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CENTRALIZING AND COMMUTING GENERALIZED DERIVATIONS ON PRIME RINGS

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Abstract. Let R be a prime ring and d a derivation on R. If f is a generalized derivation on R such that f is centralizing on a left ideal U of R, then R is commutative.

A ring R is said to be prime if $aRb = 0$ implies that either $a = 0$ or $b = 0$. An additive mapping $d \colon R \to R$ is said to be a derivation if $d(xy) = d(x)\,y + xd(y)$ for all $x, y \in R$. A mapping f is said to be commuting on a left ideal U of R if $[f(x), x] = 0$ for all $x \in U$ and f is said to be centralizing if $[f(x), x] \in Z(R)$ for all $x \in U$. There has been considerable interest in commuting, centralizing and related mappings in prime and semiprime rings (see [2] for a partial bibliography).

In this note we extend some results of Bell and Martindale [1] for generalized derivations. An additive mapping $f \colon R \to R$ is said to be a generalized derivation on R if $f(xy) = f(x)\,y + xd(y)$ for all $x, y \in R$ (where d is a derivation on R). These mappings were introduced in [3].

Throughout this note R will represent a prime ring with $Z(R)$ being its centre.

In the following we state a well known fact as

REMARK 1. For a nonzero element $a \in Z(R)$, if $ab \in Z(R)$, then $b \in Z(R)$.

In order to prove the main result, we find it necessary to establish the following Lemma.

LEMMA 1. *If f is an additive mapping from R to R such that f is centralizing on a left ideal U of R, then for all $x \in U \cap Z(R)$, $f(x) \in Z(R)$.*

Proof. Since f is centralizing on U, we have $[f(x+y), x+y] \in Z(R)$, for all $x, y \in U$. This implies that

$$[f(x), y] + [f(y), x] \in Z(R). \tag{1}$$

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Now if $x \in Z(R)$, then from equation (1), $[f(x), y] \in Z(R)$. Replacing y by $f(x)y$, we get $f(x)[f(x), y] \in Z(R)$. If $[f(x), y] = 0$, then $f(x) \in C_R(U)$, the centralizer of U in R, and hence ([1, Identity IV]) belongs to $Z(R)$. But on the other hand if $[f(x), y] \neq 0$, it again follows from Remark 1 that $f(x) \in Z(R)$. ∎

Next we prove the result which generalizes [1, Theorem 4].

THEOREM 1. *Let R be a prime ring. Let $d\colon R \to R$ be a nonzero derivation and f be a generalized derivation on a left ideal U of R. If f is commuting on U then R is commutative.*

Proof. Since f is commuting on U, we have $[f(x), x] = 0$ for all $x \in U$. Replacing x by $x+y$, we get $[f(x), y] + [f(y), x] = 0$. Now by substituting $y = yx$ and simplifying we arrive at $[yd(x), x] = 0$. Replacing y by ry, we get $[r, x]\ U\ d(x) = 0$ for all $x \in U$ and $r \in R$. Since R is a prime ring, therefore either $[r, x] = 0$ or $d(x) = 0$ for all $r \in R$. So for any element $x \in U$, either $x \in Z(R)$ or $d(x) = 0$. Since d is nonzero on R, then by [4, Lemma 2], d is nonzero on U. Suppose $d(x) \neq 0$, for some $x \in U$, then $x \in Z(R)$. Suppose $z \in U$ is such that $z \notin Z(R)$, then $d(z) = 0$ and $x + z \notin Z(R)$. This implies $d(x+z) = 0$ and so $d(x) = 0$, a contradiction. This implies $z \in Z(R)$ for all $z \in U$. Thus U is commutative and hence by [4, Lemma 3], R is commutative. ∎

Now we are ready to prove the result which involves centralizing generalized derivations on left ideals containing central elements.

THEOREM 2. *Let U be a left ideal of a prime ring R such that $U \cap Z(R) \neq 0$. Let d be a nonzero derivation and f be a generalized derivation on R such that f is centralizing on U. Then R is commutative.*

Proof. We assume that $Z(R) \neq 0$ because otherwise f is commuting on U and there is nothing left to prove. Now for a nonzero $z \in Z(R)$, we replace x by zy in (1) and get $[f(z), y]y + z[d(y), y] + z[f(y), y] \in Z(R)$. Now by Lemma 1, $f(z) \in Z(R)$ and therefore $z[d(y), y] + z[f(y), y] \in Z(R)$. But as f is centralizing on U, we have $z[f(y), y] \in Z(R)$ and consequently $z[d(y), y] \in Z(R)$. Since z is nonzero, it follows from Remark 1 that $[d(y), y] \in Z(R)$. This implies d is centralizing on U and hence by [1, Theorem 4], we conclude that R is commutative. ∎

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