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ASCENT AND DESCENT OF WEIGHTED COMPOSITION OPERATORS ON L^p-SPACES

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Abstract. In this paper, we study weighted composition operators on L^p-spaces with finite ascent and descent. We also characterize the injective weighted composition operators.

1. Introduction

Let $\Omega = (\Omega, \Sigma, \mu)$ be a σ-finite measure space. Let $L(\mu)$ denotes the linear space of all equivalence classes of Σ-measurable functions on Ω, where we identify any two functions that are equal μ-a.e. on Ω. Let ν be another measure on the measurable space (Ω, Σ) such that $\nu(A) = 0$ for each $A \in \Sigma$ whenever $\mu(A) = 0$. Then we say that the measure ν is absolutely continuous with respect to the measure μ and we write $\nu \ll \mu$. By Radon-Nikodym Theorem, there exists a non-negative locally integrable function f_ν on Ω so that the measure ν can be represented as

$$\nu(A) = \int_A f_\nu(x)\, d\mu(x), \quad \text{for each} \quad A \in \Sigma.$$

The function f_ν is called the Radon Nikodym derivative of the measure ν with respect to the measure μ.

Let $T\colon \Omega \to \Omega$ be a non-singular measurable transformation, that is, $\mu \circ T^{-1} \ll \mu$. Let $u\colon \Omega \to \mathbf{C}$ be an essentially bounded measurable function. We assume that the support u is the domain of T. Then the linear transformation $W = W_{u,T}\colon L(\mu) \to L(\mu)$ is defined as

$$Wf = W_{u,T} f = u.f \circ T, \quad \text{for each} \quad f \in L(\mu),$$

In case W maps $L^p(\mu)$ into itself, for $p \in [1, \infty)$, we call $W = W_{u,T}$ a weighted composition operator on $L^p(\mu)$ induced by the pair (u, T).

Note that the pair (u, T) induces a weighted composition operator while T may fail to induce a composition operator on $L^p(\mu)$. For example if $u(y) = 0$, for

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each $y \in \Omega$, then $W_{u,T}$ induces a weighted composition operator whether T induces the corresponding composition operator or not.

Now, we define a measure $\mu^1_{u,T}$ on Σ as

$$\mu^1_{u,T} = \int_{T^{-1}(A)} |u(x)|^q \, d\mu(x), \quad \text{for each} \quad A \in \Sigma.$$

Clearly $\mu^1_{u,T} \ll \mu \circ T^{-1} \ll \mu$. Let $f^1_{u,T}$ denotes the Radon-Nikodym derivative of $\mu^1_{u,T}$ with respect to μ and let $h_1 = (f^1_{u,T})^{\frac{1}{p}} : \Omega \to \mathbf{C}$.

Note that W is a continuous weighted composition operator on $L^p(\mu)$, for $p \in [1, \infty)$ if and only if $h_1 \in L^\infty(\mu)$. For details on the study of weighted composition operators on L^p-spaces, see [5, p. 51]. The study of weighted composition operators between two L^p-spaces has been initiated in [3]. The interesting study of composition operators on Banach function spaces with finite ascent and finite descent has been initiated in [2].

We also define a measure $\mu^2_{u,T}$ on Σ as

$$\mu^2_{u,T} = \int_{T^{-1}(A)} |u(x)|^q \, d\mu^1_{u,T}(x), \quad \text{for each} \quad A \in \Sigma.$$

Clearly $\mu^2_{u,T} \ll \mu^1_{u,T} \circ T^{-1} \ll \mu$. Let $f^2_{u,T}$ denotes the Radon-Nikodym derivative of $\mu^2_{u,T}$ with respect to μ and let $h_2 = (f^2_{u,T})^{\frac{1}{p}} : \Omega \to \mathbf{C}$.

Definition 1. For a bounded operator $A \colon F \to F$ on a Banach space F, the ascent $\alpha(A)$ of A is the least non-negative integer such that $\ker(A^k) = \ker(A^{k+1})$ and the descent $d(A)$ of A is the least non-negative integer such that $\overline{\mathrm{Ran}(A^k)} = \overline{\mathrm{Ran}(A^{k+1})}$.

Note that $\ker(A^k) \subseteq \ker(A^{k+1})$ and $\mathrm{Ran}(A^{k+1}) \subseteq \mathrm{Ran}(A^k)$, for each $k \geq 0$. In case $\alpha(A) < \infty$ and $d(A) < \infty$, then $d = \alpha(A) = d(A)$ on $L^p(\mu)$-spaces, for $p \in [1, \infty)$.

