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INCREASING SOLUTIONS OF $(r(x)y^{(n)})^{(n)} = yf(x)$

Julka Knežević-Miljanović

Abstract. We study the existence of positive, monotonic, unbounded solutions of the equation $(r(x)y^{(n)})^{(n)} = yf(x)$. We obtain necessary and sufficient conditions for the existence of different classes of these solutions.

Previous investigations of the equation

$$(r(x)y^{(n)})^{(n)} = \pm yf(x,y) \tag{1. $\pm$ 1}$$

include that of Kusano and Naito [1], who studied $(1. -1)$ with $n = 2$, and Kreith [2], who studied $(1. \pm 1)$ with $r(x) = 1$. We consider the equation

$$(r(x)y^{(n)})^{(n)} = yf(x) \tag{2}$$

where $f(x)$ and $r(x)$ are positive and continuous on $[\tau, \infty)$ and $\int_\tau^\infty du/r(u) = \infty$.

Definition. Denote $E_k(x,y) = y^{(k)}$, $0 \leqslant k \leqslant n-1$, $E_k(x,y) = (r(x)y^{(n)})^{(k-n)}$ for $n \leqslant k \leqslant 2n$, and $E_k(x) = E_k(x, y(x))$. A solution $y(x)$ of the equation (2) is said to be of the type $2j$, $0 \leqslant j \leqslant n$ if $E_k(x) > 0$, $0 \leqslant k \leqslant 2j$ and $(-1)^k E_k(x) > 0$, for $2j \leqslant k \leqslant 2n$ and $\sigma \leqslant x < \infty$ for some $\sigma \geqslant \tau$. Denote also

$$R_k(x,t) = \begin{cases} \dfrac{(t-x)^k}{k!}, & \text{for } 0 \leqslant k \leqslant n-1, \\ \displaystyle\int_x^t \frac{(u-x)^{n-1}(t-u)^{k-n}}{(n-1)!\,(k-n)!\,r(u)}\,du, & \text{for } n \leqslant k \leqslant 2n-1, \end{cases} \qquad \tau \leqslant t, x < \infty.$$

We observe that $R_k(x,t) > 0$ for $\tau \leqslant t < s < \infty$, and $(-1)^k R_k(x,t) > 0$ for $x > t$.

The following facts are known:

1) A solution of (2) which is positive on $[\tau, \infty)$ must be of the type $2j$ for some j, $0 \leqslant j \leqslant n$.

2) Equation (2) has solutions of the type $2j$ for $j = 0$ and $j = n$ [3].

Unlike earlier work on this subject, which considered solutions satisfying the asymptotic condition $0 < \lim_{x\to\infty} y(x)R_m^{-1}(x) < \infty$, we impose a stronger asymptotic condition

$$\lim_{x\to\infty} \left| y(x) - \sum_{k=0}^{m} A_k R_k(x) \right| = 0.$$

THEOREM. *$y(x)$ is a solution of the type $2j$, $0<j<n$, of (2) if and only if $E_{2j}(x)\downarrow A_{2j}\geqslant 0$ as $x\to\infty$. In this case there exist positive constants α, β such that*

$$\alpha R_{2j-1}(\rho,x)<y(x)<\beta R_{2j}(\rho,\tau),\qquad \rho\leqslant x<\infty,$$

ρ sufficiently large. Further, $y(x)\approx R_{2j}(\tau,x)$ if and only if $A_{2j}>0$; $y(x)\approx R_{2j-1}(\tau,x)$ if and only if $A_{2j}=0$ and $E_{2j-1}(x)\uparrow A_{2j-1}>0$, $x\to\infty$.

Proof. That $E_{2j}\downarrow A_{2j}\geqslant 0$ follows from the definition of the solutions of type $2j$, since E_{2j} is a positive decreasing function. It follows that for sufficiently large t, $0\leqslant A_{2j}\leqslant E_{2j}(x)<A_{2j}+\varepsilon$. If $2j\leqslant n-1$, we can integrate these inequalities $2j$ times and obtain positive constants α, β such that

$$\alpha R_{2j-1}(\tau,x)<y(x)<\beta R_{2j}(\tau,x)$$

for sufficiently large t. If $2j\geqslant n$, we integrate $2j-n$ times obtaining $\alpha_1 R_{2j-n-1}(\tau,x)\leqslant r(x)y^{(n)}(x)\leqslant \beta_1 R_{2j-n}(\tau,x)$, then dividing by $r(x)$ and integrating n times we obtain

$$\alpha R_{2j-1}(\tau,x)<y(x)<\beta R_{2j}(\tau_1,x).$$

If A_{2j} is strictly positive, then the preceding argument in fact gives $\alpha R_{2j}(\tau,x)<y(x)<\beta R_{2j}(\tau,x)$ and $y(x)$ is in fact asymptotically equivalent to R_{2j}. If $A_{2j}=0$ then E_{2j-1} is bounded and increasing shows that $y(x)$ is asymptotically equivalent to $R_{2j-1}(\tau_1,x)$.

We note that a solution of (2) can be written as

$$y(x)=E_0(x)=\sum_{k=0}^{2n-1}(-1)^k E_k(b)R_k(x,b)+\int_x^b R_{2n-1}(x,t)p(t)x(t)\,dt. \tag{3}$$

Formula (3) follows from the Taylor's theorem:

$$y(x)=\sum_{k=0}^{n-1}\frac{(-1)^k(b-x)^k y^{(k)}(b)}{k!}+(-1)^k\int_t^b\frac{(t-x)^{n-1}r(t)y^{(n)}(t)}{(n-1)!\,r(t)}\,dt \tag{a}$$

$$r(t)y^{(n)}(t)=\sum_{k=0}^{n-1}\frac{(-1)^k(b-t)^k(r(b)y^{(n)}(b))^{(k)}}{k!}+(-1)^n\int_x^b\frac{(t-n)^{n-1}p(u)y(u)}{(n-1)!}\,du; \tag{b}$$

if we substitute (b) into (a) and interchange the order of integration, (3) results.

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Faculty of Mathematics, P.O.B. 550, 11000 Beograd, Yugoslavia