МАТЕМАТИЧКИ ВЕСНИК
60 (2008), 59–78

UDK 515.122
оригинални научни рад
research paper

HIT-AND-FAR-MISS TOPOLOGIES

Giuseppe Di Maio and Somashekhar Naimpally

Dedicated to the memory of prof. Robert A. Wijsman

Abstract. We offer a unified approach to investigate hypertopologies. In this setting the proofs are simple and transparent. New problems are raised.

1. Introduction

About ninety years back Hausdorff and Vietoris introduced hyperspace topologies. Over the years a lot of research was done on this topic especially after the seminal paper of Michael [Mi]. Then twenty five years ago the subject exploded with many new hyperspace topologies and applications to Convex Analysis, Optimization, Economics, Image Processing, Sound Analysis and Synthesis, etc.

Recently, it was shown that with the use of proximity all hypertopologies known so far are of the type hit-and-miss [Na2]. This approach led to unification of all hypertopologies under one topology called the Bombay Hypertopology, which uses two proximities [DMN1]. In this setting many known results, concerning comparisons of hypertopologies, are included just in one theorem and the conceptual proof of this result is short and simple. These new developments have led to many new problems and we plan to include them also in this document. Obviously, the first important and general problem is the following:

(GP1) Is there a hypertopology which is not of the hit-and-miss type?

In order to keep the subject simple, we explain the main hypertopologies in the context of a metric space. And later we explain how the ideas and results are generalized in the context of uniform and proximity spaces. So the paper is organized in four parts (each divided into sections). The first part is dedicated to preliminaries. The second introduces six fundamental hypertopologies (*the bricks*) and investigates the so called hit-and-far-miss (or HAFM)-topology $\tau_w(\Delta, \delta)$, a generalizations of the Wijsman topology, rediscovering known results. In the third new hypertopologies are constructed via the fundamental bricks in metric and uniform settings. Finally, in the fourth open questions and problems are raised.

AMS Subject Classification: Hypertopology, Bombay hypertopology, hit-and-miss, hit-and-far-miss topology, locally finite, discrete, uniformly discrete family, Hausdorff metric, Wijsman topology.

Keywords and phrases: Primary: 54B20; Secondary: 54E05, 54E15.

Part 1

1. Preliminaries

Let X be a T_1 topological space and δ a compatible proximity on X [NW]. We write $\underline{\delta}$ for the negation of δ and $A \ll_\delta E$ (or simply $\ll$ if the proximity is obvious) for $A\, \underline{\delta}\, E^c$.

(X, d) denotes a metric space with the topology $\mathrm{T} = \mathrm{T}(d)$. Let $\mathrm{S}(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ and $\mathrm{B}(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$ be the open and the closed balls with the centre x and radius $\epsilon > 0$.

For any $A \subset X$, $\mathrm{S}_\epsilon(A) = \bigcup\{\mathrm{S}(x, \epsilon), x \in A\}$ is the ϵ-parallel body of A.

Let δ_m or simply δ denote the metric proximity introduced by d:

$$A\delta B \Leftrightarrow D(A, B) = \inf\{\, d(a, b) : a \in A, b \in B\,\} = 0. \tag{1.1}$$

We use also the Lodato proximity δ_0, i.e.

$$A\delta_0 B \Leftrightarrow \mathrm{CL}\, A \cap \mathrm{CL}\, B \neq \emptyset. \tag{1.2}$$

Let $\mathrm{CL}(X)$ (respectively $\mathrm{K}(X)$) denote the family of all non-empty closed (respectively compact) subsets of X. Δ denotes a non-empty subfamily of $\mathrm{CL}(X)$. Without any loss of generality and to simplify the exposition, we assume that Δ is closed under finite unions and contains all singletons. We call Δ a *cobase*. An important example of a cobase is the family $\mathbf{B}$ of the finite unions of all closed balls in (X, d) with non-negative radii. The concept of a cobase, without any name, was first used by Poppe [Po].

For an open set $E \subset X$, we use standard notation:

$E^- = \{A \in \mathrm{CL}(X) : A \cap E \neq \emptyset\}$ = closed sets that *hit* the open set E.

$\mathrm{E}^- = \{A \in \mathrm{CL}(X) : A \cap E \neq \emptyset$ for each $E \in \mathrm{E}\}$, $\mathrm{E} \subset \mathrm{T}$.

$E^+ = \{A \in \mathrm{CL}(X) : A \subset E\}$ = closed sets that *miss* the closed set E^c.

$E^{++} = \{A \in \mathrm{CL}(X) : A \ll_\delta E\}$ = closed sets that are *far from* the closed set E^c.

We list lower hypertopologies, namely:

(i) The *Lower Vietoris Topology* $\tau(\mathrm{V}^-)$ has a basis $\{\mathrm{E}^- : \mathrm{E} \subset \mathrm{T}$ is finite$\}$;

(ii) The *Lower Locally Finite Topology* $\tau(\mathrm{LF}^-)$ has a basis $\{\mathrm{E}^- : \mathrm{E} \subset \mathrm{T}, \mathrm{E}$ a locally finite$\}$.

We recall that for $A \in \mathrm{CL}(X)$, $\epsilon > 0$, the set

$$\{S_\epsilon(x) : x \in Q \subset A, \text{ where } Q \text{ is } \epsilon\text{-discrete}\}$$

is the *discrete (mobile) ϵ-grid of A centered at Q* and is denoted by $\mathrm{S}_\epsilon(Q, A)$. Furthermore, if Q^* is a *maximal* ϵ-discrete subset of A, then $A \subset \mathrm{S}_\epsilon(Q^*)$ and $\mathrm{S}_\epsilon(Q^*, A)$ is also called a *maximal ϵ-grid* of A. Obviously, Q^* is finite for every A if and only if $\mathcal{U}$ is totally bounded (TB).

Let UD be the collection of families of all *ϵ-grids*. The family UD allows us to define a lower discrete "hit" topology.

(iii) The Lower *Uniformly Discrete* (UD) *Topology* $\tau(\mathrm{UD}^-)$ on $\mathrm{CL}(X)$ consists of $\{\mathrm{E}^- : \mathrm{E} \in \mathrm{UD}\}$. We note that each member of UD is locally finite.

We list upper hypertopology, namely:

(iv) *The Upper Proximal* Δ*-Topology* (w.r.t. δ) σ $(\delta\Delta^+)$ is generated by the basis $\{E^{++} : E^c \in \Delta\}$. We write $\sigma(\Delta^+)$ when the proximity δ is obvious.

(v) The *Upper* Δ*-Topology* $\tau(\Delta^+) = \sigma(\delta_0\Delta^+)$ is generated by the basis $\{E^+ : E^c \in \Delta\}$.

(vi) The *Upper Wijsman* Δ*-Topology* $\tau(W\Delta^+)$ is generated by neighborhoods of $A \in \mathrm{CL}(X)$ given by sets of the form $\{\mathrm{E}^+ : A \in \mathrm{E}^{++}, \mathrm{E}^c \in \Delta\}$ [Na2]. In other words, a typical neighborhood of A consists of closed sets which *miss* a member of Δ that is *far* from A, i.e. it is a *far-miss* topology.

We call the above six (families of) hypertopologies, the three lower ones and the three upper (also labelled by Δ), *the fundamental (or basic) bricks* of the lattice of hypertopologies on X, since most known hypertopologies can be expressed as appropriate joins (or in their terms).

The *Proximal-*Δ*-Topology* (w.r.t. δ) $\sigma(\delta\Delta) = \sigma(\delta\Delta^+) \vee \tau(\mathrm{V}^-)$.
The Δ-Topology $\tau(\Delta) = \tau(\Delta^+) \vee \tau(\mathrm{V}^-)$.

The *Proximal Locally Finite* Δ*-Topology* (w.r.t. δ)

$$\sigma(\mathrm{LF}\delta\Delta) = \sigma(\delta\Delta^+) \vee \tau(\mathrm{LF}^-)$$

The *Proximal-Uniformly-Discrete-*Δ*-Topology* (w.r..t. δ)

$$\sigma(\mathrm{UD}\delta\Delta) = \sigma(\delta\Delta^+) \vee \tau(\mathrm{UD}^-).$$

We omit δ if it is obvious from the context and write, for example, $\sigma(\Delta)$ for $\sigma(\delta\Delta)$.

Well known special cases are:

(a) $\Delta = \mathrm{CL}(X)$

Vietoris or Finite Topology $\tau(\mathrm{V}) = \tau(\mathrm{CL}(X))$ [Mi];
Proximal Topology $\sigma(\delta) = \sigma(\mathrm{CL}(X))$ [DCNS];
Locally Finite Topology $\tau(\mathrm{LF}) = \tau(\mathrm{LF}\ \mathrm{CL}(X))$, [Ma], [Wa], [BHPV], [NS];
Proximal Locally Finite Topology $\sigma(\mathrm{LF}\sigma) = \sigma(\mathrm{LF}\ \delta\ \mathrm{CL})$, [DCNS].

We pause to give some notations and comments. We recall that given a metric space (X, d), the ϵ-parallel body $\mathrm{S}_\epsilon(B)$ of $B \in \mathrm{CL}(X)$, is the set of all points X whose distance from B is less than ϵ, or equivalently the union of all open spheres of radius ϵ centered at points of B.

