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# ON $b$-OPEN SETS

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**Abstract.** A new class of generalized open sets in a topological space, called $b$-open sets, is introduced and studied. This class is contained in the class of semi-preopen sets and contains all semi-open sets and all preopen sets. It is proved that the class of $b$-open sets generates the same topology as the class of preopen sets.

## 1. Introduction

In recent years a number of generalizations of open sets have been considered. Four of these notions were defined similarly using the closure operator cl and the interior operator int in the following way.

DEFINITION 1. A subset $S$ of a space $X$ is called

(1) *an $\alpha$-set* if $S \subset \operatorname{int}(\operatorname{cl}(\operatorname{int} S))$,

(2) *semi-open* if $S \subset \operatorname{cl}(\operatorname{int} S)$,

(3) *preopen* if $S \subset \operatorname{int}(\operatorname{cl} S)$,

(4) *semi-preopen* if $S \subset \operatorname{cl}(\operatorname{int}(\operatorname{cl} S))$.

The first three notions are due to Njåstad [10], Levine [8] and Mashhour et al. [9], respectively. The concept of a preopen set was introduced by Corson and Michael [6] who used the term "locally dense". The fourth concept was introduced by Abd El-Monsef et al. [1] under the name "$\beta$-open", and in [3] these sets were called semi-preopen sets. We denote the classes of these sets in a space $(X, \mathcal{T})$ by $\mathcal{T}_\alpha$, $SO(X)$, $PO(X)$ and $SPO(X)$ respectively. All of them are larger than $\mathcal{T}$ and closed under forming arbitrary unions. Njåstad [10] showed that $\mathcal{T}_\alpha$ is a topology on $X$. In general, $SO(X)$ need not be a topology on $X$, but the intersection of a semi-open set and an open set is semi-open. The same holds for $PO(X)$ and $SPO(X)$. The complement of a semi-open set is called *semi-closed*. Thus $S$ is semi-closed if and only if $\operatorname{int}(\operatorname{cl} S) \subset S$. *Preclosed* and *semi-preclosed* sets are similarly defined. For a subset $S$ of a space $X$ the *semi-closure* (resp. *preclosure,*

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*semi-preclosure*) of $S$, denoted by $\operatorname{scl} S$ (resp. $\operatorname{pcl} S$, $\operatorname{spcl} S$) is the intersection of all semi-closed (resp. preclosed, semi-preclosed) subsets of $X$ containing $S$. Dually, the *semi-interior* (resp. *preinterior, semi-preinterior*) of $S$, denoted by $\operatorname{sint} S$, (resp. $\operatorname{pint} S$, $\operatorname{spint} S$), is the union of all semi-open (resp. preopen, semi-preopen) subsets of $X$ contained in $S$. By $\mathrm{cl}_\alpha$ and $\mathrm{int}_\alpha$ we denote the closure and the interior operator in $(X, \mathcal{T}_\alpha)$.

An extensive study of these operators was done in [3]. We recollect some of the relations that, together with their duals, we shall use in the sequel.

PROPOSITION 1.1. *Let $S$ be a subset of a space $X$. Then:*

(1) $\mathrm{cl}_\alpha S = S \cup \operatorname{cl}(\operatorname{int}(\operatorname{cl} S))$, $\mathrm{int}_\alpha S = S \cap \operatorname{int}(\operatorname{cl}(\operatorname{int} S))$,

(2) $\operatorname{scl} S = S \cup \operatorname{int}(\operatorname{cl} S)$, $\operatorname{sint} S = S \cap \operatorname{cl}(\operatorname{int} S)$,

(3) $\operatorname{pcl} S = S \cup \operatorname{cl}(\operatorname{int} S)$, $\operatorname{pint} S = S \cap \operatorname{int}(\operatorname{cl} S)$,

(4) $\operatorname{spcl} S = S \cup \operatorname{int}(\operatorname{cl}(\operatorname{int} S))$, $\operatorname{spint} S = S \cap \operatorname{cl}(\operatorname{int}(\operatorname{cl} S))$.

