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ON OPTIMALITY OF THE INDEX OF SUM, PRODUCT, MAXIMUM, AND MINIMUM OF FINITE BAIRE INDEX FUNCTIONS

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Abstract. Chaatit, Mascioni, and Rosenthal defined finite Baire index for a bounded real-valued function f on a separable metric space, denoted by $i(f)$, and proved that for any bounded functions f and g of finite Baire index, $i(h) \leq i(f) + i(g)$, where h is any of the functions $f + g$, fg, $f \vee g$, $f \wedge g$. In this paper, we prove that the result is optimal in the following sense : for each $n, k < \omega$, there exist functions f, g such that $i(f) = n$, $i(g) = k$, and $i(h) = i(f) + i(g)$.

1. Introduction

A real-valued function f defined on a separable metric space X is called a difference of bounded semicontinuous functions if there exist bounded lower semicontinuous functions u and v on X such that $f = u - v$. The class of all such functions is denoted by $DBSC(X)$. Some authors have studied this class and some of its subclasses (see, e.g. [1,3]). Chaatit, Mascioni, and Rosenthal [1] defined finite Baire index for functions belonging to $DBSC(X)$, whose definition we now recall.

Let X be a separable metric space. For a given bounded function $f : X \to \mathbb{R}$, the upper semicontinuous envelope $\mathcal{U}f$ of f is defined by

$$\mathcal{U}f(x) = \overline{\lim}_{y \to x} f(y) = \inf\{\sup_{y \in U} f(y) : U \text{ is a neighborhood of } x\}$$

for all $x \in X$. The lower oscillation $\underline{\text{osc}}f$ of f is defined by

$$\underline{\text{osc}}f(x) = \overline{\lim}_{y \to x} |f(y) - f(x)|$$

for all $x \in X$. Finally, the oscillation $\text{osc}\, f$ of f is defined by $\text{osc}\, f = \mathcal{U}\underline{\text{osc}}f$. Next, for any $\varepsilon > 0$, let $os_0(f, \varepsilon) = X$. If $os_j(f, \varepsilon)$ has been defined for some $j \geq 0$, let $os_{j+1}(f, \varepsilon) = \{x \in L : \text{osc}\, f|_L(x) \geq \varepsilon\}$, where $L = os_j(f, \varepsilon)$. A bounded function

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$f : X \to \mathbb{R}$ is said to be of finite Baire index if there is an $n < \omega$ such that $os_n(f, \varepsilon) = \emptyset$ for all $\varepsilon > 0$. Then the finite Baire index of f is defined by

$$i(f) = \max_{\varepsilon>0} i(f, \varepsilon),$$

where $i(f, \varepsilon) = \sup\{n : os_n(f, \varepsilon) \neq \emptyset\}$.

Clearly, if $f \in DBSC(X)$ then f is a Baire-1 function, that is, the pointwise limit of a sequence of continuous functions. Based on the Baire Characterization Theorem, Kechris and Louveau [4] defined the oscillation index of real-valued Baire-1 functions. The study on oscillation index of real-valued Baire-1 functions was continued by several authors (see, e.g., [3, 5, 6]). We recall here the definition of oscillation index. Let $\mathcal{C}$ denote the collection of all closed subsets of a Polish space X. Now, let $\varepsilon > 0$ and a function $f : X \to \mathbb{R}$ be given. For any $H \in \mathcal{C}$, let $\mathcal{D}^0(f, \varepsilon, H) = H$ and $\mathcal{D}^1(f, \varepsilon, H)$ be the set of all $x \in H$ such that for every open set U containing x, there are two points x_1 and x_2 in $U \cap H$ with $|f(x_1) - f(x_2)| \geq \varepsilon$. For all $\alpha < \omega_1$ (ω_1 is the first uncountable ordinal number), set

$$\mathcal{D}^{\alpha+1}(f, \varepsilon, H) = \mathcal{D}^1(f, \varepsilon, \mathcal{D}^\alpha(f, \varepsilon, H)).$$

If α is a countable limit ordinal, let

$$\mathcal{D}^\alpha(f, \varepsilon, H) = \bigcap_{\alpha'<\alpha} \mathcal{D}^{\alpha'}(f, \varepsilon, H).$$

The *ε-oscillation index* of f on H is defined by

$$\beta_H(f, \varepsilon) = \begin{cases} \text{the smallest ordinal } \alpha < \omega_1 \text{ such that } \mathcal{D}^\alpha(f, \varepsilon, H) = \emptyset \\ \text{if such an } \alpha \text{ exists,} \\ \omega_1, \quad \text{otherwise.} \end{cases}$$

The *oscillation index* of f on the set H is defined by $\beta_H(f) = \sup\{\beta_H(f, \varepsilon) : \varepsilon > 0\}$. We shall write $\beta(f, \varepsilon)$ and $\beta(f)$ for $\beta_X(f, \varepsilon)$ and $\beta_X(f)$ respectively.