We also note that if $d = \alpha(A) = d(A) < \infty$, then $V = \ker(A^d)$ and $W = \mathrm{Ran}(A^d)$, is the only reducing pair for the operator A such that A is nilpotent on V and invertible on W, see [1, p. 81]. In particular, we take $A = W = W_{u,T}$, a weighted composition operator induced by the pair (u, T).

Definition 2. A standard Borel space Ω is a Borel subset of a complete metric space (S, d), where d is a metric on a set S. The class Σ will consist of all sets of the form $\Omega \cap E$, where E is a Borel subset of S.

In this paper, we give a necessary and sufficient condition for weighted composition operators with ascent 1 and descent 1. We also give a necessary and sufficient condition for the injective weighted composition operators.

2. Main results

In this section, we prove our main result with the help of the following lemma.

LEMMA 2.1. *Let $W = W_{u,T}$ be a continuous weighted composition operator on $L^p(\mu)$, for $p \in [1, \infty)$. Then, we have* $\ker(W) = L^p(\Omega_\circ)$, *where* $\Omega_\circ = \{x \in \Omega : f^1_{u,T}(x) = 0\}$ *and*

$$L^p(\Omega_\circ) = \{ f \in X : f(x) = 0 \text{ a.e. } x \in \Omega \setminus \Omega_\circ \}.$$

Proof. For $f \in L^p(\mu)$, the support of f is $\operatorname{supp}(f) = \{x \in \Omega : f(x) \neq 0\}$. Clearly, we have

$$L^p(\Omega_\circ) = \{ f \in L^p(\mu) : \operatorname{supp}(f) \subseteq \Omega_\circ \text{ a.e.} \} = \{ f \in L^p(\mu) : f^1_{u,T}|_{\operatorname{supp}(f)} = 0 \}.$$

For $f \in L^p(\Omega_\circ)$, we have

$$\begin{aligned} \|Wf\|_p^p &= \int_\Omega |Wf(x)|^p \, d\mu(x) = \int_\Omega |f(x)|^p f^1_{u,T}(x) \, d\mu(x) \\ &= \int_{\Omega \setminus \Omega_\circ} |f(y)|^p f^1_{u,T}(x) \, d\mu(y) + \int_{\Omega_\circ} |f(y)|^p f^1_{u,T}(x) \, d\mu(y) = 0. \end{aligned}$$

Thus $f \in \ker(W)$ so that $L^p(\Omega_\circ) \subseteq \ker(W)$.

Conversely, let $f \in \ker(W)$. Then $u.f \circ T = 0$ a.e.. We have

$$0 = \int_\Omega |u(x)|^p |f(T(x))|^p \, d\mu(x) = \int_\Omega |f(x)|^p f^1_{u,T}(x) \, d\mu(x)$$

which implies that $f^1_{u,T}|_{\operatorname{supp}(f)} = 0$ a.e., so that $f \in L^p(\Omega_\circ)$. This proves the reverse inclusion. ∎

The next result characterizes the injective weighted composition operators. For this we need the following definition.

DEFINITION 3. A measurable transformation $T : \Omega \to \Omega$ is said to be essentially surjective if $\mu(\Omega \setminus T(\Omega)) = 0$.

THEOREM 2.2. *Let $W = W_{u,T}$ be a continuous weighted composition operator on $L^p(\mu)$, for $1 \leq p < \infty$. Then W is injective if and only if T is essentially surjective.*

Proof. If W is injective, then using Lemma 2.1, we see that $L^p(\Omega_\circ) = \{0\}$. Thus $f^1_{u,T}(x) \neq 0$ a.e.. This implies that $\mu(\Omega_\circ) = 0$. Therefore T is essentially surjective.

Now we show that $\Omega \setminus \Omega_\circ = T(\Omega)$. Clearly, $\Omega \setminus \Omega_\circ = \operatorname{supp}(f^1_{u,T}) \supseteq T(\Omega)$. Also, for each $E \in \Sigma$ such that $E \subseteq \Omega \setminus T(\Omega)$, we have

$$0 = \mu^1_{u,T}(E) = \int_E f^1_{u,T}(x) \, d\mu(x),$$

which implies that $f^1_{u,T}|_E = 0 \Rightarrow E \subseteq \Omega_\circ$. This shows that $\Omega \setminus T(\Omega) \subseteq \Omega_\circ \Rightarrow \Omega \setminus \Omega_\circ \subseteq T(\Omega)$. This proves that $T(\Omega) = \Omega \setminus \Omega_\circ$.

Note that we have used the fact that $\mu^1_{u,T} \ll \mu \circ T^{-1}$. ∎

Corollary 2.3. *If (Ω, Σ, μ) is a non-atomic measure space, then the nullity of W is either zero or infinite.*

Remark. The above results in this section has been proved for composition operators on Orlicz spaces in [4].

