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# SOLUTION OF ONE PROBLEM OF G. PÓLYA

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**Abstract.** We present the complete solution of the following problem of G. Pólya: Circular forest has a center at the origin and radius $R \geqslant 1$. A person is staying at the center and the trees of the radius $r$ are planted at all other lattice points of the forest. Determine the maximal value $\rho$ of the radius $r$ for which the person can see out of the forest and, in the case $r = \rho$, determine the directions in which he/she should look in order to see out of the forest.

## 1. Formulation of the problem

Circular forest has a center at the origin and radius $R \geqslant 1$. A person is staying at the center and the trees of the radius $r$ are planted at all other lattice points (points with integer coordinates) of the forest.

1) Determine the maximal value $\rho$ of the radius $r$ for which the person staying at the center can see out of the forest.

2) In the case $r = \rho$, determine the directions in which he/she should look in order to see out of the forest. (We assume that the ray tangent to a tree is not blocked by that tree.)

The above problem was studied for the first time by G. Pólya in [1], where he proved that

$$\lim_{R\to\infty} \rho R = 1.$$

In [2] (problem 239) the following estimate

$$\frac{1}{\sqrt{R^2+1}} \leqslant \rho < \frac{1}{R}$$

in the case when $R$ is integer is obtained by method of A. Speiser.

In some more recent books (see [3], [4]) the same estimate is obtained using Minkowski theorem concerned with the existence of the lattice point in the set which is convex and symmetric about the origin.

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We give here the complete solution of Pólya problem: we determine the exact value of $\rho$ and directions in which to look in order to see out of the forest in the case when $r = \rho$.

## 2. Solution of the problem

Let $m$ be the smallest integer gretaer than $R^2$ which can be represented as the sum of the squares of two relatively prime integers and let $k$ be a circle which has center at the origin and radius $\sqrt{m}$.

In the sequel we shall prove the following two statements and thus our problem will be comletely solved:

a) If $r > 1/\sqrt{m}$, one cannot see out of the forest no matter in what direction one looks.

b) If $r = 1/\sqrt{m}$, the points from the circle $k$ which can be seen from the origin are those whose coordinates are relatively prime integers.

Hence, $\rho = 1/\sqrt{m}$. When $R$ is integer we have that $\rho = 1/\sqrt{R^2+1}$.

## 3. Proofs

Denote by $A$ and $B$ points with coordinates $(R, 0)$ and $(0, R)$. Let $l$ be a half-line starting from the origin and avoiding every lattice point from the forest. Without loss of generality we can suppose that $l$ belongs to the first quadrant. Denote by $L$ the intersection point between $l$ and the forest border. Suppose $U$ is the lattice point from the region $OAL$ with the least positive distance from the half-line $l$ and $V$ is the lattice point from the region $OBL$ having the least positive distance from $l$. Let $W$ be the lattice point symmetric to the origin about the midpoint of the segment $UV$. Having in mind the extremal properties of the points $U$ and $V$, we shall prove the following facts:

1) Parallelogram $OUVW$ has unit area.

2) The point $W$ lies outside of the forest.

3) Coordinates of $W$ are relatively prime.

We can prove 1) using the following theorem:

PICK'S THEOREM. *The area of a polygom whose vertices are lattice points is given by* $S = i + b/2 - 1$, *where* $i$ *is the number of lattice points lying inside the polygon and* $b$ *is the number of lattice points lying on the boundary of the polygon.*

The proof of Pick's theorem can be found in [5], [6] and in many other books as well.

Suppose that the triangle $OUV$ has a lattice point $M$, different from its vertices. $M$ does not lie on $l$, because the only lattice point from the forest which lies on $l$ is the origin. If $M$ belonged to the region $OAL$, it would be closer to $l$ than the point $U$, contrary to the assumption that $U$ is the lattice point from the region $OAL$ with the least positive distance from $l$. Therefore $M$ does not belong to the region $OAL$. In the similar way we conclude that $M$ does not belong to the region $OBL$ either. Since the starting assumption brings us to a contradiction, we conclude that it is not correct, i.e. we conclude that the triangle $OUV$ has no lattice points except its vertices. An application of the Pick's theorem yields that the area of $OUV$ equals $1/2$ and therefore the area of the parallelogram $OUVW$ equals 1.

Suppose that the point $W$ belongs to the forest. $W$ does not lie on $l$, because the only lattice point from the forest which lies on $l$ is the origin. If $W$ belonged to the region $OAL$, it would be closer to $l$ than the point $U$ (because $W$ lies between $U$ and the intersection point of rays $l$ and $UW$), contrary to the assumption that $U$ is the lattice point from the region $OAL$ with the least positive distance from $l$. Therefore $W$ does not belong to the region $OAL$. In the similar way we conclude that $W$ does not belong to the region $OBL$ either. Since the starting assumption brings us to a contradiction, we conclude that it is not correct, i.e. we cocnlude that the point $W$ does not belong to the forest.

Suppose the coordinates $a$ and $b$ of the point $W$ are not relatively prime. Let $d > 2$ be their greatest common divisor. The point with coordinates $(a/d, b/d)$ is a lattice point belonging to the triangle $OUV$, different from its vertices. This contradicts the previously proved fact. Hence, coordinates of $W$ must be relatively prime.

It follows from 2) and 3) that $OW \geqslant \sqrt{m}$. Having in mind that the area of $OUWV$ equals 1, we conclude that the distances of the points $U$ and $V$ from the line $OW$ do not exceed $1/\sqrt{m}$. The equality holds if $W$ is lying on the circle $k$. If the half-line $l$ does not contain the point $W$, then one of the points $U$ and $V$ is closer to $l$ than to the line $OW$. Hence,

– when $r > 1/\sqrt{m}$, each ray will be blocked by a tree;

– when $r = 1/\sqrt{m}$, each ray whose direction intersects the circle $k$ at the point with not relatively prime integer coordinates, will be blocked by a tree.

Let $C$ be a point of the circle $k$ whose coordinates are relatively prime integers and let $l$ be a half-line starting from the origin and passing through $C$. Suppose $M$ is a lattice point from the forest with coordinates $x, y \geqslant 0$. Since the coordinates of the point $C$ are relatively prime integers, there are no lattice points lying inside the segment $OC$. So, points $O$, $C$ and $M$ are vertices of a triangle. According to the Pick's theorem, the area of the triangle $OCM$ is greater or equal to $1/2$. Since $OC = \sqrt{m}$, the distance between the point $M$ and the half-line $l$ is graeter or equal to $1/\sqrt{m}$. Therefore, for $r = 1/\sqrt{m}$, the ray whose direction is determine by $l$ will not be blocked by any tree. Thus we have proved the statements a) and b).

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