In fact, a function f is of finite Baire index if and only if $\beta(f) < \infty$ and then $\beta(f) = i(f) + 1$. Chaatit, Mascioni, and Rosenthal proved in [1] that if f and g are real-valued bounded functions of finite Baire index and h is any of the functions $f + g$, $fg, f \vee g, f \wedge g$, then $i(h) \leq i(f) + i(g)$. In this paper, we prove that the estimate $i(h) \leq i(f) + i(g)$ in [1, Theorem 1.3] is optimal in the following sense : For any $n, k < \omega$, there exist bounded real-valued functions f and g such that $i(f) = n$, $i(g) = k$, and $i(h) = i(f) + i(g)$. We process the proof by constructing functions on ordinal spaces $[1, \omega^{n+k}]$ and then we extend the construction to any compact metric space K such that $K^{(n+k)} \neq \emptyset$, where $K^{(\alpha)}$ is the α^{th} Cantor-Bendixson derivative of K. Note that for any function f on K, $\mathcal{D}^\alpha(f, \varepsilon, K) \subseteq K^{(\alpha)}$, for any $\alpha < \omega_1$.

2. Results

Before we construct functions on ordinal spaces to show that Theorem 1.3 in [1] is optimal, we prove the following fact that we will use later.

Lemma 2.1. *Let X, Y be Polish spaces and $\varepsilon > 0$ be given. If $\theta : X \to Y$ is a homeomorphism and $\rho : Y \to \mathbb{R}$, then $\mathcal{D}^\alpha(\rho, \varepsilon, Y) = \theta(\mathcal{D}^\alpha(\rho \circ \theta, \varepsilon, X))$ for all $\alpha < \omega_1$.*

Proof. We prove the lemma by induction on α. The statement in the lemma is true whenever $\alpha = 0$ since θ is surjective. By the injectivity of θ, the lemma is also true if α is a limit ordinal.

Suppose that the statement in the lemma is true for some $\alpha < \omega_1$. Let $y \in \mathcal{D}^{\alpha+1}(\rho, \varepsilon, Y)$. Since θ is bijective, there is a unique $x \in X$ such that $y = \theta(x)$. Let U be a neighborhood of x. Since θ is a homeomorphism, $\theta(U)$ is open in Y. Therefore, there exist $y_1, y_2 \in \theta(U) \cap \mathcal{D}^\alpha(\rho, \varepsilon, Y)$ such that $|\rho(y_1) - \rho(y_2)| \geq \varepsilon$. Let $x_1 = \theta^{-1}(y_1)$ and $x_2 = \theta^{-1}(y_2)$, then $x_1, x_2 \in \theta^{-1}(\theta(U) \cap \mathcal{D}^\alpha(\rho, \varepsilon, Y))$. Since

$$\begin{aligned}\theta^{-1}(\theta(U) \cap \mathcal{D}^\alpha(\rho, \varepsilon, Y)) &= \theta^{-1}(\theta(U)) \cap \theta^{-1}(\theta(\mathcal{D}^\alpha(\rho \circ \theta, \varepsilon, X)) \\ &\quad \text{by the inductive hypothesis} \\ &= U \cap \mathcal{D}^\alpha(\rho \circ \theta, \varepsilon, X),\end{aligned}$$

we have $x_1, x_2 \in U \cap \mathcal{D}^\alpha(\rho \circ \theta, \varepsilon, X)$. And also,

$$|(\rho \circ \theta)(x_1) - (\rho \circ \theta)(x_2)| = |\rho(y_1) - \rho(y_2)| \geq \varepsilon.$$

Therefore, $x \in \mathcal{D}^{\alpha+1}(\rho \circ \theta, \varepsilon, X)$, which implies that $y = \theta(x) \in \theta(\mathcal{D}^{\alpha+1}(\rho \circ \theta, \varepsilon, X))$.

