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SOME THEOREMS ON COMMON FIXED POINTS

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Abstract. Three general theorems on common fixed points of non-commuting selfmaps of a metric space are given. These results generalize the recent results of Naidu and Prasad [7], Leader [5] and a number of earlier results.

1. Introduction

S.V.R. Naidu and J.R. Prasad, in [7], obtained a number of results on common fixed points for a pair of selfmaps of a metric space, where the maps satisfied a variety of generalised contraction definitions governed by a control function. The purpose of this note is to show that their contractive conditions seem to be still restricted.

2. Results

Let (X,d) be a metric space. For a subset A of X, denote $\operatorname{diam}(A) = \sup\{\, d(x,y) \,:\, x,y \in A\,\}$. For any selfmap h of X and $x_0 \in X$, the set $O_h(x_0) = \{\, h^n x_0 \,:\, n \geqslant 0\,\}$ is called the h-orbit of x_0. For any pair of selfmaps f and g of X and any x, $y \in X$, denote

$$\alpha(x,y) = \operatorname{diam}\{O_f(x) \cup O_g(y)\}; \quad \beta(x,y) = \sup\{\, d(f^i x, g^j y) \,:\, i \geqslant 0.j \geqslant 0\,\}.$$

Definition. We will say that a real-valued function $F\colon X \to [0,\infty)$ is h-orbitally weaker lower semicontinuous (w.l.s.c.) relative to x_0, if $\{x_n\}$ is a sequence in $O_h(x_0)$ and $x_n \to x^*$ implies that $F(x^*) \leqslant \limsup F(x_n)$.

The following result is main.

Theorem 2.1. *Let X be a metric space, f and g a pair of selfmaps of X, and $x_0 \in X$ with* $\operatorname{diam}[O_f(x_0)] < \infty$ *or* $\operatorname{diam}[O_g(x_0)] < \infty$. *Suppose that*

(A) for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\varepsilon \leqslant \alpha(x,y) < \varepsilon + \delta \quad \textit{implies} \quad \inf_{n\geqslant 0} \beta(f^n x, g^n y) < \varepsilon$$

AMS (MOS) Subject Classification (1980): Primary 54H25 *for all* $x \in O_f(x_0)$ *and* $y \in O_g(x_0)$. *Suppose also that* $\{d(f^n x_0, g^n x_0)\}$ *converges to zero. Then*

(a) $\{f^n x_0\}$ *and* $\{g^n x_0\}$ *are Cauchy sequences, and if one of them converges then the other also converges to the same limit. Furthermore, if either* f *or* g *has a fixed point* u, *then the two sequences converge to* u.

(b) If one of the two sequences $\{f^n x_0\}$ *and* $\{g^n x_0\}$ *convergees to some* x^* *in* X, *then* x^* *is a fixed point of* f *(resp.* g*) if a function* $F_1(x) = d(x, fx)$ *or* $F_2(x) = d(x, f^2 x)$, *[resp.* $G_1(x) = d(x, gx)$ *or* $G_2(x) = d(x, g^2 x)$*] is* f*-orbitally (resp.* g*-orbitally) w.l.s.c. relative to* x_0.

(c) If $F_2(x)$ *[or* $F_1(x)$*] and* $G_2(x)$ *[or* $G_1(x)$*] are orbitally w.l.s.c. relative to* x_0, *then* x^* *is a common fixed point of* f *and* g.

Proof. Put $\alpha_n = \alpha(f^n x_0, g^n x_0)$, $\beta_n = \beta(f^n x_0, g^n x_0)$. Since $\{d(f^n x_0, g^n x_0)\}$ converges to zero and one of sequences $\{f^n x_0\}$, $\{g^n x_0\}$ is bounded, it follows that $\alpha_0 = \alpha(x_0, x_0) < \infty$. It is clear that $\alpha_{n+1} \leqslant \alpha_n$ and $\beta_{n+1} \leqslant \beta_n$. Hence $\lim \alpha_n = \varepsilon$ and $\lim \beta_n = \beta$ exist.