PROPOSITION 1.2. *Let $S$ be a subset of a space $X$. Then:*

(1) $\operatorname{scl}(\operatorname{sint} S) = \operatorname{sint} S \cup \operatorname{int}(\operatorname{cl}(\operatorname{int} S))$.

(2) $\operatorname{pcl}(\operatorname{pint} S) = \operatorname{pint} S \cup \operatorname{cl}(\operatorname{int} S)$.

(3) $\operatorname{spcl}(\operatorname{spint} S) = \operatorname{spint}(\operatorname{spcl} S)$.

(4) $\operatorname{int}(\operatorname{scl} S) = \operatorname{pint}(\operatorname{cl} S) = \operatorname{pint}(\operatorname{scl} S) = \operatorname{scl}(\operatorname{pint} S) = \operatorname{int}(\operatorname{cl} S)$.

(5) $\operatorname{int}(\operatorname{pcl} S) = \operatorname{scl}(\operatorname{int} S) = \operatorname{spcl}(\operatorname{int} S) = \operatorname{int}(\operatorname{spcl} S) = \operatorname{int}(\operatorname{cl}(\operatorname{int} S))$.

## 2. Some properties of $b$-open sets.

Now we consider a new class of generalized open sets.

DEFINITION 2. A subset $S$ of a space $X$ is called *$b$-open* if $S \subset \operatorname{cl}(\operatorname{int} S) \cup \operatorname{int}(\operatorname{cl} S)$. The class of all $b$-open sets in $X$ will be denoted by $B(X)$.

It is obvious that $PO(X) \cup SO(X) \subset BO(X) \subset SPO(X)$ and we shall show that the inclusions cannot be replaced with equalities.

EXAMPLE. Consider the set $\mathbf{R}$ of real numbers with the usual topology, and let $S = [0,1] \cup ((1,2) \cap \mathbf{Q})$ where $\mathbf{Q}$ stands for the set of rational numbers. Then $S$ is $b$-open but neither semi-open nor preopen. On the other hand, let $T = [0,1) \cap \mathbf{Q}$. Then $T$ is semi-preopen but not $b$-open.

PROPOSITION 2.1. *For a subset $S$ of a space $X$ the following are equivalent:*

(a) *$S$ is $b$-open.*

(b) $S = \operatorname{pint} S \cup \operatorname{sint} S$.

(c) $S \subset \operatorname{pcl}(\operatorname{pint} S)$.

*Proof.* (a) $\Rightarrow$ (b): Let $S$ be $b$-open, that is $S \subset \operatorname{cl}(\operatorname{int} S) \cup \operatorname{int}(\operatorname{cl} S)$. Then by Proposition 1.1 we have $\operatorname{pint} S \cup \operatorname{sint} S = (S \cap \operatorname{int}(\operatorname{cl} S)) \cup (S \cap \operatorname{cl}(\operatorname{int} S)) = S \cap (\operatorname{int}(\operatorname{cl} S) \cup (\operatorname{cl}(\operatorname{int} S)) = S$.

(b) $\Rightarrow$ (c): Propositions 1.1 and 1.2 imply $S = \operatorname{pint} S \cup \operatorname{sint} S = \operatorname{pint} S \cup (S \cap \operatorname{cl}(\operatorname{int} S)) \subset \operatorname{pint} S \cup \operatorname{cl}(\operatorname{int} S) = \operatorname{pcl}(\operatorname{pint} S)$.

(c) $\Rightarrow$ (a): By Propositions 1.2 and 1.1 we have $S \subset \operatorname{pcl}(\operatorname{pint} S) = \operatorname{pint} S \cup \operatorname{cl}(\operatorname{int} S) \subset \operatorname{int}(\operatorname{cl} S) \cup \operatorname{cl}(\operatorname{int} S)$, and so $S$ is $b$-open. ■

REMARK 1. It follows from (b) that every $b$-open set can be represented as a union of a preopen set and a semi-open set. Since $\operatorname{pint} S \setminus \operatorname{sint} S = \operatorname{pint} S \setminus (S \cap \operatorname{cl}(\operatorname{int} S)) = \operatorname{pint} S \setminus \operatorname{cl}(\operatorname{int} S)$ is preopen, we can always have a disjoint union.