For $A, B \in \mathrm{CL}(X)$ the *lower Hausdorff distance.* $d_l(A, B)$ between A and B is $d_l(A, B) = \inf\{\lambda > 0 : A \subset S_\lambda(B)\}$ or ∞ if the infimum does not exist. Similarly, the *upper Hausdorff distance* $d_u(A, B)$ between A and B is $d_u(A, B) = \inf\{\lambda > 0 : B \subset S_\lambda(A)\}$ or ∞ if the infimum does not exist. It is clear that $d_u(A, B) = d_l(B, A)$.

The *Hausdorff distance* $d_H(A, B)$ between $A, B \in \mathrm{CL}(X)$ is given by

$$d_H(A, B) = \max\{\epsilon_1 = d_l(A, B), \epsilon_2 = d_u(A, B)\}. \qquad (*)$$

Recently, one of the authors proved this unexpected result:

THEOREM 1.1. [Hausdorff Metric Topology [Na2]] *The Hausdorff metric topology equals the Uniformly discrete hit-and-far topology*

$$\tau(d_H) = \tau(\mathrm{UD}^-) \vee (\sigma(\delta\,\mathrm{CL}(X)^+)).$$

With the above representation of the Hausdorff metric topology $\tau(d_H)$ as *hit-and-far* hypertopology, it is easy to compare it with the Vietoris topology $\tau(\mathrm{V})$. In fact, $\tau(d_H)$ is coarser than $\tau(\mathrm{V})$ if and only if X is totally bounded. $\tau(\mathrm{V})$ is coarser than $\tau(d_{\mathrm{H}})$ if and only if X is UC. (A metric space X is UC if and only if every real valued continuous function is uniformly continuous, or equivalently if and only if disjoint closed subsets are a positive distance apart, see [DMN3]). Consequently, they are equal if and only if X is UC (which implies it is complete) and totally bounded, i.e. X is compact.

We recall that the Hausdorff metric topology $\tau(d_H)$ is sensitive to the change of compatible metrics on X. Two metrics induce the same Hausdorff topology if and only they are uniformly equivalent.

(b) $\Delta = \mathrm{K}(X)$

Fell Topology $\tau(\mathrm{F}) = \sigma(\mathrm{F}) = \tau(\mathrm{K}(X))$ [Fe],

(c) $\Delta = \mathbf{B}$

Ball Topology $\tau(\mathbf{B}) = \tau(\mathbf{B}^+) \vee \tau(\mathrm{V}^-)$ [BE]

Proximal Ball Topology $\sigma(\mathbf{B}) = \sigma(\delta\mathbf{B}) = \sigma(\delta\mathbf{B}^+) \vee \tau(\mathrm{V}^-))$ [DN1].

We remind that the formula $(*)$, for the Hausdorff metric, can also be expressed as

$$d_H(A, B) = \sup\{|d(x, A) - d(x, B)| : x \in X\}. \qquad (**)$$

From the above expression, it is clear that a sequence $\{A_k\}$ of closed sets converges to a closed set A in the Hausdorff metric topology if and only if the sequence $\{d(x, A_k)\}$ converges *uniformly* on X to $d(x, A)$. For tackling problems in Convex Analysis, Wijsman replaced uniform convergence by *pointwise* convergence [Wi].

But, the Wijsmann convergence $\tau(\mathrm{W}_d)$ or simply $\tau(\mathrm{W})$ is also an hit-and-far-miss (briefly) HAFM topology as shown in [Na2] (see (vi)):

Wijsman Topology $\tau(\mathrm{W}_d) = \tau(\mathrm{W}\mathbf{B}^+) \vee \tau(\mathrm{V}^-)$, [Wi].

We recall that also the Wijsmann $\tau(\mathrm{W}_d)$ topology is sensitive to the change of compatible metrics on X and the condition on the compatible metrics to have the same Wijsman topologies is rather involved: a transparent and simple proof is offered in Section 2 (compare with the one in [CLZ].

We point out that the ball and proximal ball topologies were discovered in attempts to express the Wijsman topology as a kind of hit-and-miss topology. Actually, we stress again that a typical neighborhood of a closed set A in $\tau(\mathrm{W}\mathbf{B}^+)$ is E^+ where $E^c \in \mathbf{B}$ and $A \in E^{++}$. Thus, $\tau(\mathrm{W}\mathbf{B}^+)$ is a far-miss topology and it is indeed a mixed kind of topology compared with, the upper ball topology $\tau(\mathbf{B})^+$, which deals with miss $\{E^+ : E^c \in \mathbf{B}\}$, and the upper proximal ball topology $\sigma(\mathbf{B}, \delta)^+$, which deals with far sets $\{E^{++} : E^c \in \mathbf{B}\}$. We will find this formulation much simpler to use than the original definition. It also yields a generalization to T_1 spaces. Results obtained by prominent researchers need long proofs using epsilonetics (cf. [Be]), and the same results follow in a few lines with the use of proximity.

It easily follows from the fact $E^{++} \subset E^{+}$, that the Wijsman topology is coarser than both the ball and the proximal ball topologies. It is also obvious that a sufficient (but not necessary) condition for all the three topologies to coincide is $\delta = \delta_0$, a proximity in which disjoint closed sets are far. If this happens in the case a metric space (X, d), then it is a UC-space, i.e. each continuous function on X to $\mathbf{R}$ is uniformly continuous.

(d) $\Delta = \mathrm{TB}(X)$ (the family of all closed totally bounded subsets of X) or $\Delta = \mathrm{B}(X)$ (the family of all closed bounded subsets of X)

Totally Bounded Topology $\tau(\mathrm{TB}(X)) = \tau(\mathrm{TB}(X)^{+}) \vee \tau(\mathrm{V}^{-})$ [DH3];

Proximal Totally Bounded Topology $\sigma(\mathrm{TB}(X)) = \sigma(\mathrm{TB}(X)^{+}) \vee \tau(\mathrm{V}^{-})$, [DH3];

Bounded Topology $\tau(\mathrm{B}(X)) = \tau(\mathrm{B}(X)^{+}) \vee \tau(\mathrm{V}^{-})$, [BL1], [LP];

Proximal Bounded Topology $\sigma(\mathrm{B}(X)) = \sigma(\mathrm{B}(X)^{+}) \vee \tau(\mathrm{V}^{-})$, [BL1], [BL2].

During the last decades Wijsman topologies have been intensively investigated. They are the *building blocks* in the lattice of all hypertopologies on a given metrizable space (cf. [BLLN]). In Part II we propose to study a generalized form of the Wijsman topology by working with a *compatible proximity* δ and a *cobase* $\Delta \subset \mathrm{CL}(X)$. The aim is also to discover simple and conceptual proofs of classical results.

Part II

2. The HAFM topology

First, we give the following:

Definition 2.1. The *hit-and-far-miss* (or *HAFM*)-*topology* $\tau_w(\Delta, \delta)$ on $\mathrm{CL}(X)$ is generated by the join of the lower Vietoris and the upper Wijsman-Δ-Topology $\tau(W\Delta^{+})$, see (vi) in Preliminaries.

Here we define a useful concept for later use [DN1]:

Definition 2.2. Let (X, d) be a metric space and δ the metric proximity.

(a) If A and B are subset of X, then we say that A is Δ-*totally bounded w.r.t* δ *in* B if and only if there is a $D \in \Delta$, with $A \subset D \ll_\delta B$.

(b) If A and B are subset of X, then we say that A is *weakly* Δ-*totally bounded in* B if and only if there is a $D \in \Delta$, with $A \subset D \subset B$.

Remark 2.3. (a) If (X, d) is a metric space, δ the metric proximity and $\Delta = \mathbf{B}$ the family of finite unions of all proper closed balls in (X, d) with non-negative radii, the definition 2.2(a) (respectively (b)) is equivalent to the definition of A is strictly d-included (respectively d-included) in B [Be], p. 38.

(b) We recall A is *strictly d-included* in B if and only if there is a $D \in \mathbf{B}$ such that $A \subset D \ll_\delta B$; this equivalent to $A \ll_\delta D' \ll_\delta B$ for some $D' \in \mathbf{B}$. A is *d-included* in B if and only if there is a $D \in \mathbf{B}$ such that $A \subset D \subset B$.

(c) We note that the use of *finite unions of balls* simplifies the original definition as well as statements of theorems and their proofs, as will be shown below.

3. Comparisons among the HAFM topology and (proximal) Δ-Topologies

In this section we compare the HAFM-topology and Δ-(proximal) hypertopologies, when the involved proximities and cobases are different. The proofs are, for the most part, easy and follow from the definitions. We have already observed that it follows directly from the definitions that

$$\tau_w(\Delta,\delta)^+ \subset \tau(\Delta)^+ \wedge \sigma(\Delta,\delta)^+. \qquad (+)$$

THEOREM 3.1. *Let δ, δ' be two compatible proximities on X, and let Δ, Δ' be two cobases in X. Then*

(a) $\tau_w(\Delta,\delta)^+ \subset \tau_w(\Delta',\delta')^+$ if and only if for each $D\in\Delta$, $D\ll_\delta U$, where U is open, implies D is Δ'-totally bounded w.r.t. δ' in U.

(b) $\tau(\Delta)^+ \subset \tau(\Delta')^+$ if and only if for each $D\in\Delta$, $D\subset U$, where U is open, implies D is weakly Δ'-totally bounded in U.

(c) $\sigma(\Delta,\delta)^+ \subset \sigma(\Delta',\delta')^+$ if and only if for each $D\in\Delta$, $D\ll_\delta U$, where U is open, implies D is Δ'-totally bounded w.r.t δ' in U.

