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SOME CURVATURE CONDITIONS OF THE TYPE 4×2 ON THE SUBMANIFOLDS SATISFYING CHEN'S EQUALITY

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Abstract. Submanifolds of the Euclidean spaces satisfying equality in the basic Chen's inequality have, as is known, many interesting properties. In this paper, we discuss the curvature conditions of the form $E \cdot S = 0$ on such submanifolds, where E is any of the standard 4-covariant curvature operators, S is the Ricci curvature operator, and E acts on S as a derivation.

1. Introduction

1. Let M^n be an n-dimensional submanifold of a Euclidean space E^m of dimension $m = n + p$ ($p \geq 1$, $n \geq 2$). Let g be the Riemannian metric induced on M^n from the standard metric on E^m, ∇ the corresponding Levi Civita connection on M^n, and R, S and τ respectively the Riemann-Christoffel curvature tensor, the Ricci tensor and the scalar curvature of M^n. We use the sign convention given by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and the normalization of the scalar curvature given by $\tau = \sum_{i,j=1}^{n} K(e_i \wedge e_j)$, where K denotes the sectional curvature and $e_i \wedge e_j$ is the plane section of TM^n spanned by the vectors e_i and e_j for $i \neq j$ of an orthonormal tangent frame field $e_1, \dots, e_n$ on M^n.

Consider the real function $\inf K$ on M^n defined for every $x \in M$ by

$$(\inf K)(x) := \inf\{\, K(\pi) : \pi \text{ is a 2-plane in } T_x(M^n)\,\}.$$

Since the set of 2-planes at a certain point is compact, this infimum is actually a minimum. B. Y. Chen proved in [5] the following basic inequality between the intrinsic scalar invariants $\inf K$ and τ of M^n, and the extrinsic scalar invariant $|H|$, being the length of the mean curvature vector field H of M^n in E^m.

THEOREM A. ([5]). *Let M^n ($n \geq 2$), be any submanifold of E^m ($m = n + p$, $p \geq 1$). Then*

$$\inf K \geq \frac{1}{2}\left\{\tau - \frac{n^2(n-2)}{n-1}\,|H|^2\right\}. \tag{1}$$

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Equality holds in (1) *at a point x if and only if with respect to suitable local orthonormal frames $e_1, \dots, e_n \in T_x M^n$ and $e_{n+1}, \dots, e_{n+p} \in T_x^\perp M^n$, the Weingarten maps A_t with respect to the normal sections $\xi_t = e_{n+t}$ $(t = 1, \dots, p)$ are given by*

$$A_1 = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix}, \quad A_t = \begin{pmatrix} c_t & d_t & 0 & \dots & 0 \\ d_t & -c_t & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (t > 1),$$

where $\mu = a + b$. For any such frame, $\inf K(x)$ *is attained by the plane $e_1 \wedge e_2$.*

The purpose of the present paper is to study submanifolds M^n of E^m for which the basic inequality (1) at all points is actually an equality. Such submanifolds are called *ideal submanifolds*.

If S is the Ricci operator of a manifold M^n in a Euclidean space, the Ricci curvatures $\mathrm{Ric}_i = S_{ii}$ $(i = 1, \dots, n)$ of M^n are given by

$$\mathrm{Ric}_i = \mathrm{Ric}(e_i) = \sum_{j=1}^{n} K_{ij},$$

where $K_{ij} = K(e_i \wedge e_j)$ $(i, j = 1, \dots, n)$ are the corresponding sectional curvatures. If, in particular, M^n is satisfying the basic equality in (1), then we have

$$K_{12} = ab - \sigma K_{1j} = a\,\mu, \quad K_{2j} = b\,\mu, \quad K_{ij} = \mu^2,$$

for $i, j > 2$, where $\sigma = \sum_{t=2}^{p} (c_t^2 + d_t^2)$. In this case the Ricci curvatures Ric_i of M^n $(i = 1, \dots, n)$ are given by

