МАТЕМАТИЧКИ ВЕСНИК
59 (2007), 75–84

UDK 515.122
оригинални научни рад
research paper

SOME SUBSETS OF IDEAL TOPOLOGICAL SPACES

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Abstract. In ideal topological spaces, $\star$-dense in itself subsets are used to characterize ideals and mappings. In this note, properties of $\mathcal{A}_{\mathcal{I}}$-sets, $\mathcal{I}$-locally closed sets and almost strong $\mathcal{I}$-open sets are discussed. We characterize codense ideals by the collection of these sets. Also, we give a decomposition of continuous mappings and deduce some well-known results.

1. Introduction and preliminaries

Ideals in topological spaces have been considered since 1930. This topic has won its importance by the paper of Vaidyanathaswamy [19]. In this note, we discuss the properties of the $\star-$dense in itself sets, namely, $\mathcal{A}_{\mathcal{I}}$-sets, regular $\mathcal{I}$-closed sets and almost strong $\mathcal{I}$-open sets in ideal topological spaces.

An *ideal* $\mathcal{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies: (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X,τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(\cdot)^\star\colon \mathcal{P}(X) \to \mathcal{P}(X)$, called a *local function* [13] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^\star(\mathcal{I},\tau) = \big\{\, x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x) \,\big\}$ where $\tau(x) = \big\{\, U \in \tau \mid x \in U \,\big\}$. We will make use of the basic facts about the local functions [9, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\star(\cdot)$ for a topology $\tau^\star(\mathcal{I},\tau)$, called the $\star$-*topology*, finer than τ is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I},\tau)$ [19]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I},\tau)$ and $\tau^\star$ or $\tau^\star(\mathcal{I})$ for $\tau^\star(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an *ideal space.* $\mathcal{I}$ is said to be *codense* [4] if $\tau \cap \mathcal{I} = \{\emptyset\}$. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X,τ).

By a space, we always mean a topological space (X,τ) with no separation properties assumed. If $A \subset X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of A in (X,τ) and $cl^\star(A)$ and $int^\star(A)$ will, respectively, denote the closure and interior of A in $(X,\tau^\star)$. An open subset A of a space (X,τ) is

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AMS Subject Classification: Primary: 54 A 05, 54 A 10; Secondary: 54 C 08, 54 C 10.

Keywords and phrases: Codense ideal, semiopen set, preopen set, $\mathcal{I}$-locally closed set, $f_{\mathcal{I}}$-set, regular $\mathcal{I}$-closed set, $\mathcal{A}_{\mathcal{I}}$-set, semicontinuity, $\mathcal{A}_{\mathcal{I}}$-continuity, $f_{\mathcal{I}}$-continuity.

said to be *regular open* if $A = int(cl(A))$. The complement of a regular open set is *regular closed*. The family of all regular open (resp. regular closed) set is denoted by $RO(X,\tau)$ (resp. $RC(X,\tau)$). A subset A of a space (X,τ) is an *α-open* [16] (resp. *semiopen* [14], *preopen* [15], *β-open* or *semipreopen* [1]) set if $A \subset int(cl(int(A)))$ (resp. $A \subset cl(int(A))$, $A \subset int(cl(A))$, $A \subset cl(int(cl(A)))$). The complement of a semiopen (resp. preopen) set is *semiclosed* (resp. *preclosed*). The family of all α-open (resp. semiopen, preopen) sets in (X,τ) is denoted by τ^α (resp. $SO(X,\tau)$, $PO(X,\tau)$). The smallest preclosed set containing A is called the *preclosure* of A and is denoted by $pcl(A)$. Also, $pcl(A) = A \cup cl(int(A))$ [1, Theorem 1.5(e)]. A subset A of a space (X,τ) is *locally closed* [2] (resp. $\mathcal{A}$-set [18]) if $A = U \cap V$ where U is open and V is closed (resp. regular closed). A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be *$\mathcal{I}$-open* [10] if $A \subset int(A^\star)$. The largest $\mathcal{I}$-open set contained in A is called the $\mathcal{I}$-interior of A and is denoted by $\mathcal{I}int(A)$. The family of all $\mathcal{I}$-open sets is denoted by $IO(X,\tau)$. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is *$\tau^\star$-closed* [8] (resp. *$\star$-dense in itself* [7], *$\star$-perfect* [7]) if $A^\star \subset A$ (resp. $A \subset A^\star$, $A = A^\star$). Clearly, A is $\star$-perfect if and only if A is $\tau^\star$-closed and $\star$-dense in itself. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is *$\mathcal{I}$-locally closed* [3] if $A = G \cap V$, where G is open and V is $\star$-perfect. We will denote the collection of all $\mathcal{I}$-locally closed sets in $(X,\tau,\mathcal{I})$ by $\mathcal{I}LC(X,\tau)$. Clearly, every $\star$-perfect set is $\mathcal{I}$-locally closed. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is called an *$f_{\mathcal{I}}$-set* [12] (resp. *regular $\mathcal{I}$-closed* [11]) if $A \subset (int(A))^\star$ (resp. $A = (int(A))^\star$). The family of all $f_{\mathcal{I}}$-sets in a space $(X,\tau,\mathcal{I})$ will be denoted by $f_{\mathcal{I}}(X,\tau)$. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is called *pre-$\mathcal{I}$-open* [3] (resp. *$\alpha-\mathcal{I}$-open* [6], *semi-$\mathcal{I}$-open* [6])if $A \subset int(cl^\star(A))$ (resp. $A \subset int(cl^\star(int(A)))$, $A \subset cl^\star(int(A))$). The family of all pre-$\mathcal{I}$-open (resp. α-$\mathcal{I}$-open, semi-$\mathcal{I}$-open) sets is denoted by $P_{\mathcal{I}}O(X,\tau)$ (resp. $\alpha_{\mathcal{I}}O(X,\tau)$, $S\mathcal{I}O(X,\tau)$). Given a space (X,τ) and ideals $\mathcal{I}$ and $\Im$ on X, the *extension* of $\mathcal{I}$ via $\Im$ [10], denoted by $\mathcal{I} \star \Im$, is the ideal given by $\mathcal{I} \star \Im = \{ A \subset X \mid A^\star(\mathcal{I}) \in \Im \}$. In particular, $\mathcal{I} \star \mathcal{N} = \{ A \subset X \mid int(A^\star(\mathcal{I})) = \emptyset \}$ is an ideal containing both $\mathcal{I}$ and $\mathcal{N}$ and $\mathcal{I} \star \mathcal{N}$ is usually denoted by $\widetilde{\mathcal{I}}$. The following lemmas will be useful in the sequel.