Proof. (a) $\tau_w(\Delta,\delta)^+ \subset \tau_(\Delta',\delta') \Leftrightarrow D\in\Delta$, $A\in \mathrm{CL}(X)$, $A\in(D^c)^{++}$ w.r.t. δ, i.e. $D\ll_\delta A^c=U \Rightarrow$ there is a $P\in\Delta'$ such that $A\in(P^c)^{++}$ w.r.t. δ' and $(P^c)^+\subset(D^c)^+ \Leftrightarrow D\subset P\ll_{\delta'} U$.

(b) $\tau(\Delta)^+\subset\tau(\Delta')^+ \Leftrightarrow D\in\Delta, A\in\mathrm{CL}(X), A\in(D^c)^+$, i.e. $D\subset A^c=U\Rightarrow$ there is a $P\in\Delta'$ such that $A\in(P^c)^+$ and $(P^c)\subset(D^c)^+\Leftrightarrow D\subset P\subset U$.

(c) $\sigma(\Delta,\delta)^+ \subset \sigma(\Delta',\delta')^+ \Leftrightarrow D\in\Delta$, $A\in\mathrm{CL}(X)$, $A\in(D^c)^{++}$ w.r.t δ, i.e. $D\ll_\delta A^c=U \implies$ there is a $P\in\Delta'$ such that $A\in(P^c)^{++}$ w.r.t δ' and $(P^c)^{++}\subset(D^c)^{++}\Leftrightarrow D\subset P\ll{}'_\delta U$. ■

COROLLARY 3.2. ([Be] p. 39) *Let d,ρ be two equivalent metrics on X, and let $\mathbf{B}$, $\mathbf{B}'$ be the respective families of finite unions of proper closed balls. Then using* (2.3)*(b) and* (3.1)*, we have the following known results.*

(a) $\tau_w(\mathbf{B},\delta)\subset\tau_w(\mathbf{B}',\delta')$ if and only if each closed d-ball is ρ-strictly included in each of its open $\epsilon-d$-enlargements.

(b) $\tau(\mathbf{B})\subset\tau(\mathbf{B}')$ if and only id each closed d-ball is ρ-included in each of its open $\epsilon-d$-enlargements.

(c) $\sigma(\mathbf{B},\delta)\subset\sigma(\mathbf{B}',\delta')$ if and only if each closed d-ball is ρ-strictly included in each of its open $\epsilon-d$-enlargements.

THEOREM 3.3. *Let δ be a compatible proximity on X, and let Δ be a cobase in X. Then $\sigma(\Delta)^+=\tau_w(\Delta,\delta)^+$ if and only if for each $D\in\Delta$, $D\ll_\delta U$, where U is open, implies D is Δ-totally bounded w.r.t. δ in U.*

Proof. In view of (+) we need to consider only the case $\sigma(\Delta)^+\subset\tau_w(\Delta,\delta)^+$. $\sigma(\Delta)^+\subset\tau_w(\Delta,\delta)^+\Leftrightarrow D\in\Delta, A\in\mathrm{CL}(X)$, $A\in(D^c)^{++}$, i.e. $D\ll_\delta A^c=U\Rightarrow$ there is a $P\in\Delta$ such that $A\in(P^c)^{++}$ and $(P^c)^+\subset(D^c)^{++}\Leftrightarrow D\subset P\ll_\delta U$. ■

COROLLARY 3.4. ([Be], p. 45) *Let* $\mathbf{B}$ *the family of finite unions of proper closed ball of a metric space* (X, d). *Then* $\sigma(\mathbf{B}) \subset \tau_w(\mathbf{B}, \delta)$ *if and only if each closed ball in* X *is* d*-strictly included in each of its open* ϵ*-enlargements.*

THEOREM 3.5. *Let* δ *be a compatible proximity on* X, *and let* Δ *be a cobase in* X. *Then the following are equivalent:*

(a) $\tau(\Delta)^+ = \tau_w(\Delta, \delta)^+$.

(b) $\tau(\Delta)^+ = \sigma(\Delta)^+ = \tau_w(\Delta, \delta)^+$.

(c) For each $D \in \Delta$, $D \subset U$, *where* U *is open, implies* D *is* Δ*-totally bounded w.r.t.* δ *in* U.

Proof. (a)⇔(c) $\tau(\Delta)^+ \subset \tau_w(\Delta, \delta)^+ \Leftrightarrow D \in \Delta$, $A \in \mathrm{CL}(X)$, $A \in (D^c)^+$, i.e. $D \subset A^c = U \Rightarrow$ there is a $P \in \Delta$ such that $A \in (P^c)^{++}$ and $(P^c)^+ \subset (D^c)^+ \Leftrightarrow D \subset P \ll_\delta U$.

(a)⇒(b) follows from (2.3) and (a).

(b)⇒(a) is trivial. ■

COROLLARY 3.6. ([Be], p. 53) *Let* $\mathbf{B}$ *be the family of finite unions of proper closed balls of a metric space* (X, d). *Then the following are equivalent:*

(a) $\tau(\mathbf{B})^+ = \tau_w(\mathbf{B}, \delta)^+$.

(b) $\tau(\mathbf{B})^+ = \sigma(\mathbf{B})^+ = \tau_w(\mathbf{B}, \delta)^+$.

(c) For each $D \in \mathbf{B}$, $D \subset U$, *where* U *is open, implies* D *is strictly* d*-included in* U.

THEOREM 3.7. *Let* δ *be a compatible proximity on* X, *and let* Δ *be a cobase in* X. *Then the following are equivalent:*

(a) $\tau(\Delta) \subset \sigma(\Delta, \delta)$;

(b) $\tau(\Delta) = \sigma(\Delta, \delta)$;

(c) If $A \in \Delta$, $B \in \mathrm{CL}(X)$, *and* $A \cap B = \emptyset \Rightarrow A \underline{\delta} B$.

THEOREM 3.8. *Let* δ *be a compatible proximity on* X, *and let* Δ *be a cobase in* X *which is* δ*-Urysohn* [DMN1], p. 31, *i.e.* $A \in \mathrm{CL}(X)$, $B \in \Delta$ *with* $A \underline{\delta} B$ *implies there exists* $E \in \Delta$ *such that* $A \underline{\delta} E$ *and* $E^c \underline{\delta} B$. *Then* $\sigma(\Delta, \delta) \subset \tau(\Delta)$.

Proof. Suppose $A \in \mathrm{CL}(X)$, $B \in \Delta$ and $A \in (B^c)^{++} \in \sigma(\Delta)^+$. Then $A \underline{\delta} B$ and so there exists $E \in \Delta$ such that $A \underline{\delta} E$ and $E^c \underline{\delta} B$. Then $A \in (E^c)^+ \subset (B^c)^{++}$. ■

4. Comparisons among classical topologies

In this section we compare the Wijsman, Ball and Proximal Ball Topologies with the Fell, the Proximal, the Vietoris, the Hausdorff metric topology (see [Be], [DHo]). We have the following diagram

$$\tau(\mathrm{F}) \subset \tau(\mathrm{W}_d) \subset \sigma(\delta_m) \subset \begin{matrix} \tau(\mathrm{V}) \\ \tau(d_H) \end{matrix} \qquad (++)$$

First, we offer a very simple proof (a few lines instead of pages) of a celebrated theorem showed in [BLLN] (see also [DMN1]).

THEOREM 4.1. *Let (X, d) be a metric space and δ the metric proximity. The following are equivalent:*

(i) $\sigma(\delta) = \tau(\mathrm{W}_d)$;

(ii) (X, d) *is totally bounded;*

(iii) $\tau(d_H) = \tau(\mathrm{W}_d)$.

Proof. (i)⇒(ii) Suppose $\sigma(\delta) = \tau(\mathrm{W}_d)$, i.e $\sigma(\delta)^+ \subset \tau(\mathrm{W}_d)^+$. We show that X is totally bounded. If not, there is a sequence of distinct points $\{a_n\}$ which has no Cauchy subsequence. So, there is an $\epsilon > 0$ such that for each pair a_n, a_m $d(a_n, a_m) > 2\epsilon$. Let A be the range of the sequence $\{a_n\}$. Set $E = [\mathrm{S}(A, \epsilon)]^c$. Then $E \in (A^c)^{++} \in \sigma(\delta)^+$. Thus there is a $B \in \mathbf{B}$ such that $E \in (B^c)^{++} \subset (B^c)^+ \in \tau(W_d)$ and $(B^c)^+ \subset (A^c)^{++}$. So $E \ll B^c \ll A^c$, i.e. $A \ll B \ll E^c$. Since B is a finite union of closed balls of radii less than ϵ, there is one ball containing infinitely many points of A, a contradiction.

(ii)⇒(iii) It suffices to show that (X, d) is totally bounded implies $\sigma(\delta)^+ \subset \tau(\mathrm{W}_d)^+$. Suppose $A \in (E^c)^{++} \in \sigma(\delta)^+$ and E is totally bounded. Then $d(A, E) = 2\epsilon > 0$. Since E is totally bounded there is a finite set $\{x_k : k \le n \in \mathbf{N}\} \subset E$ such that $E \subset \bigcup\{\mathrm{S}(x_k, \epsilon/3) : k \le n \in \mathbf{N}\} \ll \bigcup\{\mathrm{S}(x_k, \epsilon/2) : k \le n \in \mathbf{N}\} \ll A^c$. Then

$$E \ll \bigcup\{\mathrm{S}(x_k, \epsilon) : k \le n \in \mathbf{N}\} \ll A^c.$$

If we set $B = \bigcup\{\mathrm{S}(x_k, \epsilon) : k \le n \in \mathbf{N}\} \in \mathbf{B}$, then $A \ll B \ll E^c$. So $\sigma(\delta)^+ \subset \tau(\mathrm{W}_d)^+$ and they are equal. Since (X, d) is totally bounded implies $\tau(d_H) = \sigma(\delta)$ we are done.