$$\begin{aligned} &\mathrm{Ric}_1 = (n-2)\,a\,\mu + K_{12}, \quad \mathrm{Ric}_2 = (n-2)\,b\,\mu + K_{12}, \\ &\mathrm{Ric}_3 = \dots = \mathrm{Ric}_n = (n-2)\,\mu^2, \end{aligned}$$

and we also have $S(e_i, e_j) = 0$ if $i \neq j$. The scalar curvature $\tau = 2ab - 2\sigma + (n-1)(n-2)\,\mu^2$. In the sequel, we shall also denote $\mathrm{Ric}_1 = \alpha$, $\mathrm{Ric}_2 = \beta$, $\mathrm{Ric}_3 = \gamma$.

2. Next, we recall several curvature operators which we shall use in the sequel.

The *concircular curvature operator* $Z(X, Y)$ is defined for $n \geq 2$ by

$$Z(X, Y) = R(X, Y) - \frac{\tau}{n(n-1)}\, B(X, Y),$$

where $\tau = \tau(x)$ is the scalar curvature of M^n, and the operator $B(X, Y)U = (X \wedge Y)U = g(U, Y)X - g(U, X)Y$ $(X, Y, U \in T_x(M^n))$. Note that, in components, $B_{ijkl} = g_{il} g_{jk} - g_{ik} g_{jl}$.

The *projective curvature operator* $P(X, Y)$ is defined for $n \geq 2$ by

$$P(X, Y)U = R(X, Y)U - \frac{1}{n-1} B(X, Y)(SU).$$

The *Weyl's conformal curvature operator* C is defined for $n \geq 3$ by

$$C(X,Y) = R(X,Y) - \frac{1}{n-2}\{SX \wedge Y + X \wedge SY\} + \frac{\tau}{(n-1)(n-2)}B(X,Y).$$

As is well known, every submanifold of dimension $n = 3$ is conformally flat, that is $C = 0$ identically holds on M^3.

The *conharmonic curvature operator* $K(X,Y)$ is defined for $n \geq 3$ by

$$K(X,Y) = R(X,Y) - \frac{1}{n-2}\{SX \wedge Y + X \wedge SY\}.$$

The *Einstein curvature operator* is defined for $n \geq 2$ by $G = S - \frac{\tau}{n}I$, where I is the identity operator on $T_x(M^n)$.

As is well-known, every manifold M^2 is of constant sectional curvature (that is, its curvature tensors Z and P vanish), and a manifold M^n $(n \geq 3)$ is conharmonically flat if and only if it is conformally flat, and its scalar curvature τ identically vanishes.

3. Now, assume that M^n is a submanifold of the Euclidean space E^m $(m = n+p, p \geq 1, n \geq 2)$ satisfying Chen's basic equality. If we denote $R_{ijk} = R(e_i,e_j)e_k$ $(i,j,k = 1,\ldots,n)$, then a straightforward calculation of the curvature operator $R(X,Y)U$ gives that, in the orthonormal basis $\{e_1,\ldots,e_n\}$ of the tangent space $T_x(M^n)$, we have

$$\begin{cases} R_{121} = R_1\, e_2, \quad R_{122} = -R_1\, e_1, \\ R_{1p1} = R_2\, e_p \quad (p \geq 3), \quad R_{1pp} = -R_2\, e_1 \quad (p \geq 3), \\ R_{2p2} = R_3\, e_p \quad (p \geq 3), \quad R_{2pp} = -R_3\, e_2 \quad (p \geq 3), \\ R_{pqp} = R_0\, e_q \quad (p,q \geq 3, p \neq q), \end{cases} \tag{2}$$

where $R_1 = -K_{12} = \sigma - ab = \sum_{t=2}^{p}(c_t^2 + d_t^2) - ab$, $R_2 = -K_{1j} = -a\,\mu$, $R_3 = -K_{2j} = -b\,\mu$ and $R_0 = -K_{ij} = -\mu^2$ $(i,j > 2)$. Besides, $R_{ijk} = 0$ if all i,j,k are mutually distinct.

