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SOME ESTIMATIONS OF POSITIVE AND NEGATIVE EIGENVALUES FOR AN INHOMOGENEOUS MEMBRANE

Milutin Dostanić

Abstract. In this paper we give some simple estimates of positive and negative eigenvalues for the inhomogeneous membrane boundary problem

$$-\Delta u = \lambda m u, \quad u\,|_{\partial D=0}$$

if $D \subset \mathbf{R}^2$ is a simple connected domain and m is a real bounded function (possibly changing the sign).

Introduction

There exist lower bounds [1] for the smallest eigenvalue λ_1 for an inhomogeneous membrane problem

$$-\Delta u = \lambda m u, \quad u\,|_{\partial D=0} \tag{1}$$

where m is a bounded *postive* function on D and $D \subset \mathbf{R}^N$ is a simple connected domain. They are expressed in terms of the smallest positive zero of the Bessel function $J_{\frac{N}{2}-1}$ and the maximum of the function m.

In some cases it is possible to deduce more precise estimates. Namely, if $N = 2$ and $\log m$ is a subharmonic function, Nehari [5] proved that

$$\lambda_1 \geqslant \pi j_0^2 / \textstyle\int_D m\,dx\,dy$$

(j_0 is the smallest zero of the Bessel function J_0). This result is a generalization of the well known Faber-Krahn inequality [1].

In the case when $D \subset \mathbf{R}^2$ and when $\log m$ is not a subharmonic function, the known inequality of Schwartz [1; Th. 3.5, p. 105] gives a lower bound for λ_1 in terms of the smallest eigenvalue for a boundary value problem of the same type as (1), which is obtained from the previous one by the Schwartz symmetrization. But it is not explicitely seen how that lower bound depends on m.

More explicite estimation is given by Banks-Krein theorem [1]. Namely, if $0 \leqslant m \leqslant p_1$, $M = \int_D m\,dx\,dy$ and numbers R and r_1 are defined by $|D| = \pi R^2$, $M = p_1 \pi r_1^2$, then $\lambda_1 \geqslant C$ where C is the smallest positive zero of the equation

$$J_0(r_1\sqrt{\lambda p_1}) = r_1\sqrt{\lambda p_1}\,J_0'(r_1\sqrt{\lambda p_1}) \log \frac{r_1}{R}$$

($|D|$ denotes the area of D, $J_0(x) = \sum\limits_{k=0}^{\infty} \left(\frac{x}{2}\right)^k \frac{(-1)^k}{(k!)^2}$).

In this paper we give some simple estimates of positive and negative eigenvalues for the inhomogeneous membrane boundary problem (1) if $D \subset \mathbf{R}^2$ is a simple connected domain and m is a real bounded function (possibly changing the sign). In that case the spectrum of the problem (1) consists of two series $0 \leqslant \lambda_1^+ \leqslant \lambda_2^+ \leqslant \cdots$ and $\cdots \leqslant \lambda_2^- \leqslant \lambda_1^- < 0$.

Result

Denote by $G(z,\xi)$ the Green function for Dirichlet Laplacian for domain $D \subset \mathbf{C}$. Let d be the diameter of the domain D. It is well known [4] that the following inequality holds:

$$0 \leqslant G(z,\xi) \leqslant \ln \frac{d}{|z-\xi|}. \tag{2}$$

Let $M = (\int_D \int_D \ln^4 d|z-\xi|^{-1}\, dA(z)\, dA(\xi))^{1/4}$, $dA(z) = dz\, dy$, $z = x+iy$. Consider first the case when m is a positive function. Then all eigenvalues are positive.

Lemma 1. *If $m \in L^\infty(D)$ and $m \geqslant 0$ then*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \leqslant M^2 \int_D m^2(z)\, dA(z)$$

where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ are the eigenvalues of the problem (1).

Proof. Consider the operator $A = M\Gamma M$ where M, $\Gamma\colon L^2(D) \to L^2(D)$ are the operators defined by

$$Mf(z) = \sqrt{m(z)}\, f(z), \quad \Gamma f(z) = \int_D G(z,\xi) f(\xi)\, dA(\xi) \ (= (-\Delta)^{-1} f).$$

The operator A is positive; according to Weyl theorem [3] we have $s_n(A) \leqslant \mathrm{const}/n$ and hence A is an integral Hilbert-Schmidt operator. The operator A has the kernel

$$H(z,\xi) = \sqrt{m(z)} G(z,\xi) \sqrt{m(\xi)}.$$

As the eigenvalues of A are reciprocal to the corresponding eigenvalues of the problem (1) we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} &= \int_D \int_D |H(z,\xi)|^2\, dA(\xi)\, dA(z) \\
&= \int_D \int_D m(\xi) m(z) |G(z,\xi)|^2\, dA(\xi)\, dA(z) \\
&\leqslant \int_D \int_D m(\xi) m(z) \ln^2 \frac{d}{|z-\xi|}\, dA(z)\, dA(\xi) \quad \text{(according to (2))} \\
&\leqslant \left(\int_D \int_D m^2(z) m^2(\xi)\, dA(\xi)\, dA(z) \right)^{1/2} \left(\int_D \int_D \ln^4 \frac{d}{|z-\xi|}\, dA(\xi)\, dA(z) \right)^{1/2} \\
&= M^2 \int_D m^2(z)\, dA(z). \quad \blacksquare
\end{aligned}$$

Now, consider the case when the function m changes sign. Let

$$D^+ = \{\, z \in D \;:\; m(z) > 0 \,\}, \qquad m_+(z) = \begin{cases} m(z); & z \in D^+ \\ 0; & z \in D \setminus D^+ \end{cases}$$
$$D^- = \{\, z \in D \;:\; m(z) < 0 \,\}, \qquad m_-(z) = \begin{cases} -m(z); & z \in D^- \\ 0; & z \in D \setminus D^-. \end{cases}$$

Clearly, $m_+ \geqslant 0$, $m_- \geqslant 0$ on D and $m = m_+ - m_-$.