(iii)⇒(i) is an easy consequence of (++). ■

Now, we consider in more details the Fell Topology [Fe]. In proximities that are R or EF (see [NW], [En]), a compact set is far from each disjoint closed set and hence, in these cases, $\tau(\mathrm{F}) = \sigma(\mathrm{F})$ a proximal Δ-topology! We assume that in all our work $\mathrm{K}(X) \subset \Delta$. We have the following:

THEOREM 4.2. *Let δ be an R or EF proximity (see* [NW]*).*

(a) $\tau(\mathrm{F}) = \sigma(\mathrm{F}) \subset \tau_w(\Delta, \delta) \subset \tau(\Delta) \wedge \sigma(\Delta, \delta)$.

(b) If $X \notin \Delta$, then the following are equivalent

(i) $\Delta = \mathrm{K}(X)$;

(ii) $\tau(\mathrm{F}) = \tau(\Delta)$;

(iii) $\tau(\mathrm{F}) = \sigma(\Delta)$;

(iv) $\tau(\mathrm{F}) = \tau_w(\Delta, \delta)$.

Next, we consider the most popular hypertopologies: the *proximal topology* $\sigma(\delta) = \sigma(\Delta)$ when $\Delta = \mathrm{CL}(X)$, and the *Vietoris topology* $\tau(\mathrm{V}) = \tau(\Delta)$ when $\Delta = \mathrm{CL}(X)$. From (3.8) we get

THEOREM 4.3. *(a) if δ is* EF*, the $\sigma(\delta) \subset \sigma(\delta')$, if and only if $\delta < \delta'$.*

(b) If δ is EF*, then $\sigma(\delta) \subset \tau(\mathrm{V})$ and they are equal if and only if $\delta = \delta_0$. In the metric case X is* UC *(i.e. all real valued continuous functions on X are uniformly continuous).*

(c) $\tau_w(\Delta,\delta)^+ = \tau(\mathrm{V})^+$ *if and only if* $\Delta = \mathrm{CL}(X)$, $\delta = \delta_0$. *In the case of a metric space* $\tau_w(\mathbf{B},\delta)^+ = \tau(\mathrm{V})^+$ *if and only if* X *is compact.*

THEOREM 4.4. *Let* δ *be an R or EF-proximity.*

(a) For all Δ, $\tau(\mathrm{F}) = \sigma(\mathrm{F}) \subset \tau_w(\Delta,\delta) \subset \sigma(\Delta) \subset \sigma(\delta)$.

(b) $\sigma(\delta)^+ = \tau_w(\Delta,\delta)^+$ *if and only if for each* $D \in \mathrm{CL}(X)$, $D \ll_\delta U$, *where* U *is open, implies* D *is* Δ*-totally bounded in* U.

In the metric case with $\Delta = \mathbf{B}$, the above condition is equivalent to: (X,d) is totally bounded.

5. Uniform case

In this section we study the situation when X is a Tychonoff space with compatible uniformity $\mathcal{U}$, which, in turn, induces an EF-proximity δ (cf. [DMN1]. Let $\mathrm{TB} = \mathrm{TB}(\mathcal{U})$ be the family of all closed $\mathcal{U}$-totally bounded subsets of X. For each $x \in X$ and entourage $U \in \mathcal{U}$, we define a U-ball with centre $x \in X$ as $U(X) = \{y \in X : (x,y) \in U\}$ and $\mathbf{B}(\mathcal{U})$ = the family of finite unions of U-balls corresponding to entourages $U \in \mathcal{U}$, with centres at $x \in X$ together all singletons. Using the above setup, we can define $\mathcal{U}$-Wijsman, the $\mathcal{U}$-ball and the $\mathcal{U}$-proximal ball topologies on $\mathrm{CL}(X)$ as in Section 1 viz:

DEFINITION 5.1.

The *$\mathcal{U}$-Wijsman topology*: $\tau_w(\mathbf{B}(\mathcal{U}),\delta) = \tau(\mathrm{V})^- \vee \tau_w(\mathbf{B}(\mathcal{U}),\delta)^+$;

the *$\mathcal{U}$-ball topology*: $\tau(\mathbf{B}(\mathcal{U})) = \tau((\mathrm{V})^- \vee \tau(\mathbf{B}(\mathcal{U}))^+$;

the *$\mathcal{U}$-proximal ball topology*: $\sigma(\mathbf{B}(\mathcal{U}),\delta) = \tau(\mathrm{V})^- \vee \sigma(\mathbf{B}(\mathcal{U}),\delta)^+$.

All the results of the previous sections, with $\mathbf{B}(\mathcal{U})$ substituting $\mathbf{B}$, hold good in the setting of a uniform space.

Alternately, the uniformity $\mathcal{U}$ be considered as a family of uniformly continuous pseudometrics, i.e. $\mathcal{U} = \{d \in \mathcal{U}\}$. Since a typical neighborhood of $A \in \mathrm{CL}(X)$ in the topology $\tau_w(\mathbf{B}(\mathcal{U}),\delta)^+$ is $\{E^+ : E^c \in \mathbf{B}(\mathcal{U}) \text{ and } A \in E^{++}\}$, one can separate A and E^c by a $d \in \mathcal{U}$. Thus we recover the original definition of Wijsman Convergence in a metric space modified to the uniform case.

THEOREM 5.2. *A net* A_n *in* $\mathrm{CL}(X)$ *converges to* A *in* $\tau_w(\mathbf{B}(\mathcal{U}),\delta)$ *if and only if for each* $x \in X$ *and each* $d \in \mathcal{U}$, $d(x,A_n)$ *converges to* $d(x,A)$.

We now study $\tau(\mathrm{TB})$ and $\sigma(\mathrm{TB})$ in relation to $\tau_w(\mathbf{B}(\mathcal{U}),\delta)$, $\sigma(\mathbf{B}(\mathcal{U}),\delta)$ and $\tau(\mathbf{B}(\mathcal{U}))$.

THEOREM 5.3. $\sigma(\mathrm{TB})^+ \subset \tau_w(\mathbf{B}(\mathcal{U}),\delta)^+ \subset \sigma(\mathbf{B}(\mathcal{U}),\delta)^+$.

Proof. Suppose $A \in (B^c)^{++} \in \sigma(\mathrm{TB})^+$, where $B \in \mathrm{TB}$. Then there is an entourage $V \in \mathcal{U}$ and a finite subset F of X such that $B \ll V(F) \ll A^c$. Then $V(F) \in \mathbf{B}(\mathcal{U})$, $A \in (V(F)^c)^{++}$ and $(V(F)^c)^+ \subset (B^c)^{++}$. ■

THEOREM 5.4. *If the range of every Cauchy net in* $(X,\mathcal{U})$ *is* $\mathcal{U}$*-totally bounded, then* $(X,\mathcal{U})$ *is complete* $\Leftrightarrow \tau(\mathrm{TB}) \subset \tau_w(\mathbf{B}(\mathcal{U}),\delta)$.

COROLLARY 5.5. *Let* (X,d) *be a metric space with metric proximity* δ *and* $\mathrm{TB} = \mathrm{TB}(d)$, *the family of all closed totally bounded subsets of* X. *Then* (X,d) *is complete* $\Leftrightarrow \tau(\mathrm{TB}) \subset \tau_w(\mathbf{B},\delta)$.

The next topologies that we consider for a metric space (X,d) are the *Hausdorff metric topology* $\tau(\mathrm{d}_H)$, and the *locally finite hypertopology* $\tau(\mathrm{LF})$.

THEOREM 5.6. *(a)* $\tau_w(\Delta,\delta)\subset\sigma(\Delta)\subset\sigma(\delta)\subset\tau(\mathrm{d}_H)\subset\tau(\mathrm{LF})$*;*

(b) $\tau_w(\Delta,\delta)\subset\sigma(\Delta)\subset(\delta)\subset\tau(\mathrm{V})\subset\tau(\mathrm{LF})$*;*

(c) $\tau_w(\Delta,\delta)=\tau(\mathrm{d}_H)$ *if and only if* (X,d) *is* TB.

6. Infimum and supremum

Wijsman topologies have proven to be useful as the *building blocks* of various important hypertopologies (see [BLLN]). In this section, we present results on *infima* and *suprema* of families of Wijsman topologies in a uniform or a metric space.

THEOREM 6.1. *Let X be a Tychonoff space with a compatible EF-proximity δ. Let $\prod(\delta)$ denote the family of all uniformities which are compatible with δ and* $\mathbf{U}$ *denote family of all uniformities which are compatible with the topology of X (see* [NW]). Then

(a) $\sup\{\tau_w(\mathbf{B}(\mathcal{U}),\delta):\mathcal{U}\in\prod(\delta)\}=\sigma(\delta)$.

(b) $\sup\{\tau_w(\mathbf{B}(\mathcal{U}),\delta):\mathcal{U}\in\mathbf{U}\}=\sigma(\delta_{\mathrm{F}})$.

(c) X is normal if and only if $\sup\{\tau_w(\mathbf{B}(\mathcal{U}))\}=\tau(\mathrm{V})$.

Proof. (a) It suffices to prove the result for the upper hypertopologies.