REMARK 2. It was shown in [3] that $S$ is semi-preopen if and only if $S \subset \operatorname{sint}(\operatorname{scl} S)$. It is easy to prove that the condition $S \subset \operatorname{scl}(\operatorname{sint} S)$ characterizes the semi-open sets, and the condition $S \subset \operatorname{pint}(\operatorname{pcl} S)$ characterizes the preopen sets.

The next result is obvious.

PROPOSITION 2.2. *Let $S$ be a $b$-open set such that* $\operatorname{int} S = \emptyset$. *Then $S$ is preopen.* ■

PROPOSITION 2.3. (a) *The union of any family of $b$-open sets is a $b$-open set.*

(b) *The intersection of an open and a $b$-open set is a $b$-open set.*

*Proof.* The statements are proved by using the same method as in proving the corresponding results for the other three classes of generalized open sets. (See [3].) ■

It was shown in [2] that $\operatorname{cl}_\alpha \operatorname{int}_\alpha S = \operatorname{cl}(\operatorname{int} S)$ and $\operatorname{int}_\alpha \operatorname{cl}_\alpha S = \operatorname{int}(\operatorname{cl} S)$ for any $S \subset X$. We deduce from it the following result, analogous to those established for the other three classes of generalized open sets.

PROPOSITION 2.4. *Let $(X, \mathcal{T})$ be a space. Then:*

(1) *$\mathcal{T}$ and $\mathcal{T}_\alpha$ have the same class of $b$-open sets.*

(2) *The intersection of an $\alpha$-set and a $b$-open set is a $b$-open set.* ■

DEFINITION 3. A subset $S$ of a space $X$ is called *$b$-closed* if $X \setminus S$ is $b$-open. Thus $S$ is $b$-closed if and only if $\operatorname{int}(\operatorname{cl} S) \cap \operatorname{cl}(\operatorname{int} S) \subset S$.

DEFINITION 4. If $S$ is a subset of a space $X$ the *$b$-closure* of $S$, denoted by $\operatorname{bcl} S$, is the smallest $b$-closed set containing $S$. The *$b$-interior* of $S$, denoted by $\operatorname{bint} S$, is the largest $b$-open set contained in $S$.

PROPOSITION 2.5. *Let $S$ be a subset of a space $X$. Then*:

(1) $\operatorname{bcl} S = \operatorname{scl} S \cap \operatorname{pcl} S$.

(2) $\operatorname{bint} S = \operatorname{sint} S \cup \operatorname{pint} S$.

*Proof.* We shall prove only the first statement. Since $\operatorname{bcl} S$ is a $b$-closed set, we have $\operatorname{bcl} S \supset \operatorname{int}(\operatorname{cl}(\operatorname{bcl} S)) \cap \operatorname{cl}(\operatorname{int}(\operatorname{bcl} S)) \supset \operatorname{int}(\operatorname{cl} S) \cap \operatorname{cl}(\operatorname{int} S)$ and so $\operatorname{bcl} S \supset S \cup (\operatorname{int}(\operatorname{cl} S) \cap \operatorname{cl}(\operatorname{int} S)) = \operatorname{scl} S \cap \operatorname{pcl} S$ by Proposition 1.1. The reverse inclusion is clear. ■

The last result as well as Propositions 1.1 and 1.2, enable us to relate the operators of $b$-closure and $b$-interior to the other operators defined by generalized open sets.

PROPOSITION 2.6. *Let $S$ be a subset of a space $X$. Then*:

(1) $\operatorname{bcl}(\operatorname{int} S) = \operatorname{int}(\operatorname{bcl} S) = \operatorname{int}(\operatorname{cl}(\operatorname{int} S))$.