We shall show that $\varepsilon = 0$. Suppose to the contrary that $\varepsilon > 0$. Then $\delta = \delta(\varepsilon) > 0$ and so there exists a positive integer k such that $\varepsilon \leqslant \alpha_k < \varepsilon + \delta$. From (A) with $x = f^k x_0$ and $y = g^k x_0$ we have $\inf_{n \geqslant k} \beta_n = \beta < \varepsilon$, thus $(\varepsilon - \beta)/2 > 0$ and so there exists an integer $r \geqslant k$ such that

$$\beta_r < \beta + (\varepsilon - \beta)/2. \tag{1}$$

Since $\{d(f^n x_0, g^n x_0)\}$ converges to zero, there exists an integer $s \geqslant r$ such that $d(f^n x_0, g^n x_0) < (\varepsilon - \beta)/2$ for every $r \geqslant s$. Let now $i \geqslant j \geqslant s$. Then by the triangle inequality and (1) we have:

$$\begin{aligned}
&d(f^i x_0, f^j x_0) \leqslant d(f^i x_0, g^j x_0) + d(f^j x_0, g^j x_0),\\
&d(f^i x_0, f^j x_0) \leqslant \beta_s \leqslant \beta_r,\\
&d(f^i x_0, f^j x_0) \leqslant d(f^i x_0, g^j x_0) + d(f^j x_0, g^j x_0) \leqslant \beta_r + (\varepsilon - \beta)/2,\\
&d(g^i x_0, g^j x_0) \leqslant d(f^j x_0, g^i x_0) + d(f^j x_0, g^j x_0) \leqslant \beta_r + (\varepsilon - \beta)/2.
\end{aligned}$$

Hence we get

$$\alpha_{k_2} \leqslant \beta_{k_2} + (\varepsilon - \beta)/2 < (\beta + \varepsilon)/2 + (\varepsilon - \beta)/2 = \varepsilon.$$

This is a contradiction, since $\alpha_n \geqslant \varepsilon$ for all $n \geqslant 0$. Therefore, $\lim \alpha_n = 0$. Hence we conclude that $\{f^n x_0\}$ and $\{g^n x_0\}$ are Cauchy sequences, and if one of them converges, then both sequences converge to the same limit.

Suppose now that $gu = u$. Denote

$$a_n = \alpha(f^n x_0, g^n u) = \alpha(f^n x_0, u); \quad b_n = \beta(f^n x_0, u); \quad D_n = \delta[O_f(f x_0^n)].$$

Then $a_n = \max\{b_n, D_n\}$. Since $\{f^n x_0\}$ is a Cauchy sequence, we have $\lim D_n = 0$. Let $a = \lim a_n$ and $b = \lim b_n$. If we suppose that $a > 0$, then by (A) we have $b <$

$a \leqslant a_n = \max\{b_n, D_n\}$. Taking the limit as $n \to \infty$ yields $b < b$, a contradiction. Therefore, $\lim \alpha(f^n x_0, u) = 0$. Hence, $\lim f^n x_0 = u$.

Similarly, if $fz = z$ for some $z \in X$, then it can be shown that $\{g^n x_0\}$ converges to z. The statement (a) is proved.

Suppose now that $\lim f^n x_0 = x^*$ and that a real-valued function $F_2(x) = d(x, f^2 x)$ is f-orbitally w.l.s.c. relative to x_0. Then

$$F(f^n x_0) = d(f^n x_0, f^{n+2} x_0) \to 0, \quad \text{as } n \to \infty,$$

which implies $F(x^*) = 0$. Hence $f^2 x^* = x^*$. Therefore, $O_f(x^*) = \{x^*, fx^*\}$. Since by (a) $\{g^n x_0\}$ also converges to x^*, it follows that $\operatorname{diam}[O_g(g^k x_0)] = D_k \to 0$, as $k \to \infty$. Suppose that $d^* = d(x^*, fx^*) > 0$. Since

$$\inf_{n \geqslant 0} \beta[f^n x^*, g^n(g^k x_0)] = d^*; \quad \lim_{k \to \infty} \alpha(x^*, g^k x_0) = d^*,$$

from inequality (A) we obtain $d^* < d^*$, a contradiction. Hence $d^* = d(x^*, fx^*) = 0$, i.e. x^* is a fixed point for f.

