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CONVERGENCE IN HAUSDORFF METRIC PRESERVES GEOMETRIC SHAPE

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Abstract. In mathematical approach to the pattern recognition there are some mathematical problems that are important for practical aims. One of them is invariance of geometrical shape with respect to the Hausdorff convergence.

In this paper we prove that the limit of a convergent sequence of compact sets in $\mathbf{R}^n$ of the same shape is again a set of the same shape or a singleton.

Let $\mathbf{R}^m$ be m-dimensional Euclidean space and $\exp(\mathbf{R}^m)$ the set of all its non-empty, compact subsets.

Homoteties, translations and rotations in $\mathbf{R}^m$ generate a group G_m and we say that two sets F and F' in $\exp(\mathbf{R}^m)$ have the same *geometric shape* if there exists an $h \in G_m$ such that $F' = h[F]$. In this way, an equivalence relation is evidently defined on the set $\exp(\mathbf{R}^m)$ and so, the meaning of the concept of geometrical shape is presented by the set of all properties of an $F \in \exp(\mathbf{R}^m)$ which are preserved under the action of the group G_m on the set $\exp(\mathbf{R}^m)$.

Further, for $F \in \exp(\mathbf{R}^m)$ and $\varepsilon > 0$, let

$$V_\varepsilon(F) = \{\, x \in \mathbf{R}^m \mid \operatorname{dist}(x, F) < \varepsilon \,\},$$

and for $F, F' \in \exp(\mathbf{R}^m)$ let

$$\rho(F, F') = \inf\{\, \varepsilon \mid F' \subseteq V_\varepsilon(F) \,\}.$$

Then the bigger of two numbers, $\rho(F, F'), \rho(F', F)$, i.e.

$$\mathrm{d}(F, F') = \max\{\rho(F, F'), \rho(F', F)\}$$

is a metric on $\exp(\mathbf{R}^m)$, which is called the *Hausdorff metric* and the metric space $(\exp(\mathbf{R}^m), \mathrm{d})$ is often denoted by, writting simply, $\exp(\mathbf{R}^m)$.

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We shall also use the following related fact: if $F_n \to F_0$ in $\exp(\mathbf{R}^m)$, then $\operatorname{diam}(F_n) \to \operatorname{diam}(F_0)$. (For the properties of Hausdorff metric see, for example, [1])

Recall also that a set $\{a_0, \dots, a_k\}$ of $k+1$ points od $\mathbf{R}^m$ is in *general position* if the vectors

$$a_1 - a_0, \dots, a_k - a_0$$

are lineary independent. Then, these points span a k-plane $P(a_0, \dots, a_k)$ which consists of all points x of $\mathbf{R}^m$ such that

$$x = \lambda_0 a_0 + \dots + \lambda_k a_k,$$

with $\lambda_0 + \dots + \lambda_k = 1$ and the scalars $\lambda_0, \dots, \lambda_k$ are uniquely determined by x. Thus, $x \mapsto (\lambda_0, \dots, \lambda_k)$ is one-to-one, and continuous function which maps $P(a_0, \dots, a_k)$ onto the plane $\lambda_0 + \dots + \lambda_k = 1$ in $\mathbf{R}^{k+1}$.

If T is an affine transformation of $\mathbf{R}^m$ (that is a composition of a translation and a non-singular linear transformation), then the set $\mathrm{T}a_0, \dots, \mathrm{T}a_k$ is in general position and T carries the plane $P(a_0, \dots, a_k)$ onto $P(\mathrm{T}a_0, \dots, \mathrm{T}a_k)$, (see [3]).

Recall also that the set

$$\sigma(a_0, \dots, a_k) = \{\, \lambda_0 a_0 + \dots + \lambda_k a_k \in P(a_0, \dots, a_k) \mid (\forall i) \lambda_i \geqslant 0 \,\}$$

is called *k-simplex* spanned by $a_0, \dots, a_k$. The *volume* of $\sigma(a_0, \dots, a_k)$ is a multilinear form $\operatorname{vol}(a_1 - a_0, \dots, a_k - a_0)$ and $a_0, \dots, a_k$ are in general position if and only if $\operatorname{vol}(a_1 - a_0, \dots, a_k - a_0) \neq 0$.

Each $F \in \exp(\mathbf{R}^m)$ spans a k-plane P, $0 \leqslant k \leqslant n$. Then, F contains a set $\{a_0, \dots, a_k\}$ of $k+1$ points in general position and $P(a_0, \dots, a_k) = P$. Notice also that the set

$$D_F = \{\, (\lambda_0, \dots, \lambda_k) \mid \lambda_0 a_0 + \dots + \lambda_k a_k \in F \,\}$$

is a compact subset of $\mathbf{R}^{k+1}$.

If $F_n \to F_0$ in $\exp(\mathbf{R}^m)$ and all F_n's are convex, then F_0 is also convex (proved in [2], in a more general case of a sequence of closed sets in a Banach space converging with respect to the Vietoris topology). Therefore, convexity is preserved by convergence in Hausdorff metric.