THEOREM 1. *If $m \in L^\infty(D)$ is a real function, then*

$$\begin{aligned} \sum_{n\geqslant 1} \frac{1}{(\lambda_n^+)^2} &\leqslant M^2 \int_{D^+} m^2(z)\, dA(z) \\ \sum_{n\geqslant 1} \frac{1}{(\lambda_n^-)^2} &\leqslant M^2 \int_{D^-} m^2(z)\, dA(z) \end{aligned} \tag{3}$$

where $\{\lambda_n^+\}$ and $\{\lambda_n^-\}$ are the sequences of positive and negative eigenvalues of the problem (1).

To prove Theorem 1 we need the following Lemma.

LEMMA 2. [2] *Let B be a selfadjoint compact operator on a separable Hilbert space $\mathcal{H}$ with spectral expansion*

$$B = \sum_{j\geqslant 1} \lambda_j(\cdot, \varphi_j)\varphi_j \qquad ((\varphi_i, \varphi_j) = \delta_{ij}).$$

Consider the nonnegative operators

$$B_+ = \sum_{\lambda_j>0} \lambda_j(\cdot, \varphi_j)\varphi_j \quad \textit{and} \quad B_- = -\sum_{\lambda_j<0} \lambda_j(\cdot, \varphi_j)\varphi_j.$$

Clearly, $B = B_+ - B_-$.

If $B = H_1 - H_2$ where H_1 and H_2 are nonnegative compact operators, then

$$\lambda_j(B_+) \leqslant \lambda_j(H_1), \quad \lambda_j(B_-) \leqslant \lambda_j(H_2), \qquad j = 1, 2, \ldots$$

(The symbol $\lambda_j(T)$ denotes the j-th eigenvalue of the positive compact operator T).

Proof of Theorem 1. Since the operator Γ is positive (because $\Gamma = (-\Delta)^{-1}$) we can rewrite the problem (1) in the bounded version, i.e.

$$\Gamma^{1/2} M_1 \Gamma^{1/2} f = \frac{1}{\lambda} f \tag{4}$$

where $M_1\colon L^2(D) \to L^2(D)$ is the bounded operator defined by $M_1 f(z) = m(z)f(z)$.

From (4) it follows that the eigenvalues of the operator $\Gamma^{1/2}M_1\Gamma^{1/2}$ are reciprocal to the corresponding eigenvalues of the problem (1).

Let M_+ and M_- be the operators on $L^2(D)$ defined by

$$M_+f(z) = m_+(z)f(z), \quad M_-f(z) = m_-(z)f(z).$$

Obviously $M_1 = M_+ - M_-$ and so we have $B = \Gamma^{1/2}M_+\Gamma^{1/2} - \Gamma^{1/2}M_-\Gamma^{1/2}$. Since the operators $\Gamma^{1/2}M_+\Gamma^{1/2}$ and $\Gamma^{1/2}M_-\Gamma^{1/2}$ are compact and nonnegative, then according to Lemma 2 we have

$$\begin{aligned} \sum_{n\geqslant 1} \frac{1}{(\lambda_n^+)^2} &\leqslant \sum_{j\geqslant 1} \lambda_j^2(\Gamma^{1/2}M_+\Gamma^{1/2}) \\ \sum_{n\geqslant 1} \frac{1}{(\lambda_n^-)^2} &\leqslant \sum_{j\geqslant 1} \lambda_j^2(\Gamma^{1/2}M_-\Gamma^{1/2}). \end{aligned} \tag{5}$$

Since the boundary value problem

$$-\Delta u = \lambda m_+ u, \quad u|_{\partial D} = 0$$

and the operator $\Gamma^{1/2}M_+\Gamma^{1/2}$ have the mutually reciprocal eigenvalues then from Lemma 1 we obtain

$$\sum_{j\geqslant 1} \lambda_j^2(\Gamma^{1/2}M_+\Gamma^{1/2}) \leqslant M^2 \int_D m_+^2(z)\,dA(z) = M^2 \int_{D^+} m^2(z)\,dA(z). \tag{6}$$

Similarly we obtain

$$\sum_{j\geqslant 1} \lambda_j^2(\Gamma^{1/2}M_-\Gamma^{1/2}) \leqslant M^2 \int_{D^-} m^2(z)\,dA(z). \tag{7}$$

From (5), (6) and (7) it follows (3). ■

Corollary. From (3) it follows that

$$\frac{1}{(\lambda_1^+)^2} \leqslant M^2 \int_{D^+} m^2(z)\,dA(z), \quad \frac{1}{(\lambda_1^-)^2} \leqslant M^2 \int_{D^-} m^2(z)\,dA(z)$$

i.e. $\lambda_1^+ \geqslant \dfrac{1}{M(\int_{D^+} m^2(z)\,dA(z))^{1/2}}$; $\lambda_1^- \leqslant -\dfrac{1}{M(\int_{D^-} m^2(z)\,dA(z))^{1/2}}$.

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Faculty of Mathematics, P.O.B. 550, 11000 Beograd, Yugoslavia