Conversely, let $y \in \theta(\mathcal{D}^{\alpha+1}(\rho \circ \theta, \varepsilon, X))$. Then there exists $x \in \mathcal{D}^{\alpha+1}(\rho \circ \theta, \varepsilon, X)$ such that $\theta(x) = y$. Let V be any neighborhood of y. Since $\theta^{-1}(V)$ is open in X and $x \in \theta^{-1}(V)$, there exist $x_1, x_2 \in \theta^{-1}(V) \cap \mathcal{D}^\alpha(\rho \circ \theta, \varepsilon, X)$ such that

$$|(\rho \circ \theta)(x_1) - (\rho \circ \theta)(x_2)| \geq \varepsilon.$$

Let $y_1 = \theta(x_1)$ and $y_2 = \theta(x_2)$, then $y_1, y_2 \in \theta(\theta^{-1}(V) \cap \mathcal{D}^\alpha(\rho \circ \theta, \varepsilon, X))$. By the inductive hypothesis,

$$\theta(\theta^{-1}(V) \cap \mathcal{D}^\alpha(\rho \circ \theta, \varepsilon, X)) = \theta(\theta^{-1}(V)) \cap \theta(\mathcal{D}^\alpha(\rho \circ \theta, \varepsilon, X)) = V \cap \mathcal{D}^\alpha(\rho, \varepsilon, Y).$$

Therefore, $y_1, y_2 \in V \cap \mathcal{D}^\alpha(\rho, \varepsilon, Y)$ and

$$|\rho(y_1) - \rho(y_2)| = |\rho(\theta(x_1)) - \rho(\theta(x_2))| \geq \varepsilon.$$

Thus, $y \in \mathcal{D}^{\alpha+1}(\rho, \varepsilon, Y)$. □

Besides the lemma above, we will use the following useful lemma that can be found in [5].

Lemma 2.2 ([5], Lemma 2.1). *Let U be a clopen subset of X and $f : X \to \mathbb{R}$ is Baire-1. Then for any $\varepsilon > 0$ and $\alpha < \omega_1$, we have $\mathcal{D}^\alpha(f, \varepsilon, X) \cap U = \mathcal{D}^\alpha(f, \varepsilon, U)$.*

Now we are ready to give the construction. For any $m \in \mathbb{N}$, denote the clopen ordinal interval $[1, \omega^m]$ by I_m. Note that for any nonzero countable ordinal α can be uniquely written in the Cantor normal form

$$\alpha = \omega^{r_1} \cdot j_1 + \omega^{r_2} \cdot j_2 + \ldots + \omega^{r_\ell} \cdot j_\ell$$

where $m \geq r_1 > \ldots > r_\ell \geq 0$ and $\ell, j_1, \ldots, j_\ell \in \mathbb{N}$ (see, e.g., [7]). In the sequel we use the following function. Let a, b be any real numbers such that $a \neq b$ and $m \in \mathbb{N}$.

We define $\varphi_{a,b,m} : I_m \to \{a,b\}$ by

$$\varphi_{a,b,m}(\omega^{r_1} \cdot j_1 + \omega^{r_2} \cdot j_2 + \ldots + \omega^{r_\ell} \cdot j_\ell) = \begin{cases} a & \text{if } j_\ell \text{ is odd} \\ b & \text{if } j_\ell \text{ is even,} \end{cases}$$

where $m \geq r_1 > \ldots > r_\ell \geq 0$ and $\ell, j_1, \ldots, j_\ell \in \mathbb{N}$. The following lemma is related to the function $\varphi_{a,b,m}$.

Lemma 2.3. *For sufficiently small $\varepsilon > 0$, if we let $\varphi = \varphi_{a,b,m}$, then $\mathcal{D}^m(\varphi, \varepsilon, I_m) = \{\omega^m\}$ and $\varphi(\omega^m) = a$.*