Suppose the sup is denoted by S. Since $\sigma(\delta)^+$ depends on δ and, for each $\mathcal{U}\in\prod(\delta)$, $\tau_w(\mathbf{B}(\mathcal{U}),\delta)^+\subset\sigma(\delta)^+$, it follows that $S\subset\sigma(\delta)^+$.

Conversely, suppose $A\in(E^c)^{++}\in\sigma(\delta)^+$, where $E\in\mathrm{CL}(X)$. Then for the totally bounded uniformity $\mathcal{W}\in\prod(\delta)$ [NW], there is an entourage W in $\mathcal{W}$ such that $W^3(A)\subset(E^c)^{++}$. Since A is totally bounded, $A\subset W(F)$, where F is a finite subset of A. Then $A\subset W(F)\in W^2(A)^{++}\subset[W^2(A)]^+\subset(E^c)^{++}$. So, $(E^c)^{++}\in\mathcal{S}$, i.e. $\sigma(\delta)^+\subset\mathcal{S}$.

(b) This follows from (a) and the fact: if δ, δ' are EF-proximities with $\delta<\delta'$, then $\sigma(\delta)^+\subset\sigma(\delta')^+$ (see [DCNS]).

(c) This follows from the facts $\tau(\mathrm{V})=\tau(\delta_0)$ and in a normal space δ_0 is EF. ∎

THEOREM 6.2. *Let (X,d) be a metric space with metric proximity δ. Here it would be convenient to denote $\tau_w(\mathbf{B},\delta)$ by $\tau_w(d)$. Then*

(a) $\sup\{\tau_w(e):e$ *is uniformly equivalent to* $d\}=\sigma(\delta)$*;*

(b) $\sup\{\tau_w(e):e$ *is topologically equivalent to* $d\}=\tau(\mathrm{V})$.

THEOREM 6.3. *Let X be a Tychonoff space with a compatible EF-proximity δ. Let $\prod(\delta)$ denote the family of all uniformities which are compatible with δ and $\mathcal{U}^*$, the coarsest totally bounded member of $\prod(\delta)$. Then* $\sigma(\mathrm{TB})^+=\inf\{\tau_w(\mathbf{B}(\mathcal{U}),\delta)^+:\mathcal{U}\in\prod(\delta)\}=\inf\{\sigma(\mathbf{B}(\mathcal{U}),\delta)^+:\mathcal{U}\in\prod(\delta)\}=\tau_w(\mathbf{B}(\mathcal{U}^*),\delta)^+=\sigma(\mathbf{B}(\mathcal{U}^*),\delta)^+$.

Proof. The result follows from Theorem (4.3). ∎

COROLLARY 6.4. ([DH1], [DH3]) *Under the conditions of the previous result the following are equivalent:*

(a) $\sigma(\mathrm{F})^+ = \sigma(\mathrm{TB})^+$*;*

(b) $\sigma(\mathrm{F})^+ = \inf\{\tau_w(\mathbf{B}(\mathcal{U}), \delta)^+ : \mathcal{U} \in \prod(\delta)\}$*;*

(c) $\sigma(\mathrm{F})^+ = \inf\{\sigma(\mathbf{B}(\mathcal{U}), \delta)^+ : \mathcal{U} \in \prod(\delta)\}$*.*

7. Countability properties and metrization

PROPOSITION 7.1. *Let δ be a compatible proximity on X, and let Δ be a cobase in X. Then the following are equivalent:*

(1) $\tau_w(\Delta, \delta) = \tau(\mathrm{V})^- \vee \tau_w(\Delta, \delta)^+$ *is first countable;*

(2) $\tau(\mathrm{V})^-$ *and* $\vee\tau_w(\Delta, \delta)^+$ *are first countable.*

PROPOSITION 7.2. *Let δ be a compatible proximity on X, and let Δ be a cobase in X.*

(1) $\tau(\mathrm{V})^-$ is first countable if and only if X is first countable and each closed set of X is separable [DH1]);

(2) $\tau_w(\Delta, \delta)^+$ is first countable if and only if $\sigma(\Delta)^+$ is first countable, i.e. for each closed set $A \in \mathrm{CL}(X)$ of X there exists a countable family $\mathcal{L}(A) = L_n = ((D_n)^c)^+$, with $D_n \in \Delta$, such that for each $D \in \Delta$, with $A \ll_\delta D^c$, there exists D_n such that $A \ll_\delta {D_n}^c \ll_\delta D^c$, or equivalently if every open set U hemi(Δ, δ), i.e. in the family $\{D \in \Delta : D \ll_\delta U\}$ there is a countable cofinal subfamily w.r.t. the strong inclusion $\ll_\delta$.

PROPOSITION 7.4. [HL] *Let (X, τ) be an Hausdorff space. The following are equivalent:*

(a) $\tau(\mathrm{V})^-$ *is first countable;*

(b) X is first countable and each closed set is separable.

PROPOSITION 7.4. *Let δ be a compatible proximity on X, and let Δ be a cobase in X. Then the following are equivalent:*

(a) $\tau_w(\Delta, \delta) = \tau(\mathrm{V})^- \vee \tau_w(\Delta, \delta)^+$ *is second countable;*

(b) $\tau(\mathrm{V})^-$ *and* $\vee\tau_w(\Delta, \delta)^+$ *are second countable.*

PROPOSITION 7.5. *Let δ be a compatible proximity on X, and let Δ be a cobase in X. Then the following are equivalent:*

(1) $\tau_w(\Delta, \delta)^+$ *is second countable;*

(2) $\sigma(\Delta)^+$ is second countable, i.e. X there exists a countable family $\mathcal{L}(\Delta) = L = ((D_n))$, with $D_n \in \Delta$, such that for each $D \in \Delta$, and every open set U with $D \ll_\delta U$, there exists D_n such that $D \ll_\delta D_n \ll_\delta U$, or equivalently the space X is hemi(Δ, δ), i.e. in $\{D \in \Delta : D \ll_\delta U\}$ there is a countable cofinal subfamily $L = ((D_n))$ w.r.t. the strong inclusion $\ll_\delta$.

PROPOSITION 7.6. *Let δ be a compatible EF-proximity on a Tychonoff space X, and let Δ be a cobase which is δ-Urysohn* [DMN1], p. 31, *i.e. $A \in \mathrm{CL}(X)$, $B \in \Delta$ with $A\underline{\delta}B$ implies there exists $E \in \Delta$ such that $A\underline{\delta}E$ and $E^c\underline{\delta}B$. Then $\sigma(\Delta, \delta) = \tau(\mathrm{V})^- \vee \sigma(\Delta, \delta)^+$ is Tychonoff (in fact it coincides with the Δ-Attouch-Wets Uniformity, see Theorem 2.1* [DMN1]*.)*

Proposition 7.7 *Let $(X,\mathcal{U})$ be a uniform space. Suppose that $\mathcal{U}$ is described by the family of uniformly continuous pseudometrics, i.e. $\mathcal{U} = \{d \in \mathcal{U}\}$. Let $\Delta = \mathbf{B}$, the family of finite unions of all proper closed balls. Then the Wijsman topology $\tau_w(\Delta, \delta)$ is Tychonoff.*

Proof. We can use the above proposition, since the proximity on $\mathrm{CL}(X)$ associated to $\mathcal{U}$ is EF. The result can be obtained in a more transparent way. In fact, $\tau_w(\Delta, \delta)$ is the weak topology on $\mathrm{CL}(X)$ determined by the family $d(x, \cdot)$, $x \in X$ and $d \in \mathcal{U}$. ∎

Corollary 7.8. *Let (X, d) be a metric space $\Delta = \mathbf{B}$, the family of finite unions of all proper closed balls. Then the following are equivalent:*

(i) *the Wijsman topology $\tau_w(\Delta, \delta)$ is first countable;*

(ii) *$\tau(\mathrm{V})$ is first countable;*

(iii) *X is second countable;*

(iv) *$\tau_w(\Delta, \delta)$ is second countable;*

(v) *$\tau_w(\Delta, \delta)$ is metrizable.*

Proof. Use the Urysohn Metrization Theorem. ∎

Part III

8. New topologies

From the expression of the Wijsman Topology as a hit-and-far-miss topology and the Hausdorff Metric topology as the Uniformly Discrete Proximal Topology in [Na2], many new topologies are naturally born by combining the new lower topologies with the upper ones, i.e. the six fundamental bricks (labelled by the different cobases Δ). It would be worthwhile to investigate properties of these new hypertopologies which we list below. Before we present the list of problems concerning new hypertopologies, we give two examples to show the importance of studying them. We note that the Locally Finite Topology was first studied in [Ma] and Uniformly Discrete Topology has recently been studied in [DMN2]. In a metrizable space X, the Locally Finite Topology on $\mathrm{CL}(X)$ is the supremum of Hausdorff metric topologies generated by all topologically equivalent metrics [Wa], [BHPV]. Moreover, it is the topology on $\mathrm{CL}(X)$) induced by the Hausdorff-Bourbaki uniformity corresponding to the fine uniformity on X [NS]. The last result characterizes a normal space as follows:

Propostion 8.1. *A Tychonoff space X is normal if and only if the Locally Finite Topology on $\mathrm{CL}(X)$ is induced by the Hausdorff-Bourbaki uniformity corresponding to the fine uniformity on X.*