(2) $\operatorname{bint}(\operatorname{cl} S) = \operatorname{cl}(\operatorname{bint} S) = \operatorname{cl}(\operatorname{int}(\operatorname{cl} S))$.

(3) $\operatorname{bcl}(\operatorname{sint} S) = \operatorname{scl}(\operatorname{sint} S)$.

(4) $\operatorname{bint}(\operatorname{scl} S) = \operatorname{sint}(\operatorname{scl} S)$.

(5) $\operatorname{sint}(\operatorname{bcl} S) = \operatorname{scl} S \cap \operatorname{cl}(\operatorname{int} S)$.

(6) $\operatorname{scl}(\operatorname{bint} S) = \operatorname{sint} S \cup \operatorname{int}(\operatorname{cl} S)$.

(7) $\operatorname{pint}(\operatorname{bcl} S) = \operatorname{bcl}(\operatorname{pint} S) = \operatorname{pint}(\operatorname{pcl} S)$.

(8) $\operatorname{pcl}(\operatorname{bint} S) = \operatorname{bint}(\operatorname{pcl} S) = \operatorname{pcl}(\operatorname{pint} S)$.

(9) $\operatorname{spint}(\operatorname{bcl} S) = \operatorname{bcl}(\operatorname{spint} S) = \operatorname{sint}(\operatorname{scl} S) \cap \operatorname{pcl} S$.

(10) $\operatorname{spcl}(\operatorname{bint} S) = \operatorname{bint}(\operatorname{spcl} S) = \operatorname{scl}(\operatorname{sint} S) \cup \operatorname{pint} S$. ■

We conclude this section with

PROPOSITION 2.7. *Let $S$ be a subset of a space $X$. Then* $\operatorname{bint}(\operatorname{bcl} S) = \operatorname{bcl}(\operatorname{bint} S)$.

*Proof.* The statement follows from Propositions 1.2, 2.5 and 2.6. ■

### 3. On the topology generated by $b$-open sets.

Although none of $SO(X)$, $PO(X)$, $SPO(X)$ and $BO(X)$ is a topology on $X$, each of these classes generates a topology in a natural way. Let $\mathcal{T}(\mathcal{S}) = \{V \subset X \mid V \cap S \in \mathcal{S}$ whenever $S \in \mathcal{S}\}$, where $\mathcal{S}$ stands for $SO(X)$, $PO(X)$, $SPO(X)$ and $BO(X)$ respectively. Clearly $\mathcal{T}(\mathcal{S})$ is a topology on $X$ larger than $\mathcal{T}$. Njåstad [10] showed that $\mathcal{T}(\mathcal{S}) = \mathcal{T}_\alpha$ for $\mathcal{S} = SO(X)$. The topology generated by $PO(X)$ was studied in [4] and denoted by $\mathcal{T}_\gamma$. The closure and the interior of a set $S$ in $(X, \mathcal{T}_\gamma)$ are denoted by $\operatorname{cl}_\gamma S$ and $\operatorname{int}_\gamma S$. It was proved in [7] that $\mathcal{T}(\mathcal{S}) = \mathcal{T}_\gamma$ for $\mathcal{S} = SPO(X)$. The topology generated by $b$-open sets will be denoted by $\mathcal{T}_b$ and we shall prove that $\mathcal{T}_b = \mathcal{T}_\gamma$. We first recollect some results.

PROPOSITION 3.1. [7] *The intersection of a semi-open and a preopen set is a semi-preopen set.*

PROPOSITION 3.2. [7] *For a space $(X, \mathcal{T})$ and $x \in X$ the following are equivalent:*

(a) $\{x\} \in SPO(X)$.

(b) $\{x\} \in PO(X)$.

(c) $\{x\} \in \mathcal{T}_\gamma$.

