of the simultaneous approximation problem. In case $M = X$ we write $\mathrm{Cent}_M(B) = \mathrm{Cent}(B)$ and call it the set of Chebyshev centres of B. When B is a singleton, say $\{f\}$, then $\mathrm{rad}_M(B) = \inf_{m \in M} d(f, m) = \mathrm{dist}(f, M)$ and $\mathrm{Cent}_M(B) = \{\, x \in M : d(f, x) = \mathrm{dist}(f, M) \,\}$ is the set of best approximations to f in M.

The elements of $\mathrm{Cent}(B)$ best represent the set B as if x is any particular element of X chosen to represent the set B, then error incurred will be $\sup\{\, d(f, x) : f \in B \,\}$ and $x_0 \in X$ best represents the set B when this error is minimum.

We say that M has the *relative Chebyshev centre property* in X if $\mathrm{Cent}_M(B) \neq \emptyset$ for all non-empty bounded sets B in X. When $M = X$, and $\mathrm{Cent}(B) \neq \emptyset$ for every non-empty bounded subset B of X, i.e. X has the relative Chebyshev centre property in X, we say that X *admits Chebyshev centres.* Since $B = \{f\}$ is bounded, any set M which has the relative Chebyshev centre property in X is proximinal in X.

A mapping $P\colon A \to X$, where A is a non-empty subset of X, is said to be *non-expansive* if $d(Px, Py) \leqslant d(x, y)$ for all $x, y \in A$. If $K \subset A$ is such that $P(K) \subset K$ then K is said to be *P-invariant* or *invariant under P.*

The study of Chebyshev centres, initiated by Garkavi has attracted much attention. Question concerning their existence and uniqueness have been analyzed by many researchers. The following result on the existence and uniqueness of Chebyshev centres was proved in [9].

THEOREM 6. *If X is a reflexive strictly convex Banach space then every convex remotal set A in X has a unique Chebyshev centre.*

This theorem can be formulated somewhat stronger as under.

THEOREM 7. *Let X be a strictly convex dual Banach space then every convex remotal set A in X has a unique centre.*

Proof. Since A is remotal, it is bounded, i.e. $\|y\| \leqslant m$ for all $y \in A$ and for some m. Let $B(0, m)$ be a ball in X with centre 0 and radius m. Then

$$\inf_{x \in X} \sup_{y \in A} \|x - y\| = \inf_{x \in B} \sup_{y \in A} \|x - y\|.$$

Since X is a dual Banach space, $B(0, m)$ is w^*-compact and so $g\colon X \to \mathbf{R}$ defined by $g(x) = \sup_{y \in A} \|x - y\|$ is weakly lower semi-continuous and therefore it attains its infimum at some x_0, i.e. $g(x_0) = \inf_{x \in X} g(x)$, i.e. $\sup_{y \in A} \|x_0 - y\| = \inf_{x \in X} \sup_{y \in A} \|x - y\|$. This proves the existence. The uniqueness part is the same as given in [9]. ∎

NOTE 1. The existence part also follows from Garkavi's theorem: Every dual Banach space admits Chebyshev centres to bounded sets.

2. It will be interesting to study the existence and uniqueness of Chebyshev centres in metric spaces.

The following result deals with the invariance of the set $\mathrm{Cent}_M(B)$ under a non-expansive mapping.

THEOREM 8. *Let (X,d) be a metric space, $P\colon X \to X$ be a non-expansive mapping and B a non-empty bounded subset of X such that $B \subset P(B)$. If M is a P-invariant non-empty subset of X then $\mathrm{Cent}_M(B)$ is P-invariant.*

In normed linear spaces this result was proved by J. B. Prolla [11] (Proposition 1) and the proof given in [11] can easily be extended to metric spaces.

The following result gives conditions under which subsets of a metric space admitting centres also admit centres.

THEOREM 9. *If a metric space (X,d) admits centres and $M \subset X$ is the range of an idempotent non-expansive mapping $p\colon X \to X$, then M admits centres.*

This result was proved by Prolla [11] in normed linear spaces (Proposition 2) and that proof can be easily extended to metric spaces.

Now suppose (X,d) is an ultrametric space, i.e. metric space in which strong triangle inequality $d(x,y) \leqslant \max\{d(x,z), d(z,y)\}$ is satisfied for all $x,y,z \in X$. An ultrametric space (X,d) is said to be *spherically complete* if every nest of closed spheres has a non-empty intersection. The following theorem deals with Chebyshev centre property in ultrametric spaces.

THEOREM 10. *Every spherically complete subspace of an ultrametric space has the Chebyshev centre property.*

In non-archimedian normed linear spaces, this theorem was proved by Prolla [12] and the same proof can easily be extended to ultrametric spaces.

COROLLARY 1. *Every spherically complete ultrametric space admits Chebyshev centres.*

COROLLARY 2. *Every spherically complete subspace of an ultrametric space is proximinal.*

REMARKS 1. Uniqueness of elements of b.s.a. is also guaranteed if the functional ϕ defined in Lemma 1 attains its infimum at exactly one $k^* \in K$.

2. When F is a singleton, say $\{f\}$, then $\delta(F,K) = \inf_{k\in K} d(f,k) = \mathrm{dist}(f,K)$. So the problem of b.s.a. (relative Chebyshev centres) reduces to the problem of best approximation and consequently, results proved in this paper extends known results on best approximation.

3. When K is a singleton, say $\{k\}$, then $D(F,K) = \sup_{f\in F} d(f,k)$. So the simultaneous furthest point problem reduces to the farthest point problem and consequently, results proved in this paper extends known resulta on farthest points.

4. The proof of Theorem 9 (see [11], Proposition 2) uses the fact that the range of an idempotent mapping is the set of its fixed points. So, it is natural to ask when the set of fixed points of a non-expansive mapping $P\colon X \to X$ admits

Chebyshev centres assuming that X admits Chebyshev centres. Prolla [11] gave a solution to this problem when X is an (AL)-space. This problem in metric spaces is yet to be discussed.

5. It will be interesting to study elements of ε-b.s.a. to F in K (*ε-simultaneous furthest points* to K in F) for any given $\varepsilon > 0$, i.e. elements $k^* \in K$ satisfying $\sup_{y\in F} d(y,k^*) \leqslant \delta(F,K) + \varepsilon$ (i.e. elements $f^* \in F$ satisfying $\inf_{k\in K} d(f^*,k) \geqslant D(F,k) - \varepsilon$).

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