The next theorem characterises weighted composition operators with ascent 1.

Theorem 2.4. *Let $W = W_{u,T}$ be a continuous weighted composition operator on $L^p(\mu)$. Then W has ascent 1 if and only if the measures $\mu^1_{u,T}$ and $\mu^2_{u,T}$ are equivalent.*

Proof. Since W is bounded, we have $\mu^2_{u,T} \ll \mu^1_{u,T} \circ T^{-1} \ll \mu$. Then, we have

$$\mu^2_{u,T} = \int_E f^2_{u,T}(x)\, d\mu(x) = \int_E |u(x)|^p \, d\mu^1_{u,T}(x), \text{ for each } E \in \Sigma.$$

Now, suppose $\mu^1_{u,T} \ll \mu^2_{u,T} \ll \mu^1_{u,T}$. Then, we see that

$$\Omega_\circ = \{\, x \in \Omega : f^1_{u,T}(x) = 0 \,\} = \{\, x \in \Omega : f^2_{u,T}(x) = 0 \,\}.$$

Then, by using Lemma 2.1, we have

$$\ker(W) = \ker M_{f^1_{u,T}} = L^p(\Omega_\circ) = \ker M_{f^2_{u,T}} = \ker(W^2).$$

This shows that W is a weighted composition operator with ascent 1.

Conversely, suppose $\ker(W) = \ker(W^2)$. Since $\ker(W) = L^p(\Omega_\circ)$, where $\Omega_\circ = \{x \in \Omega : f^1_{u,T}(x) = 0\}$ and $\ker(W^2) = L^p(\Omega'_\circ)$, where $\Omega'_\circ = \{x \in \Omega : f^2_{u,T} = 0\}$. We conclude that $\Omega_\circ = \Omega'_\circ$. Since $\mu^1_{u,T} = \int_E f^1_{u,T}(x)\, d\mu(x)$ and $\mu^2_{u,T} = \int_E f^2_{u,T}(x)\, d\mu(x)$ for each $E \in \Sigma$. Thus, we have $\mu^1_{u,T} \ll \mu^2_{u,T} \ll \mu^1_{u,T}$. This proves the theorem. ∎

Theorem 2.5. *Let (Ω, Σ, μ) be a σ-finite standard Borel space and W is a bounded operator on $L^p(\mu)$, for $p \in [1, \infty)$. Then the operator W has ascent 1 if and only if $T[\Omega_1] \supseteq \Omega_1$, where $\Omega_1 = \Omega \setminus \Omega_\circ$ and $\Omega_\circ = \{x \in \Omega : f^1_{u,T}(x) = 0\}$.*

Proof. Suppose $T[\Omega_1] \supseteq \Omega_1$. By Lemma 2.1, we have $\ker(W) = L^p(\Omega_\circ)$. Then $L^p(\Omega) = L^p(\Omega_\circ) \oplus L^p(\Omega_1)$. Thus each $f \in \ker(W^2)$ can be written as $f = f_1 + g_1$, where $f_1 \in \ker(W)$ and $g_1 \in L^p(\Omega_1)$. Since

$$0 = W^2 f = W^2(f_1 + g_1) = W^2 g_1 = u.u \circ T.g_1 \circ T^2$$

and $T[\Omega_1] \supseteq \Omega_1$, we see that $g_1 = 0$ a.e. on Ω_1. Then $f = f_1$. Thus $\ker(W^2) \subseteq \ker(W)$. Therefore, we have $\ker(W) = \ker(W^2)$. This implies that W has ascent 1.

Conversely, suppose that $T[\Omega_1] \not\supseteq \Omega_1$. Suppose $E \in \Sigma$ with $E \subseteq \Omega_1 \setminus T[\Omega_1]$ of non zero finite measure such that $W^2 \chi_E = 0$. Since $E \subseteq \Omega_1$, we have $W\chi_E \neq 0$, which contradicts the fact that W has ascent 1. ∎

COROLLARY 2.6. *Let W be as above. Then W is of ascent* 1 *if and only if* $(T \circ T)[\mathbf{N}] = T(\mathbf{N})$, *where* $T[\mathbf{N}]$ *is the range of* $\mathbf{N}$.

The following theorem characterises composition operators with descent 1.

THEOREM 2.7. *Let $W = W_{u,T}$ be a continuous weighted composition operator on $L^p(\mu)$, for $1 \le p < \infty$. Then W has descent* 1 *if and only if the measures $\mu^1_{u,T}$ and $\mu^2_{u,T}$ are equivalent.*

Proof. Using Theorem 2.4 and the arguments following the definition 1, the proof is through. ■

REMARK. For the examples of composition operators on L^p spaces with finite ascent and finite descent, see [2].

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