Proof. Take any $0 < \varepsilon < |a - b|$. We prove the lemma by induction on m. Clearly, the assertion is true for $m = 1$ since ω is the only limit ordinal in $[1, \omega]$. Suppose that the assertion is true for some $m \in \mathbb{N}$. If $\varphi = \varphi_{a,b,m+1}$, it is clear that $\varphi(\omega^{m+1}) = a$. For each $k < \omega$, let $L_k = [\omega^m \cdot k + 1, \omega^m \cdot (k+1)]$. Clearly, $\theta : I_m \to L_k$ defined by $\theta(\xi) = \omega^m \cdot k + \xi$ is a homeomorphism. Therefore, by Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned}\mathcal{D}^m(\varphi, \varepsilon, I_{m+1}) \cap L_k &= \mathcal{D}^m(\varphi|_{L_k}, \varepsilon, L_k) = \theta(\mathcal{D}^m(\varphi|_{L_k} \circ \theta, \varepsilon, I_m)) \\ &= \theta(\mathcal{D}^m(\varphi_{a,b,m}, \varepsilon, I_m)) = \theta(\{\omega^m\}) = \{\omega^m \cdot (k+1)\}.\end{aligned}$$

Thus $\{\omega^m \cdot k : 0 < k < \omega\} \subseteq \mathcal{D}^m(\varphi, \varepsilon, I_{m+1})$.

Recall that $\mathcal{D}^m(\varphi, \varepsilon, I_{m+1}) \subseteq \mathcal{D}^j(\varphi, \varepsilon, I_{m+1})$ for all $j < m$. Let $j \leq m$ and take any neighborhood U of ω^{m+1}. Then there exists an even $k < \omega$ such that $\omega^m \cdot k \in U \cap \mathcal{D}^j(\varphi, \varepsilon, I_{m+1})$ and $|\varphi(\omega^{m+1}) - \varphi(\omega^m \cdot k)| = |a - b| \geq \varepsilon$. Thus $\omega^{m+1} \in \mathcal{D}^{m+1}(\varphi, \varepsilon, I_{m+1})$, and therefore $\mathcal{D}^m(\varphi, \varepsilon, I_{m+1}) = \{\omega^m \cdot k : 0 < k < \omega\} \cup \{\omega^{m+1}\}$. Since $(\mathcal{D}^m(\varphi, \varepsilon, I_{m+1}))' = \{\omega^{m+1}\}$, then it follows that $\mathcal{D}^{m+1}(\varphi, \varepsilon, I_{m+1}) = \{\omega^{m+1}\}$. □

The ordinal interval $I_{n+k} = [1, \omega^{n+k}]$, $n, k \in \mathbb{N}$, can be written as a disjoint union

$$\bigcup_{0 \leq \alpha < \omega^k} [\omega^n \cdot \alpha + 1, \omega^n \cdot (\alpha + 1)] \cup \{\omega^n \cdot \xi : \xi \leq \omega^k, \xi \text{ is a limit ordinal}\}.$$

We use the function $\varphi_{a,b,m}$ to prove the following lemma.

Lemma 2.4. *Let $n \in \mathbb{N}$ be fixed and $a, b \in \mathbb{R}$ with $a \neq b$. If for any $k \in \mathbb{N}$ we define $g_k : I_{n+k} \to \{a,b\}$ by*

$$g_k(\tau) = \begin{cases} \varphi(\alpha + 1) & \textit{if } \tau = \omega^n \cdot \alpha + \xi, \quad \xi \in [1, \omega^n],\ 0 \leq \alpha < \omega^k \\ \varphi(\xi) & \textit{if } \tau = \omega^n \cdot \xi,\ \xi \leq \omega^k \quad \textit{is a limit ordinal,} \end{cases}$$

where $\varphi = \varphi_{a,b,k} : I_k \to \{a,b\}$, then $\mathcal{D}^k(g_k, \varepsilon, I_{n+k}) = \{\omega^{n+k}\}$ for any sufficiently small $\varepsilon > 0$.

Proof. Take any $0 < \varepsilon < |b - a|$. We prove the lemma by induction on k. First we prove for $k = 1$. Take any neighborhood U of ω^{n+1}, then there is an odd number $\ell < \omega$ such that $\omega^n \cdot \ell + 1 \in U$, therefore

$$|g_1(\omega^n \cdot \ell + 1) - g_1(\omega^{n+1})| = |\varphi(\ell + 1) - \varphi(\omega)| = |b - a| \geq \varepsilon.$$

Thus $\omega^{n+1} \in \mathcal{D}^1(g_1, \varepsilon, I_{n+1})$. Furthermore, for all $\tau < \omega^{n+1}$ can be written as $\tau = \omega^n \cdot \ell + \xi$, where $\ell < \omega$ and $1 \leq \xi \leq \omega^n$. Therefore $g_1(\tau) = \varphi(\ell + 1)$ and since $[1, \omega)' = \emptyset$, it follows that $\mathcal{D}^1(g_1, \varepsilon, I_{n+1}) = \{\omega^{n+1}\}$.