Moreover, if $E = E_4$ is any of the curvature operators R, Z, B, C, K, we get the similar equations for the values $E_{ijk} = E(e_i,e_j)e_k$ $(i,j,k = 1,\ldots,n)$, with the corresponding functions E_i $(i = 0,1,2,3)$. We only give in short the exact values for the functions E_i $(i = 0,1,2,3)$ in each of these cases.

(B): $B_1 = B_2 = B_3 = B_0 = -1$ at any point $x \in M^n$.

$$(Z): \quad \begin{cases} Z_1 = R_1 + \frac{\tau}{n(n-1)}, \quad Z_2 = -a\,\mu + \dfrac{\tau}{n(n-1)}, \\ Z_3 = -b\,\mu + \dfrac{\tau}{n(n-1)}, \quad Z_0 = -\mu^2 + \dfrac{\tau}{n(n-1)}. \end{cases}$$

$$(K): \quad K_1 = \frac{n-4}{n-2}R_1 + \mu^2,\ K_2 = K_3 = -\frac{R_1}{n-2} + \mu^2,\ K_0 = \mu^2.$$

$$(C): \quad \begin{cases} C_1 = \dfrac{n-3}{n-1}R_1,\ C_2 = C_3 = -\dfrac{n-3}{(n-1)(n-2)}R_1, \\ C_0 = \dfrac{2\,R_1}{(n-1)(n-2)}. \end{cases}$$

We note that each of the functions R_0, B_0, Z_0, K_0, C_0 exists only if $n \geq 4$.

We also note that the corresponding equations for the projective curvature operator P differ of the previous because the corresponding curvature tensor is not antisymmetric in the last two indices. In fact, denoting $P_{ijk} = P(e_i, e_j)e_k$ $(i, j, k = 1, \dots, n)$, we find the following equations:

$$\begin{cases} P_{121} = P_1\, e_2, \quad P_{122} = \widetilde{P}_1\, e_1, \quad P_{1p1} = P_2\, e_p \quad (p \geq 3), \\ P_{1pp} = \widetilde{P}_2\, e_1 \quad (p \geq 3), \quad P_{2p2} = P_3\, e_p \quad (p \geq 3), \\ P_{2pp} = \widetilde{P}_3\, e_2\ (p \geq 3), \quad P_{pqp} = P_0\, e_q\ (p, q \geq 3, p \neq q), \end{cases} \tag{3}$$

where

$$\begin{cases} P_1 = \frac{n-2}{n-1}\,(R_1 + a\,\mu), \quad \widetilde{P}_1 = -\frac{n-2}{n-1}\,(R_1 + b\,\mu), \\ P_2 = -\frac{R_1 + a\,\mu}{n-1}, \quad \widetilde{P}_2 = \frac{\mu}{n-1}[a - (n-2)b], \\ P_3 = -\frac{R_1 + b\,\mu}{n-1}, \quad \widetilde{P}_3 = \frac{\mu}{n-1}\,[b - (n-2)a], \\ P_0 = -\frac{\mu^2}{n-1}. \end{cases}$$

Besides, $P_{ijk} = 0$ if all i, j, k are mutually distinct, and P_0 exists only if $n \geq 4$.

Finally, denoting any of the tensors S, G by E, we find that the Ricci operator S and the Einstein tensor G act as follows:

$$Ee_1 = \alpha\, e_1, \quad Ee_2 = \beta\, e_2, \quad Ee_p = \gamma\, e_p \qquad (p \geq 3),$$

where, if $E = S$: $\alpha = (n-2)\,a\,\mu - R_1$, $\beta = (n-2)\,b\,\mu - R_1$, $\gamma = (n-2)\,\mu^2$, and, if $E = G$: $\alpha_0 = (n-2)\,a\,\mu - R_1 - \tau/n$, $\beta_0 = (n-2)\,b\,\mu - R_1 - \tau/n$, $\gamma_0 = (n-2)\,\mu^2 - \tau/n$.