Note that if $F_1(x) = d(x, fx)$ is f-orbitally w.l.s.c., then it is easy to see that $d(x^*, fx^*) = 0$. Hence $fx^* = x^*$.

Similarly, g-orbitally weakly lower semi-continuity of $G_1(x) = d(x, gx)$, $G_2(x) = d(x, g^2 x)$ implies $gx^* = x^*$. So we have showed (b). Note that (c) is clear. ■

COROLLARY 1. *Theorem 1 holds if the condition (A) is replaced by the following condition:*

(B) $\inf_{1 \leqslant n < \infty} \beta(f^n x, g^n y) \leqslant \varphi[\alpha(x, y)]$

for all x, y in X, where $\varphi\colon [0, \infty) \to [0, \infty)$ is an increasing function with the property that $\varphi(t+) < t$ for every $t > 0$.

Proof. It is well known that the conditions of the type (B) imply the conditions of the type (A) (see [6] and [4]). ■

REMARK 1. Corollary 1 is a slightly generalization of Theorem 1 of Naidu and Prasad [7], since they suppose in b) that f or f^2 (resp. g or g^2) is orbitally continuous at x^*.

REMARK 2. In Corollary 1 (and Theorem 1) one cannot drop the condition: a sequence $\{d(f^n x_0, g^n x_0)\}$ converges to zero. Examples 5 and 6 in [8] and Example 4 in [7] show it.

THEOREM 2. *Let X be a metric space, f, $g\colon X \to X$ selfmaps of X and $x_0 \in X$ with* $\operatorname{diam}[O_f(x_0) \cup O_g(x_0)] < \infty$. *Suppose that*

(C) for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $x \in O_f(x_0)$ and $y \in O_g(x_0)$

$$\varepsilon \leqslant \alpha(x, y) < \varepsilon + \delta \quad \textit{implies} \quad \inf_{n \geqslant 0} \alpha(f^n x, g^n y) < \varepsilon.$$

Then conclusions of Theorem 1 follow. Furthermore, the assumptions of continuity in (b) and (c) of Theorem 1 can be relaxed as follows: there exists a positive integer k (resp. m) such that the function $P(x) = d(x, f^k x)$ *[resp.* $Q(x) = d(x, g^m x)$*] is f-orbitally (resp. g-orbitally) w.l.s.c. relative to* x_0.

The proof of Theorem 2 is omitted, since it follows the same arguments as those of Theorem 1.

COROLLARY 2. *Theorem 2 holds if the condition (C) is replaced by the following condition:*

(D) $\inf_{1 \leqslant n < \infty} \alpha(f^n x, g^n y) \leqslant \varphi[\alpha(x, y)]$

for all x, $y \in X$, *where* φ *is as in Corollary 1. Furthermore, each of f and g has at most one fixed point.*

REMARK 3. Corollary 2 is a slightly generalization of Theorem 2 of Naidu and Prasad [7] like Remark 1. The second part of Theorem 6 of Ding [2] also follows from Theorem 2.

REMARK 4. If in Theorem 2 the condition (C) holds for all x, y in X and $f = g$, then we derive the main fixed point theorem of Leader [5] as a corollary.

THEOREM 3. *Theorem 2 holds with the contractive condition (E) below in the place of condition (B):*

(E) for any $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\varepsilon \leqslant \beta(x, y) < \varepsilon + \delta \quad \textit{implies} \quad \inf_{n \geqslant 0} \beta(f^n x_0, g^n x_0) < \varepsilon$$

for each $x \in O_f(x_0)$ *and* $y \in O_g(x_0)$.

The proof of Theorem 3 parallels that of Theorem 1.

COROLLARY 3. *Theorem 3 holds if the condition (E) is replaced by the following condition:*

(F) $\inf_{1 \leqslant n < \infty} \beta(f^n x, g^n y) \leqslant \varphi[\beta(x, y)]$

for all x, y *in* X, *where* φ *is as in Corollary 1. Furthermore, each of f and g has at most one fixed point.*

REMARK 5. Corollary 3 is a slightly modification of theorem 3 of Naidu and Prasad [7] like Remark 1.

REMARK 6. Corollaries 1, 3, 4 and 5 of Naidu and Prasad [7] follow from our corresponding Theorems 1, 2 or 3.

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