Now we assume that $\mathcal{D}^k(g_k,\varepsilon,I_{n+k})=\{\omega^{n+k}\}$ and we will prove that $\mathcal{D}^{k+1}(g_{k+1},\varepsilon, I_{n+k+1})=\{\omega^{n+k+1}\}$. For each $j<\omega$, let $L_j:=[\omega^{n+k}\cdot j+1,\omega^{n+k}\cdot(j+1)]$ and $g^j=g_{k+1}|_{L_j}$. Let $\theta: I_{n+k}\to L_j$ be defined by $\theta(\xi)=\omega^{n+k}\cdot j+\xi$, clearly that θ is a homeomorphism and $g_k=g^j\circ\theta$. Therefore, by Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned}\mathcal{D}^k(g_{k+1},\varepsilon,I_{n+k+1})\cap L_j &= \mathcal{D}^k(g^j,\varepsilon,L_j)=\theta(\mathcal{D}^k(g^j\circ\theta,\varepsilon,I_{n+k}))\\ &=\theta(\mathcal{D}^k(g_k,\varepsilon,I_{n+k}))=\theta(\{\omega^{n+k}\})=\{\omega^{n+k}\cdot(j+1)\}.\end{aligned}$$

Thus,$\{\omega^{n+k}\cdot j:0<j<\omega\}\subseteq\mathcal{D}^k(g_{k+1},\varepsilon,I_{n+k+1})\subseteq\mathcal{D}^\ell(g_{k+1},\varepsilon,I_{n+k+1})$, for all $\ell<k$. Since $g_{k+1}(\omega^{n+k+1})=\varphi(\omega^{k+1})=a$ and there exists an even $j<\omega$ which implies $g_{k+1}(\omega^{n+k}\cdot j)=\varphi(j)=b$, it follows that $\omega^{n+k+1}\in\mathcal{D}^{k+1}(g_{k+1},\varepsilon,I_{n+k+1})$. Since $\{\omega^{n+k}\cdot j:0<j<\omega\}'=\emptyset$, we obtain $\mathcal{D}^{k+1}(g_{k+1},\varepsilon,I_{n+k+1})=\{\omega^{n+k+1}\}$. The proof is completed. □

Theorem 2.5 below shows that Theorem 1.3 in [1] is optimal.

THEOREM 2.5. *For any $n,k\in\mathbb{N}$, there exist $f,g:I_{n+k}\to\mathbb{R}$ such that $i(f)=n$, $i(g)=k$, and $i(h)=n+k$, where h is any of the functions $f+g$,fg,$f\vee g$, $f\wedge g$.*

Proof. Let $a,b\in\mathbb{R}$ with $a\neq b$, $n\in\mathbb{N}$, and $\varphi=\varphi_{a,b,n}:I_n\to\{a,b\}$. Define $f:I_{n+k}\to\{a,b\}$ by

$$f(\tau)=\begin{cases}\varphi(\xi) & \text{if}\quad \tau=\omega^n\cdot\alpha+\xi,\quad \xi\in[1,\omega^n],\ 0\le\alpha<\omega^k\\ a & \text{if } \tau=\omega^n\cdot\xi,\ \ \xi\le\omega^k \quad \text{is a limit ordinal.}\end{cases}.$$

We are to prove that $i(f)=n$. Take any $0<\varepsilon<|b-a|$. For any $\alpha<\omega^k$, let $L_\alpha=[\omega^n\cdot\alpha+1,\omega^n\cdot(\alpha+1)]$ and $f_\alpha=f|_{L_\alpha}$. Let $\theta:I_n\to L_\alpha$ be defined by $\theta(\xi)=\omega^n\cdot\alpha+\xi$. Then it is clear that θ is a homeomorphism from I_n to L_α. Also, by the definition of f, clearly $f_\alpha\circ\theta=\varphi$.