In [15] it was proved that a submanifold M^n $(n \geq 3)$ satisfying Chen's equality is flat if and only if it is totally geodesic. The last condition means that $a = b = \sigma = 0$, thus that $a = b = c_t = d_t = 0$ $(t = 2, \dots, p)$. There, it was also proven that M^n is Einstein if and only if $n = 2$, or it is flat.

In the next simple proposition we collect the conditions for a manifold M^n satisfying Chen's equality to be flat with respect to different curvature operators.

Proposition 1. *If $n \geq 2$, then M^n is flat if and only if $n = 2$ and $\sigma = ab$, or $n \geq 3$ and $\sigma = a = b = 0$.*

If $n \geq 2$, then M^n is of constant curvature if and only if $n = 2$, or it is flat.

If $n \geq 3$, then M^n is conformally flat if and only if $n = 3$, or $n \geq 4$ and $\sigma = ab$.

If $n \geq 3$, then M^n if conharmonically flat if and only if it is flat, or $n = 3$ and $\sigma = ab + \mu^2$.

2. Main results

Throughout this section we suppose that M^n is a submanifold in the Euclidean space E^m $(m = n + p,\ p \geq 1,\ n \geq 2)$ satisfying the basic Chen's equality, and we investigate on such a submanifold several curvature conditions of the form $E \cdot F_2 = 0$, where E is any of the curvature operators R, Z, P, K, C, F_2 is any of the operators S, G, and the operation $E \cdot F_2$ is defined as

$$(E(X,Y) \cdot F_2)U = E(X,Y)(F_2 U) - F_2(E(X,Y)U),$$

for all tangent vectors $X, Y, U \in T_x(M^n)$.

Similar curvature conditions of the form $E_4 \cdot F_2 = 0$, and of the form $E_4 \cdot F_4 = 0$ have been investigated in many papers (see e.g. [1–3],[7–10], [14–15], [18–20], [22–33], etc.).

Since obviously $E \cdot G = E \cdot S$, we shall consider only the case $F_2 = S$, thus we shall discuss exactly the operations $R \cdot S$, $Z \cdot S$, $P \cdot S$, $C \cdot S$ and $K \cdot S$.

In the most simple case $n = 2$, we get that $S = \frac{\tau}{2} I$ (moreover, $Z = P = 0$), so that $R \cdot S = Z \cdot S = P \cdot S = 0$. Hence, this case is trivial and we can suppose that $n \geq 3$.

If $n \geq 3$ and E is any of the curvature operators R, Z, K, C, then obviously $E \cdot S = 0$ if and only if $(E \cdot S)_{ijk} = (E(e_i, e_j) \cdot S)e_k = 0$ for all indices $i, j, k = 1, \ldots, n$. It is also not difficult to see that the above condition is satisfied if and only if the following three equations hold:

$$(E \cdot S)_{121} = (E \cdot S)_{131} = (E \cdot S)_{232} = 0.$$

Hence, $E \cdot S = 0$ if and only if the next system of equations is satisfied:

$$(\alpha - \beta)\, E_1 = 0, \quad (\alpha - \gamma)\, E_2 = 0, \quad (\beta - \gamma)\, E_3 = 0. \tag{4}$$

By an easy discussion of the corresponding system in any of the cases $E = R, Z, K, C$ we get the following theorems.

Theorem 1. *If $n \geq 3$, then $R \cdot S = 0$ if and only if one of the following three cases occurs*: (1^0) $\mu = 0$; (2^0) $\sigma = a = 0$, $b \neq 0$; (3^0) $\sigma = b = 0$, $a \neq 0$.