There exist some particular cases where preservation of geometric shape is proved: cases of balls (or spheres) in $\mathbf{R}^n$ (associated with proof of existance of Jung ball of a compact set), case of described regular n-simplex etc. The following theorem includes these particular cases and states the preservation of geometric shape in general.

THEOREM. *Let (F_n) be a convergent sequence in $\exp(\mathbf{R}^m)$ and F_0 its limit. If all F_n, ($n \in \mathbf{N}$) have the same geometrical shape, then F_0 either has that same shape or is a point.*

Proof. Since F_1 and F_n have the same geometric shape, there is an $h_n \in G_n$ such that $F_n = h_n[F_1]$. Let F_1 span a k-dimensional plane and let $\{a_0^1, \dots, a_k^1\} \in F_1$ be in general position. Let $h_n(a_i^1) = a_i^n$, then since h_n is affine, we have

$$h_n(\lambda_0 a_0^1 + \dots + \lambda_k a_k^1) = \lambda_0 a_0^n + \dots + \lambda_k a_k^n,$$

for $\lambda_0 a_0^1 + \cdots + \lambda_k a_k^1 \in F_1$. Each sequence $(a_i^n), i = 0, \ldots, k$ is bounded and thus, there exists a convergent subsequence $(a_i^{k(n)})$. Let $a_i^{k(n)} \to a_i^0$. For the sake of simplicity, we can suppose $k(n) = n$.

If
$$\|\overline{\lambda_0} a_0^1 + \cdots + \overline{\lambda_k} a_k^1 - \overline{\overline{\lambda_0}} a_0^1 - \cdots - \overline{\overline{\lambda_k}} a_k^1\| = \operatorname{diam}(F_1)$$
then
$$\|\overline{\lambda_0} a_0^n + \cdots + \overline{\lambda_k} a_k^n - \overline{\overline{\lambda_0}} a_0^n - \cdots - \overline{\overline{\lambda_k}} a_k^n\| = \operatorname{diam}(F_n)$$
and for each $i, j,\ i \neq j$
$$\frac{\|a_i^n - a_j^n\|}{\|a_i^1 - a_j^1\|} = \frac{\operatorname{diam}(F_n)}{\operatorname{diam}(F_1)} = k_n,$$
what easily follows by checking these relations in the case of generating elements of G_m.

Let $k_n \to k_0$. Consider two cases.

(I) $k_0 = 0$: In this case $\operatorname{diam}(F_n) \to 0$ and it follows that $F_n \to \{a_0^0\}$. Hence, F_0 is a one point set.

(II) $k_0 > 0$: From the relation (which is again easily checked for generating elements of G_m)
$$\frac{|\operatorname{vol}(a_1^n - a_0^n, \ldots, a_k^n - a_0^n)|}{\|a_1^n - a_0^n\| \cdots \|a_k^n - a_0^n\|} = c > 0,$$
letting $n \to \infty$, we get
$$\frac{|\operatorname{vol}(a_1^0 - a_0^0, \ldots, a_k^0 - a_0^0)|}{\|a_1^0 - a_0^0\| \cdots \|a_k^0 - a_0^0\|} = c$$
and therefore
$$\operatorname{vol}(a_1^0 - a_0^0, \ldots, a_k^0 - a_0^0) \neq 0.$$
Thus, the set $\{a_0^0, \ldots, a_k^0\}$ is in general position. From
$$\|a_i^n - a_j^n\| = k_n \|a_i^1 - a_j^1\|,$$
we get
$$\|a_i^0 - a_j^0\| = k_0 \|a_i^1 - a_j^1\|.$$
Let $h_0 \in G_m$ be such that $h_0(a_i^1) = a_i^0$, $i = 0, \ldots, k$. Then,
$$h_0[F_1] = \{\, \lambda_0 a_0^0 + \cdots + \lambda_k a_k^0 \mid (\lambda_0, \ldots, \lambda_k) \in D_{F_1} \,\}$$
and $h_0[F_1]$ and F_1 have the same geometrical shape.

Let $m = \max\{\, |\lambda_0| + \cdots + |\lambda_k| \mid (\lambda_0, \ldots, \lambda_k) \in D_{F_1} \,\}$ and let $\varepsilon > 0$ be arbitrary. Let $n_0 \in \mathbf{N}$ be such that
$$\|a_i^n - a_i^0\| < \varepsilon / m$$
for $n \geqslant n_0$. Then, for $n \geqslant n_0$,
$$\begin{aligned} &\|\lambda_0 a_0^n + \cdots + \lambda_k a_k^n - \lambda_0 a_0^0 - \cdots - \lambda_k a_k^0\| \\ &\qquad \leqslant |\lambda_0| \cdot \|a_0^n - a_0^0\| + \cdots + |\lambda_k| \cdot \|a_k^n - a_k^0\| < \frac{\varepsilon}{m} \cdot m = \varepsilon, \end{aligned}$$
which proves that $F_n \to h_0[F_1]$. ■

This short paper was motivated by some considerations related to pattern recognition. We hope that the theorem we have proved has also some independent meaning.

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