Since $\theta:I_n\to L_\alpha$ is a homeomorphism, then by Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned}\mathcal{D}^n(f,\varepsilon,I_{n+k})\cap L_\alpha &= \mathcal{D}^n(f_\alpha,\varepsilon,L_\alpha)=\theta(\mathcal{D}^n(f_\alpha\circ\theta,\varepsilon,I_n))\\ &=\theta(\mathcal{D}^n(\varphi,\varepsilon,I_n))=\theta(\{\omega^n\})=\{\omega^n\cdot(\alpha+1)\}.\end{aligned}$$

and $f_\alpha(\omega^n\cdot(\alpha+1))=(\varphi\circ\theta^{-1})(\omega^n\cdot(\alpha+1))=\varphi(\omega^n)=a$. It follows that

$$\mathcal{D}^n(f,\varepsilon,I_{n+k})\cap\left(\bigcup_{0\le\alpha<\omega^k}L_\alpha\right)=\bigcup_{0\le\alpha<\omega^k}(\mathcal{D}^n(f_\alpha,\varepsilon,L_\alpha))=\bigcup_{0\le\alpha<\omega^k}\{\omega^n\cdot(\alpha+1)\}.$$

Therefore

$$\begin{aligned}\mathcal{D}^n(f,\varepsilon,I_{n+k}) &\subseteq\left(\bigcup_{0\le\alpha<\omega^k}\{\omega^n\cdot(\alpha+1)\}\right)\cup\{\omega^n\cdot\xi:\xi\in I_k,\xi\text{ is a limit ordinal}\}\\ &=\{\omega^n\cdot\alpha:\alpha\in I_k\}.\end{aligned}$$

Since $f(\omega^n\cdot\alpha)=a$ for all $\alpha\in I_k$, then $\mathcal{D}^{n+1}(f,\varepsilon,I_{n+k})=\emptyset$. This implies that $\beta(f)=\sup_{\varepsilon>0}\beta(f,\varepsilon)=n+1$, and therefore $i(f)=n$.

Now, let $c,d\in\mathbb{R}$ with $c\neq d$ and denote $\psi=\varphi_{c,d,k}:I_k\to\{c,d\}$. Define

$g : I_{n+k} \to \{c, d\}$ by

$$g(\tau) = \begin{cases} \psi(\alpha+1) & \text{if } \quad \tau = \omega^n \cdot \alpha + \xi, \quad \xi \in [1, \omega^n], \ 0 \le \alpha < \omega^k \\ \psi(\xi) & \text{if } \tau = \omega^n \cdot \xi, \ \xi \le \omega^k \quad \text{is a limit ordinal.} \end{cases}$$

Then, by Lemma 2.4, $\mathcal{D}^k(g, \varepsilon, I_{n+k}) = \{\omega^{n+k}\}$ which implies that $i(g) = \beta(g) - 1 = \sup_{\varepsilon>0} \beta(g, \varepsilon) - 1 = k$.

Let $h = f + g$ and choose the numbers a, b, c, d such that $a + c \ne b + d$. Take any sufficiently small $\varepsilon > 0$. For each $0 \le \alpha < \omega^k$, let $L_\alpha = [\omega^n \cdot \alpha + 1, \omega^n \cdot (\alpha + 1)]$ and $h_\alpha = h|_{L_\alpha}$. For each $\alpha < \omega^k$ and $\tau = \omega^n \cdot \alpha + \xi \in L_\alpha$ we have $h_\alpha(\tau) = \varphi(\xi) + \psi(\alpha+1)$. Therefore, $h_\alpha \circ \theta = \varphi + \psi(\alpha + 1)$. Since $\theta : I_n \to L_\alpha$ is a homeomorphism, then by Lemma 2.1 and Lemma 2.2, for each $\alpha < \omega^k$ we have

$$\begin{aligned} \mathcal{D}^n(h, \varepsilon, I_{n+k}) \cap L_\alpha &= \mathcal{D}^n(h_\alpha, \varepsilon, L_\alpha) = \theta(\mathcal{D}^n(h_\alpha \circ \theta, \varepsilon, I_n)) \\ &= \theta\left(\mathcal{D}^n(\varphi + \psi(\alpha+1), \varepsilon, I_n)\right) = \theta(\{\omega^n\}) = \{\omega^n \cdot (\alpha+1)\} \end{aligned}$$

and $h(\omega^n \cdot (\alpha + 1)) = (h_\alpha \circ \theta)(\theta^{-1}(\omega^n \cdot (\alpha + 1)) = a + \psi(\alpha + 1)$.