PROPOSITION 3.3. [4] *Let $S$ be a subset of a space $X$. Then* $\mathrm{cl}_\gamma \operatorname{int} S = \mathrm{cl}(\operatorname{int} S)$.

PROPOSITION 3.4. [5] *Let $S$ be a subset of a space $X$. Then $S \in \mathcal{T}_\gamma$ if and only if $S = G \cup H$ with $G \in \mathcal{T}_\alpha$ and $\{h\} \in PO(X)$ for every $h \in H$.* ∎

The last result has an immediate consequence.

COROLLARY 3.5. *Let $V \in \mathcal{T}_\gamma$ and $G \in SO(X)$. Then $V \cap G \in BO(X)$.* ∎

Now we are in a position to prove the first part of our main result.

THEOREM 3.6. *Let $(X, \mathcal{T})$ be a space. Then $\mathcal{T}_\gamma \subset \mathcal{T}_b$.*

*Proof.* Let $V \in \mathcal{T}_\gamma$ and $S \in BO(X)$. Then $S = \operatorname{pint} S \cup \operatorname{sint} S$ by Proposition 2.1. Hence $V \cap S = (V \cap \operatorname{pint} S) \cup (V \cap \operatorname{sint} S)$ is $b$-open by Proposition 3.5, and so $V \in \mathcal{T}_b$. ∎

In order to prove the reverse inclusion we first establish two lemmas.

LEMMA 3.7. *Let $(X, \mathcal{T})$ be a space and $V \in \mathcal{T}_b$. Then $S = V \setminus \operatorname{int}(\mathrm{cl}(\operatorname{int} V))$ is a preopen set.*

*Proof.* Since $\operatorname{int}(\mathrm{cl}(\operatorname{int} V))$ is semi-closed, $S$ is $b$-open. On the other hand, $\operatorname{int} S = \emptyset$ and so $S$ is preopen by Proposition 2.2. ∎

LEMMA 3.8. *Let $(X, \mathcal{T})$ be a space and $V \in \mathcal{T}_b$. Then* $\operatorname{sint} V = \operatorname{int}_\alpha V$.

*Proof.* Let $x \in \operatorname{sint} V = V \cap \mathrm{cl}(\operatorname{int} V)$ and suppose $x \notin \operatorname{int}(\mathrm{cl}(\operatorname{int} V))$. Then $\{x\} \cup \operatorname{int} V$ is semi-open and $V \setminus \operatorname{int}(\mathrm{cl}(\operatorname{int} V))$ is preopen by Lemma 3.7. Hence $\{x\} = (\{x\} \cup \operatorname{int} V) \cap (V \setminus \operatorname{int}(\mathrm{cl}(\operatorname{int} V)))$ is semi-preopen by Proposition 3.1, and so $\{x\} \in \mathcal{T}_\gamma$ by Proposition 3.2. On the other hand, $\{x\} \in \mathrm{cl}_\gamma \operatorname{int} V$ by Proposition 3.3, a contradiction. Therefore $x \in \operatorname{int}(\mathrm{cl}(\operatorname{int} V))$ that is $x \in \operatorname{int}_\alpha V$. ∎

THEOREM 3.9. *Let $(X, \mathcal{T})$ be a space. Then $\mathcal{T}_b = \mathcal{T}_\gamma$.*

*Proof.* Let $V \in \mathcal{T}_b$ and $S \in PO(X)$. Put $G = V \cap \mathrm{cl}(\operatorname{int} V)$ and $H = V \setminus \mathrm{cl}(\operatorname{int} V)$. Then by previous lemma $V = \operatorname{int}_\alpha V \cup H$. Since $H \in \mathcal{T}_b$ and $\operatorname{int} H = \emptyset$, we have that $H \cap S \in PO(X)$ by Proposition 2.2. Therefore $V \cap S = (\operatorname{int}_\alpha V \cap S) \cup (H \cap S)$ is preopen and so $V \in \mathcal{T}_\gamma$. ∎

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