Let (X, d) be a metric space. the Uniformly Discrete Topology $\tau(\mathrm{UD}) = \tau(\mathrm{UD}^-) \vee \tau(\delta\mathrm{V}^+)$ on $\mathrm{CL}(X)$ was defined to fill a formal gap, but it provides a *missing link* in the lattice of hypertopologies [DMN2]. Observe that on studying equalities basic topological concepts occur and their strange or peculiar behaviours shed more light on their deep nature. The following is a Hasse diagram in which each arrow goes from a coarser topology to a finer one, and equality follows UC or TB as indicated.

$$\begin{tikzcd}
\text{Proximal locally finite} \arrow[r, "\text{UC}"] & \text{Locally finite} \\
\text{Hausdorff metric} \arrow[u, "\text{UC}"] \arrow[r, "\text{UC}"] & \text{Uniformly discrete} \arrow[u, "\text{UC}"'] \\
\text{Proximal} \arrow[u, "\text{TB}"] \arrow[r, "\text{UC}"] & \text{Vietoris} \arrow[u, "\text{TB}"']
\end{tikzcd}$$

It is always convenient to study the two parts of the involved hypertopologies separately also to state comparisons. Sometimes the two parts, in spite of the definition, play an unexpected symmetric role. Observe (in the above diagram) that in the upper rectangle, horizontal and vertical equalities are described by the same single property, i.e. UC. Note that in the lower rectangle the horizontal (resp. vertical) equalities occur when the space is UC (resp. TB). To squeeze the lower rectangle to a point we need two properties, namely TB and UC. Consequently, due to the UC property, also the upper rectangle reduces to a point. But, TB + UC is equivalent to compactness. So, we have that compactness is equivalent to the equality of all six hypertopologies. This is a standard technique to state equalities. We will use this tool again.

The following is a list of new hypertopologies, built by the fundamental bricks, which are worth studying when the base space is a metric space:

(P.1) Uniformly discrete Fell topology $\tau(\mathrm{UD}\mathbf{F}) = \tau(\mathbf{F}^+) \vee \tau(\mathrm{UD}^-)$,

(P.2) Locally finite Fell topology $\tau(\mathrm{LF}\mathbf{F}) = \tau(\mathbf{F}^+) \vee \tau(\mathrm{LF}^-)$,

(P.3) Uniformly discrete Ball topology $\tau(\mathrm{UD}\mathbf{B}) = \tau(\mathbf{B}^+) \vee \tau(\mathrm{UD}^-)$,

(P.4) Locally finite Ball topology $\tau(\mathrm{LF}\mathbf{B}) = \tau(\mathbf{B}^+) \vee \tau(\mathrm{LF}^-)$,

(P.5) Uniformly discrete Proximal Ball topology $\sigma(\mathrm{UD}\mathbf{B}) = \sigma(\mathbf{B}^+) \vee \tau(\mathrm{UD}^-)$,

(P.6) Locally finite Proximal Ball topology $\sigma(\mathrm{LF}\mathbf{B}) = \sigma(\mathbf{B}^+) \vee \tau(\mathrm{LF}^-)$,

(P.7) Uniformly Discrete Wijsman topology $\tau(\mathrm{UD}\mathbf{W}) = \tau(\mathrm{W}\mathbf{B}^+) \vee \tau(\mathrm{UD}^-)$

(P.8) Locally finite Wijsman topology $\tau(\mathrm{LF}\mathbf{W}) = \tau(\mathrm{W}\mathbf{B}^+) \vee \tau(\mathrm{LF}^-)$,

(P.9) Uniformly discrete Proximal topology $\sigma(\mathrm{UD}\delta) = \sigma(\delta^+) \vee \tau(\mathrm{UD}^-)$,

(P.10) Locally finite Proximal topology $\sigma(\mathrm{LF}\delta) = \sigma(\delta^+) \vee \tau(\mathrm{LF}^-)$,

(P.11) Locally finite Totally Bounded Topology $\tau(\mathrm{LFTB}(X)) = \tau(\mathrm{TB}(X)^+) \vee \tau(\mathrm{LF}^-)$,

(P.12) Uniformly discrete Totally Bounded Topology $\tau(\mathrm{UDTB}(X)) = \tau(\mathrm{TB}(X)^+) \vee \tau(\mathrm{UD}^-)$,

(P.13) Locally finite Proximal Totally Bounded Topology $\sigma(\mathrm{LFTB}(X)) = \sigma(\mathrm{TB}(X)^+) \vee \tau(\mathrm{LF}^-)$,

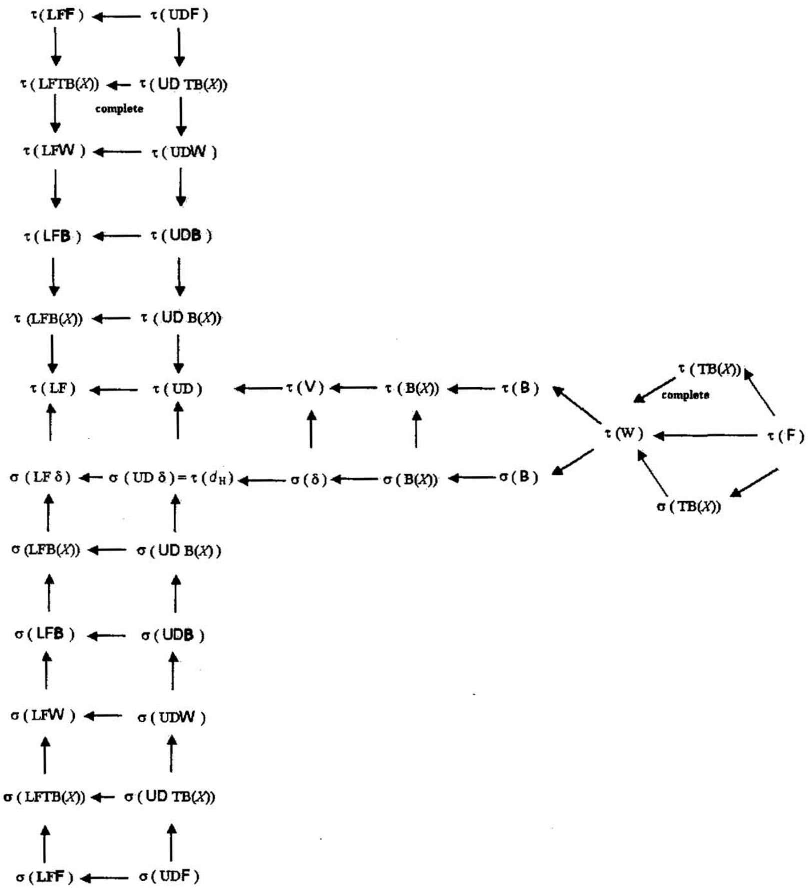
(P.14) Uniformly discrete Proximal Totally Bounded Topology $\sigma(\mathrm{UDTB}(X)) = \sigma(\mathrm{TB}(X)^+) \vee \tau(\mathrm{UD}^-)$, (P.15) Locally finite Bounded Vietoris Topology $\tau(\mathrm{LFB}(X)) = \tau(\mathrm{B}(X^+)) \vee \tau(\mathrm{LF}^-)$,

(P.16) Uniformly discrete Bounded Vietoris Topology $\tau(\mathrm{UDB}(X)) = \tau(\mathrm{B}(X^+)) \vee \tau(\mathrm{UD}^-)$,

(P.17) Locally finite Proximal Bounded Vietoris Topology $\sigma(\mathrm{B}(X)) = \sigma(\mathrm{B}(X)^+) \vee \tau(\mathrm{LF}^-)$,

(P.18) Uniformly discrete Proximal Bounded Vietoris Topology $\sigma(\mathrm{B}(X)) = = \sigma(\mathrm{B}(X)^+) \vee \tau(\mathrm{UD}^-)$.

Using the techniques in [DMN1], the following diagram can be drawn. Again, each arrow goes from a coarser topology to a finer (a necessary condition is indicated). Of course, we study special, but important, cobases, i.e. $\Delta = \mathrm{CL}(X)$, $\mathbf{B}$, $\mathrm{B}(X), \mathrm{TB(X)}, \mathrm{K}(X)$.



The reader can place conditions that force equalities by using the above diagram and the tools in Theorem 3.1, see also the Main Theorem in [DMN1]. Observe that the Ball topology and the proximal Ball topology are in general not comparable as first noticed in [DN1] (on the contrary it is obvious that the proximal topology is coarser than the Vietoris). This behaviour is due to the fact that the enlargement of a ball in general is not a ball!

9. Generalizations

A first generalization is obviously to a Tychonoff space (X, T) with a compatible uniformity $\mathcal{U}$ and the associated EF-proximity δ. Let $\mathcal{V}$ be a closed base for the uniformity $\mathcal{U}$. The family $\mathcal{B}$ (u-ball family) of finite unions of members of $(V(x) :: x \in X,\ V \in \mathcal{V})$, together with singletons, replaces the family $\mathbf{B}$ (see the previous sections) of metric space.

For $A \in \mathrm{CL}(X)$ and $V \in \mathcal{V}$, let Q^* be a maximal V-discrete subset of A. We then have a *discrete* family $\{V(x) : x \in Q^* \subset A\}$ with the properties:

(a) if $x,\ y \in Q^*$ and $x \neq y$ then $y \notin V(x)$, and

(b) $A \subset V(Q^*)$.

Obviously, Q^* is finite for every A if and only if $\mathcal{U}$ is totally bounded (TB).