Proof. By direct calculations, it is easy to check that $R \cdot S = 0$ in any of the cases (1^0), (2^0), (3^0).

Next, assume that $R \cdot S = 0$ and $\mu \neq 0$. Supposing that $a, b \neq 0$, by equations $(R \cdot S)_{131} = 0$ and $(R \cdot S)_{232} = 0$, we get

$$R_1 = -(n-2)\, a\, \mu = -(n-2)\, b\, \mu,$$

and consequently $a = b$, $\sigma = -(2n-5)\, a^2$. Therefore $\sigma = a = 0$, a contradiction. Hence, $ab = 0$. If $a = 0$, then the third equation gives $b\,\sigma = \mu\,\sigma = 0$, thus $\sigma = 0$ (because $\mu \neq 0$), so we have the case (2^0). If $b = 0$, we similarly have the case (3^0). ■

Lemma 1. *If $n \geq 3$ and $Z_1 = 0$, then M^n is a totally geodesic plane.*

Proof. Note that equation $Z_1 = 0$ in the developed form reads:

$$(n+1)\sigma = -(n-1)\, a^2 - (n-1) b^2 - (n-3)\, ab.$$

Since $n \geq 3$, the previous equality is possible only if $\sigma = a = b = 0$, which means that M^n is totally geodesic n-plane in E^m. ■

Theorem 2. *If $n \geq 3$, then $Z \cdot S = 0$ if and only if one of the following cases occurs:* (1^0) *M^n is a totally geodesic plane;* (2^0) *$n \geq 4$, $a = b$ and $\sigma = (n^2 - 5n + 5)\, a^2$.*

Proof. If $n \geq 4$, $\sigma = (n^2 - 5n + 5)\, a^2$, $a = b$, then $Z_2 = Z_3 = 0$, $\alpha = \beta$, and immediately $Z \cdot S = 0$.

Conversely, assume that $n \geq 3$, $Z \cdot S = 0$, and M^n is not totally geodesic plane. Then by Lemma 1, $Z_1 \neq 0$. By equation $(Z \cdot S)_{121} = 0$, we get $a = \pm b$. Supposing that $\mu = 0$, we have $\tau = -2\, R_1 = -2(\sigma + a^2)$, $Z_1 = \frac{(n+1)(n-2)}{n(n-1)}\, R_1$, and by $(Z \cdot S)_{131} = 0$ we get a contradiction $R_1 = \sigma + a^2 = 0$, M^n is a totally geodesic plane. If $a = b$, then

$$Z_2 = -\frac{2}{n(n-1)}\, \{\sigma - (n^2 - 5n + 5)\, a^2\},$$
$$R_1 + (n-2)\, b\, \mu = \sigma + (2n - 5)\, a^2 > 0,$$

and by equation $(Z \cdot S)_{131} = 0$, we obtain $Z_2 = 0$, i.e. $\sigma = (n^2 - 5n + 5)\, a^2$. Since M^n is not totally geodesic, the case $n = 3$ is excluded, so that $n \geq 4$, and we have the case (2^0). ∎

Lemma 2. *If $n \geq 3$ and $K_1 = K_2 = 0$, then M^n is conharmonically flat.*

Proof. If $K_1 = K_2 = 0$, then easily $R_1 = (n-2)\, \mu^2$ and $(n-3)\, \mu^2 = 0$. If $\mu = 0$, then $R_1 = 0$, and we obtain that M^n is totally geodesic plane. If $n = 3$, then $R_1 = \mu^2$, and we again get that M^n is conharmonically flat. ∎

Theorem 3. *If $n \geq 3$, then $K \cdot S = 0$ if and only if one of the following cases occurs*: (1^0) *M^n is conharmonically flat*; (2^0) $a = b \neq 0$, $\sigma = (4n - 7)\, a^2$.