Take any limit ordinal $\xi \le \omega^k$. Then $h(\omega^n \cdot \xi) = a + \psi(\xi)$. For any neighborhood U of $\omega^n \cdot \xi$, there exists $\alpha < \xi$ such that $\omega^n \cdot \alpha \in U \cap \mathcal{D}^\ell(h, \varepsilon, I_{n+k})$ for all $\ell < n$ and $h(\omega^n \cdot \alpha) \ne a + \psi(\xi)$. It follows that $\omega^n \cdot \xi \in \mathcal{D}^n(h, \varepsilon, I_{n+k})$. Thus we obtain that $\mathcal{D}^n(h, \varepsilon, I_{n+k}) = \{\omega^n \cdot \alpha : 1 \le \alpha \le \omega^k\}$ and $h(\omega^n \cdot \alpha) = a + \psi(\alpha)$ for each $1 \le \alpha \le \omega^k$. Let $q : [1, \omega^k] \to \{\omega^n \cdot \alpha : 1 \le \alpha \le \omega^k\}$ be defined by $q(\alpha) = \omega^n \cdot \alpha$. Then q is bijective and continuous (see, e.g., [7]). Since $[1, \omega^k]$ is compact and $\{\omega^n \cdot \alpha : 1 \le \alpha \le \omega^k\}$ is Haussdorf, then q is a homeomorphism (see, e.g., [2]). It can be observed that $h \circ q = \varphi_{a+c,a+d,k}$. Therefore, by Lemma 2.1,

$$\mathcal{D}^{n+k}(h, \varepsilon, I_{n+k}) = \mathcal{D}^k(h, \varepsilon, \mathcal{D}^n(h, \varepsilon, I_{n+k})) = q\left(\mathcal{D}^k(h \circ q, \varepsilon, I_k)\right) = q(\{\omega^k\}) = \{\omega^{n+k}\}.$$

It implies that $\mathcal{D}^{n+k+1}(h, \varepsilon, I_{n+k}) = \emptyset$, and therefore $i(h) = \beta(h) - 1 = n + k$.

Similarly, we can prove for $h = fg$, $h = f \wedge g$, and $h = f \vee g$ by choosing the appropriate numbers a, b, c, and d. We may choose a, b, c, d such that $ac \ne bd$, $a < b$ and $c < d$, and $a > b$ and $c > d$ for $h = fg$, $h = f \wedge g$, and $h = f \vee g$, respectively. □

Furthermore, the result in Theorem 2.5 may be extended to any compact metric space K such that $K^{(n+k)} \ne \emptyset$. For this, we use the following lemma.

Lemma 2.6 ([5], Lemma 6.8.). *Let K be a compact metric space. If $K^{(\alpha)} \ne \emptyset$ for some $0 < \alpha < \omega_1$, then there is a subspace $L \subseteq K$ such that L is homeomorphic to $[0, \omega^\alpha]$.*

Theorem 2.7. *Let K be any compact metric space such that $K^{(n+k)} \ne \emptyset$. Then there exist $f, g : K \to \mathbb{R}$ such that $i(h) = i(f) + i(g)$, where h is any of the functions $f + g$, fg, $f \wedge g$, $f \vee g$.*

Proof. By Lemma 2.6, there exists $L \subseteq K$ such that L is homeomorphic to I_{n+k}, suppose that $\theta : L \to I_{n+k}$ is the homeomorphism. By Theorem 2.5, there exist $\tilde{f}, \tilde{g} : I_{n+k} \to \mathbb{R}$ such that $i(\tilde{h}) = i(\tilde{f}) + i(\tilde{g})$, where $\tilde{h}$ is any of the functions $\tilde{f} + \tilde{g}$, $\tilde{f}\tilde{g}$, $\tilde{f} \wedge \tilde{g}$, $\tilde{f} \vee \tilde{g}$.

Define $f, g : L \to \mathbb{R}$ by $f = \tilde{f} \circ \theta$ and $g = \tilde{g} \circ \theta$. Let ψ be any of the functions $\tilde{f}$, $\tilde{g}$, $\tilde{h}$. Then by Lemma 2.1, we have $\mathcal{D}^j(\psi \circ \theta, \varepsilon, L) = \theta^{-1}(\mathcal{D}^j(\psi, \varepsilon, I_{n+k}))$, $j \le n + k$. It follows that $i(h) = i(f) + i(g)$ on L, where h is any of the functions $f + g$, fg, $f \wedge g$, $f \vee g$. Furthermore, by Theorem 3.6 of [5], f, g, and h can be extended onto K with preservation of the finite index i. □

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