Let UD be the collection of families of open sets of the form $\{V(x) : x \in Q \subset A\}$, where $A \in \mathrm{CL}(X)$, $V \in \mathcal{V}$, and Q is V-discrete. The family UD allows us to define a lower discrete "hit" topology. The Lower Uniformly Discrete (UD) Topology $\tau(\mathrm{UD}^-)$ on $\mathrm{CL}(X)$ consists of $\{\mathrm{E}^- : \mathrm{E} \in \mathrm{UD}\}$.

Then the problems given in the previous sections have their counter parts in the uniform case with the above replacements. We suffix the problems with "u" to indicate uniform case:

(P.1u) Uniformly discrete u-Fell topology $\tau(\mathrm{UD}\mathbf{F}) = \tau(\mathbf{F}^+) \vee \tau(\mathrm{UD}^-)$,

(P.2u) Locally finite u-Fell topology $\tau(\mathrm{LF}\mathbf{F}) = \tau(\mathbf{F}^+) \vee \tau(\mathrm{LF}^-)$,

(P.3u) Uniformly discrete u-Ball topology $\tau(\mathrm{UD}\mathbf{B}) = \tau(\mathbf{B}^+) \vee \tau(\mathrm{UD}^-)$,

(P.4u) Locally finite u-Ball topology $\tau(\mathrm{LF}\mathbf{B}) = \tau(\mathbf{B}^+) \vee \tau(\mathrm{LF}^-)$,

(P.5u) Uniformly discrete Proximal u-Ball topology $\sigma(\mathrm{UD}\mathbf{B}) = \sigma(\mathbf{B}^+) \vee \tau(\mathrm{UD}^-)$,

(P.6u) Locally finite Proximal u-Ball topology $\sigma(\mathrm{LF}\mathbf{B}) = \sigma(\mathbf{B}^+) \vee \tau(\mathrm{LF}^-)$,

(P.7u) Uniformly Discrete u-Wijsman topology $\tau(\mathrm{UD}\mathbf{W}) = \tau(\mathrm{W}\mathbf{B}^+) \vee \tau(\mathrm{UD}^-)$

(P.8u) Locally finite u-Wijsman topology $\tau(\mathrm{LF}\mathbf{W}) = \tau(\mathrm{W}\mathbf{B}^+) \vee \tau(\mathrm{LF}^-)$,

(P.9u) Uniformly discrete u-Proximal topology $\sigma(\mathrm{UD}\delta) = \sigma(\delta^+) \vee \tau(\mathrm{UD}^-)$,

(P.10u) Locally finite u-Proximal topology $\sigma(\mathrm{LF}\delta) = \sigma(\delta^+) \vee \tau(\mathrm{LF}^-)$,

(P.11u) Locally finite Totally Bounded Topology $\tau(\mathrm{LFTB}(X)) = \tau(\mathrm{TB}(X)^+) \vee \tau(\mathrm{LF}^-)$,

(P.12u) Uniformly discrete Totally Bounded Topology $\tau(\mathrm{UDTB}(X)) = \tau(\mathrm{TB}(X)^+) \vee \tau(\mathrm{UD}^-)$,

(P.13u) Locally finite Proximal Totally Bounded Topology $\sigma(\mathrm{LFTB}(X)) = \sigma(\mathrm{TB}(X)^+) \vee \tau(\mathrm{LF}^-)$,

(P.14u) Uniformly discrete Proximal Totally Bounded Topology $\sigma(\mathrm{UDTB}(X)) = \sigma(\mathrm{TB}(X)^+) \vee \tau(\mathrm{UD}^-)$, (P.15u) Locally finite Bounded Vietoris Topology $\tau(\mathrm{LFB}(X)) = \tau(\mathrm{B}(X^+)) \vee \tau(\mathrm{LF}^-)$,

(P.16u) Uniformly discrete Bounded Vietoris Topology $\tau(\mathrm{UDB}(X)) = \tau(\mathrm{B}(X^+)) \vee \tau(\mathrm{UD}^-)$,

(P.17u) Locally finite Proximal Bounded Vietoris Topology $\sigma(\mathrm{B}(X)) = \sigma(\mathrm{B}(X)^+) \vee \tau(\mathrm{LF}^-)$,

(P.18u) Uniformly discrete Proximal Bounded Vietoris Topology $\sigma(\mathrm{B}(X)) = \sigma(\mathrm{B}(X)^+) \vee \tau(\mathrm{UD}^-)$.

Finally, one may study hypertopologies in a Hausdorff space with a compatible LO-proximity. Using any nonempty family Δ of closed sets which are closed under finite unions and containing singletons as a cobase, we can define analogues of Ball, Proximal Ball and Wijsman Topologies. Many results that have been proven in metric spaces with elaborate long proofs can be given simple proofs [DN1], [DMN1]. Moreover, it may be possible to characterize topological, proximal or uniform properties in terms of these new hypertopologies. We give an example:

Theorem 9.1. *An LO-proximity on a Hausdorff space X is EF if and only if* $\tau(\mathrm{WCL}(X)^+) = \sigma(\delta^+)$.

Part IV

10. Graph topologies on function spaces

Let X and Y be topological spaces and let F be a family of (not necessary continuous) functions on X to Y. We identify each $f \in \mathrm{F}$ with its graph $G(f) = \{(x, f(x)) : x \in X\} \subset X \times Y$. There are many function space topologies in the literature. Most of these function space topologies can be expressed as graph topologies and graph topologies are essentially topologies on the hyperspace of $X \times Y$, restricted to functions considered as graphs.

A new topology, called *graph topology* $\mathbf{\Gamma}$ was introduced in 1966 [Na3] which is useful in Differential Topology. In a metric space, it coincides with the well known Whitney topology. In uniform spaces Γ equals the topology of uniform convergence on the family of uniformly continuous functions; and, moreover, if the range space contains an arc, then the equality of Γ and the topology of uniform convergence implies every continuous function on the domain space is uniformly continuous. The motivation for Γ came from almost continuous functions of Stallings but it was pointed out by Poppe that it is essentially the upper Vietoris hypertopology on the hyperspace of $X \times Y$ restricted to F. We emphasize that the members of F need not be closed (as graphs) in $X \times Y$ (see also [DN2]).

Definition 10.1 For each open set $U \subset X \times Y$ set

$$\mathrm{F}_U = \{f \in \mathrm{F} : G(f) \subset U\} = \{f \in \mathrm{F} : G(f) \in U^+\}$$

Then Γ is generated by the family $\{\mathrm{F}_U : U \subset X \times Y, U \text{ open}\}$ which forms a base.

The upper Vietoris topology on closed subsets has a poor record of inheriting higher separation axioms from the base space but, as remarked by Poppe, the function space F inherits them nicely. Another interesting fact is that, if X is T_1, then on the function space F, the graph topology $\Gamma =$ the Vietoris topology. There is a considerable literature on graph topologies using hypertopologies, other then the (upper) Vietoris, as for example the Hausdorff metric, proximal and Wijsman hypertopologies. and extending the study to multivalued functions or relations.

It is interesting to note that in hyperspaces the Hausdorff metric topology is not comparable with the Vietoris topology, but in $C(X, Y)$ considered as a subset of $CL(X \times Y)$, Hausdorff metric topology is coarser than the graph = the upper Vietoris topology. The lower hypertopology has no effect. The lower Hausdorff topology is a discrete family of open sets and let a function f (identified with its graph) hit this family and be contained in an open set U of $X \times Y$. Then the points of intersection form a closed set F and by removing $\{(x, Y) : (x, f(x)) \in F\}$ and substituting smaller open neighborhoods of points of F which are contained in U, we can force any function g contained in the new open set $V \subset U$ to hit the each member of the original family. So in the realm of graph topologies we have Proximal topology = Hausdorff metric topology $\subset$ topology of u.c. $\subset$ graph topology = locally finite topology.

Recall that, given a metric space (X, d), the Cauchy proximity δ_c is given by $A\delta_c B \Leftrightarrow \mathrm{cl}_{X^*} A \cap \mathrm{cl}_{X^*} B \neq \emptyset$, where X^* is the completion of X, (the proximity δ_c plays an important role in studying uniform continuity, see [DMN3]).

(GP1) What about δ_c-proximal graph topology? Note that topological properties of the "thin" subspace C(X,Y) of $CL(X \times Y)$ often characterize $CL(x \times Y)$.

The following is a loaded problem as it concerns graph topologies which come from other hypertopologies:

(GP2) Study on function spaces and relations, graph topologies which arise from the hypertopologies mentioned in Sections 8 and 9. Some results in these directions concern with Wijsmann topology, Hausdorff metric topology, [Na1], [DH2], [DN2], [Be].

(GP3) Study function space topologies on partial functions using the hypertopologies introduced in Sections 8 and 9.

Moreover, we can:

(GP4) Prove the embedding theorem for the new hypertopologies on the lines of results pertaining to Fell and Wijsman hypertopologies.

(GP5) Study distance functionals and gap functionals associated to the introduced hypertopologies.

The third (and natural) general direction of investigation is the following.

Since the Hausdorff Metric Topology $\tau(d_H)$ and he Wijsman topology $\tau(\mathrm{W})$ have new and transparent presentations as hit-and-far-miss topologies

$$\tau(d_{\mathrm{H}}) = \tau(\mathrm{UD}^-) \vee \sigma(\delta\, \mathrm{CL}(X)^+) \text{ and } \tau(\mathrm{W}) = \tau(\mathrm{W}\mathbf{B}^+) \vee \tau(\mathrm{V}^-),$$

we suggest

(GP6) Reconsideration of results scattered in the literature about these intensively studied hypertopologies (and related function spaces).