Proof. If $a = b$ and $\sigma = (4n - 7)\, a^2$, then $K_2 = 0$ and $\alpha = \beta$, so that the condition $K \cdot S = 0$ is obviously satisfied.

Conversely, assume that $n \geq 3$ and $K \cdot S = 0$. If $K_2 \neq 0$, then by equations $(K \cdot S)_{131} = 0$ and $(K \cdot S)_{232} = 0$ we obtain

$$R_1 = -(n-2)\, a\, \mu = -(n-2)\, b\, \mu,$$

and hence $a = \pm b$. If $a = b$, then by $R_1 = -2(n-2)\, a^2$, we easily get that M^n is totally geodesic, contradicting to $K_2 \neq 0$. If $a = -b$, $\mu = 0$, then $\sigma = -a^2$, and M^n is totally geodesic, again a contradiction. Hence $K_2 = 0$, i.e. $R_1 = (n-2)\, \mu^2$. If, in addition, we assume that M^n is not conharmonically flat, then by Lemma 2, $K_1 \neq 0$. By equation $(K \cdot S)_{121} = 0$, we then have $a = \pm b$. If $a = b$, then

$$K_2 = \frac{(4n - 7)\, a^2 - \sigma}{n - 2} = 0,$$

and hence $\sigma = (4n - 7)\, a^2$, so we have the case (2^0). If $a = -b$, $\mu = 0$, then by $K_2 = 0$, we get $R_1 = 0$, $\sigma = -a^2$, $\sigma = a = b = 0$, contradicting to $K_1 \neq 0$. ∎

THEOREM 4. *If $n \geq 3$, then $C \cdot S = 0$ if and only if M^n is conformally flat.*

Proof. Suppose that $C \cdot S = 0$ and M^n is not conformally flat. Then $n \geq 4$ and $R_1 \neq 0$. By equations $(C \cdot S)_{121} = 0$, $(C \cdot S)_{131} = 0$, we then get $\alpha = \beta = \gamma$, i.e. $a = \pm b$ and $R_1 = -(n-2)\, b\, \mu$. If $\mu = 0$, then we get a contradiction $R_1 = 0$. If $a = b$, then we get $\sigma = -(2n-5)\, a^2$, and hence $\sigma = a = 0$, thus again a contradiction $R_1 = 0$. ■

Next, assume that $n \geq 3$ and consider the condition $P \cdot S = 0$. It is also not difficult to see that $(P \cdot S)_{ijk} = 0$ for any choice of indices $i, j, k = 1, \dots, n$ if and only if the next equations are satisfied: $(P \cdot S)_{121} = (P \cdot S)_{122} = (P \cdot S)_{131} = (P \cdot S)_{133} = (P \cdot S)_{232} = (P \cdot S)_{233} = 0$. Hence, the complete system of equations for the operation $P \cdot S$ reads:

$$\begin{cases} (\alpha - \beta)\, P_1 = (\alpha - \beta)\, \widetilde{P}_1 = (\alpha - \gamma)\, P_2 = 0, \\ (\alpha - \gamma)\, \widetilde{P}_2 = (\beta - \gamma)\, P_3 = (\beta - \gamma)\ = \widetilde{P}_3 = 0. \end{cases} \tag{5}$$

Therefore, it is not difficult to get the following result.

THEOREM 5. *If $n \geq 3$, then $P \cdot S = 0$ if and only if M^n is a totally geodesic plane.*

Proof. Assume that $n \geq 3$ and $P \cdot S = 0$. Then, by equations $(P \cdot S)_{121} = 0$, $(P \cdot S)_{122} = 0$, we find that $a = \pm b$. If $\mu = 0$, then by equation $(P \cdot S)_{131} = 0$ we have $R_1 = 0$, and M^n is a totally geodesic plane. If $a = b$, then by the same equation we have $(\sigma + a^2)\{\sigma + (2n-5)\, a^2\} = 0$, thus $\sigma = a = b = 0$, so M^n is again totally geodesic. ■

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