Recently, the Geometry of the space $\mathcal{K}(R^n)$ of all non-empty compact subsets of R^n associated to the Hausdorff metric $\delta_H(e)$, where e is the Euclidean metric, has been introduced and investigated (see [BMPS]).

For the reader convenience we recall some basic facts. Consider elements C in $\mathcal{K}(R^n)$ which satisfy one of the following equalities:

(i) $\delta_H(e)(A, B) = \delta_H(e)(A, C) + \delta_H(e)(C, B)$;

(ii) $\delta_H(e)(A, C) = \delta_H(e)(A, B) + \delta_H(e)(B, C)$;

(iii) $\delta_H(e)(C, B) = \delta_H(e)(C, A) + \delta_H(e)(A, B)$.

Sets C satisfying (i) lie "between" A and B; sets C satisfying (ii) lie to the "right" of B, sets C satisfying (iii) lie to the "left" of A. The collection of all sets C satisfying any of the equalities (i), (ii), (iii) is the Hausdorff line $L_{H(e)}(A, B)$ through A and B. Rays, circles and segments are defined similarly. The "Strange" Geometry of the Hausdorff metric is indeed full of exciting surprises, the theorems are unexpected, exotic, seminal, but the current proofs involve tiring epsilonetics.

(GP 7) We believe that our tools based on the expression $\tau(d_h) = \tau(\mathrm{UD}^-) \vee \sigma(\delta(\mathrm{CL}(X)^+))$ will offer more transparent and deeper results as well as new directions of investigations. We have to pursue these themes since they have fruitful applications, e.g. in Sound Analysis, Image processing, Digital Geometry [BD], [DD], [KR]. Actually, our applications are based on the following easy to handle proposition.

PROPOSITION 10.2 *Let* (X, d) *be a metric space and* $A, B \in \mathrm{CL}(X)$. *Recall that* $d_u(A, B) = d_l(B, A)$.

(i) The lower Hausdorff distance $d_l(A, B)$ *between* A *and* B *is less than* ϵ *if, and only if,* B *hits some maximal* $\mathrm{S}_\delta(Q, A)$ δ*-grid of* A, *where* $0 < \delta < \epsilon/2$, *i.e.*

$$B \cap \mathrm{S}_\delta(q, A) \neq \emptyset, \text{ for each } q \in Q;$$

(ii) the upper Hausdorff distance $d_u(A, B)$ *between* A *and* B *is less than* ϵ *if, and only if* A *hits some maximal* $\mathrm{S}_\delta(Q, B)$ δ*-grid of* B, *where* $0 < \delta < \epsilon/2$, *i.e.*

$$A \cap \mathrm{S}_\delta(q, B) \neq \emptyset, \text{ for each } q \in Q;$$

(iii) the Hausdorff distance $d_H(A, B)$ *between* A *and* B *is less than* ϵ *if, and only if, both conditions (i) and (ii) are fulfilled.*

So, due to the symmetry axiom, the Hausdorff distance can be viewed to as a *hit-and-hit* phenomenon according to the previous proposition, but also as a *far-and-far* structure according to formula $(*)$ in Section 1.

Some other natural questions arise.

(P.2) Let e be a compatible metric on R^n. What are the conditions on the geometrical objects A and B and on the equivalent metrics d and e to draw the same Hausdorff lines, rays, circles?

(P.3) Discover proximity to detect distances between the whole patterns of images, i.e. a proximity *capable of imitating the Hausdorff metric* (see [LPr], [PK]).

REFERENCES

[BD] S. Busiello and G. Di Maio, *Fractal Curves and their application to Image Processing, Sound Signals and Sound Synthesis*, in progress.

[Be] G. Beer, *Topologies on closed and closed convex sets*, Kluwer Academic Pub. (1993).

[BL1] G. Beer and R. Lucchetti, *Weak topologies for the closed subsets of a metrizable space*, Trans. Amer. Math. Soc. **335** (1993), 805–822.

[BL2] G. Beer and R. Lucchetti, *Well posed optimization problems and a new topology for the closed subsets of a metric space*, Rocky Mountain J. of Maths, **23**, (1993), 1197–1220.

[BHPV] G. Beer, C. Himmelberg, C. Prikry and F. van Vleck, *The locally finite topology on* 2^X, Proc. Amer. Math. Soc. **101** (1987), 168–171.

[BLLN] G. Beer, A. Leichicki, S. Levi and S. Naimpally, *Distance Functionals and Suiprema of Hyperspace Topologies*, Ann. Mat. Pura Appl. (4) **CLXII** (1992), 367–381.

[BMPS] D. Braun, J. Mayberry, A. Powers, S. Sclicher, *The Geometry of the Hausdorff Metric*, (2004), submitted, available at `http://faculty.gsvu.edu/schlicks/Hausdorff/paper.pdf`

[CLZ] C. Costantini, S. Levi and J. Zieminska, *Metrics that generate the same hyperspace convergence*, Set Valued Analysis, **1** (1993), 141–157.

[DCNS] A. Di Concilio, S. Naimpally and P. Sharma, *Proximal Hypertopologies*, Sixth Brazilian Topology Meeting, Campinas, Brazil (1988) [unpublished].

[DD] G. Di Maio and P. Di Lorenzo, *Hausdorff metric and Music similarity*, in progress.

[DH1] G. Di Maio and L. Hola, *On hit-and-miss topologies*, Rendiconti Acc Sc. Fis. Mat. Napoli, **LXII**, 103–124.

[DH2] G. Di Maio and L. Hola, *Wijsmann topologies on function spaces*, Rendiconti Circolo Matematico di Palermo, **46** (1997), 52–70.

[DH3] G. Di Maio and L. Hola, *A hypertopology determined by the family of totally bounded sets is the infimum mof upper Wijsmann topologies*, Q & A in General Topology, **15**, (1997), 259–272.

[DHo] G. Di Maio and D. Holy, *Comparison among Wijsman and other hypertopologies*, Atti Sem. Mat. Fis. Univ. Modena, **XLVIII** (2000), 121–133.

[DMN1] G. Di Maio, E. Meccariello and S. Naimpally, *Bombay hypertopologies*, Appl. Gen. Topol. **4** (2003), 421–444.

[DMN2] G. Di Maio, E. Meccariello and S. Naimpally, *Uniformly discrete topology, a missing link in hypertopologies*, Appl. Gen. Topology. **7**, (2006), 245–252

[DMN3] G. Di Maio, E. Meccariello and S. Naimpally, *Decomposition of UC spaces*, Q & A in General Topology, **22** (2004) 140–161.

[DN1] G. Di Maio and S. Naimpally, *Comparison of hypertopologies*, Rend. Istit. Mat. Univ. Trieste **22** (1990), 140–161.

[DN2] G. Di Maio and S. Naimpally, *Proximal graph topologies*, Q & A in General Topology, **10** (1992), 97–125.

[En] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[Fe] J. Fell, *A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space*, Proc. Amer. Math. Soc. **13** (1962), 472–476.

[HL] L. Hola and S. Levi, *Decomposition properties oh Hyperspace topologies*, Set Valued Analysis, **5** (1997), 309–321.

[KR] R. Klette, A. Rosenfeld, *Digital Geometry*, Morgan Kaufmann Publishers, (2004).

[LP] R. Lucchetti and A. Pasquale, *The bounded Vietoris topology and applications*, Ricerche di Mat., **43** (1994), 61–74.

[LPr] L. Latecki, F. Prokop, *Semi-proximity continuous functions in digital images*, Pattern. Rec. Letters, **16**, (1995), 1175–1187.

[Ma] M. Marjanović, *Topologies on collections of closed subsets*, Publ. Inst. Math. (Beograd) **20** (1966), 125–130.

[**Mi**] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.

[**Na1**] S. Naimpally, *Wijsmann convergence for function spaces*, Rendiconti Circolo Matematico di Palermo, **18** (1988), 343–358.

[**Na2**] S. Naimpally, *All hypertopologies are hit-and-miss*, Appl. Gen. Top., **3** (2002), 45–53.

[**Na3**] S. Naimpally, *Graph topology for function spaces*, Trans. Amer. Math. Soc. **123** (1966), 267–272.

[**NS**] S. Naimpally and P. Sharma, *Fine uniformity and the locally finite hyperspace topology*, Proc. Amer. Math. Soc. **103** (1988), 641–646.

[**NW**] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge Tracts in Mathematics no. **59**, Cambridge University Press (1970).

[**PK**] P. Ptak and W. G. Kropatsh, *Nearness in Digital Images and Proximity Spaces*, Springer Lect. Notes in Computer Science, **1953**, (2000), 69–77.

[**Po**] K. Poppe, *Eine Bemerkung uber Trennungsaxiome im Raum der abgeschlossenen Teilmengen eines topologischen Raumes*, Arch. Math. **16** (1965), 197–199.

[**Wa**] F. Wattenberg, *Topologies on the set of closed subsets*, Pacific J. Math. **68** (1977), 537–551.

[**Wi**] R. Wijsman, *Convergence of sequences of convex sets, cones, and functions*, II, Trans. Amer. Math. Soc. **123** (1966), 32–45.

(received 15.06.2007)

Giuseppe Di Maio, Dipartimento di Matematica, Seconda Università di Napoli, Via Vivaldi, 43, 81100 Caserta, ITALY

E-mail: `giuseppe.dimaio@unina2.it`

Somashekhar Naimpally, 96 Dewson Street, Toronto, Ontario, M6H 1H3 CANADA

E-mail: `somnaimpally@yahoo.ca`