Lemma 1.1. [9, Theorem 6.1] *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then the following are equivalent.*

(a) *$\mathcal{I}$ is codense.*
(b) *$X = X^\star$.*
(c) *$G \subset G^\star$ for every open set G.*
(d) *$G \subset G^\star$ for every semiopen set G* [17, Lemma 1(c)].

Lemma 1.2. [17, Lemma 2] *Let $(X,\tau,\mathcal{I})$ be an ideal space. If A is $\star$-dense in itself, then $A^\star = cl(A) = cl^\star(A)$.*

Lemma 1.3. [17, Theorem 3.1(b)] *Let $(X,\tau,\mathcal{I})$ be an ideal space. A subset A of X is $\mathcal{I}$-locally closed if and only if $A = G \cap A^\star$ for some open set G.*

Lemma 1.4. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $\Delta = \{A \subset X \mid A \subset A^\star\}$. Then $\Delta \cap \mathcal{I} = \{\emptyset\}$.*

Proof. Suppose $A \in \Delta \cap \mathcal{I}$. Then $A \in \mathcal{I}$ implies $A^\star = \emptyset$ and $A \in \Delta$ implies that $A \subset A^\star$. Therefore, $A = \emptyset$ which implies that $\Delta \cap \mathcal{I} = \{\emptyset\}$.

2. $\mathcal{A}_\mathcal{I}$-sets.

A subset A of an ideal space $(X, \tau, \mathcal{I})$ is called an $\mathcal{A}_\mathcal{I}$*-set* [11] if $A = U \cap V$ where $U \in \tau$ and V is regular $\mathcal{I}$-closed. The family of all $\mathcal{A}_\mathcal{I}$-sets in a space $(X, \tau, \mathcal{I})$ will be denoted by $\mathcal{A}_\mathcal{I}(X, \tau)$. The following Theorem 2.1 gives some properties of $\mathcal{A}_\mathcal{I}$-sets.

THEOREM 2.1. (i) *If A is an $\mathcal{A}_\mathcal{I}$-set of an ideal space $(X, \tau, \mathcal{I})$, then the following hold.*

(a) *A and $int(A)$ are $\star$-dense in itself.*

(b) *$A^\star = cl(A) = cl^\star(A)$ and $(int(A))^\star = cl(int(A))$.*

(c) *A is an $f_\mathcal{I}$-set.*

(d) *$A^\star = (int(A))^\star = ((int(A))^\star)^\star = (A^\star)^\star$.*

(e) *$A^\star$ and $(int(A))^\star$ are $\star$-perfect and $\mathcal{I}$-locally closed.*

(f) *$A^\star(\mathcal{I}) = cl(int(A)) = A^\star(\widetilde{\mathcal{I}})$ is regular closed.*

(g) *$A^\star = pcl(A)$.*

(h) *$A^\star$ is regular $\mathcal{I}$-closed.*

(ii) *In any ideal space $(X, \tau, \mathcal{I})$, $\mathcal{A}_\mathcal{I}(X, \tau) \cap \mathcal{I} = \{\emptyset\}$.*

Proof. (i) (a) If A is an $\mathcal{A}_\mathcal{I}$-set, then $A = U \cap V$ where $U \in \tau$ and V is regular $\mathcal{I}$-closed. Therefore, $A = U \cap V = U \cap (int(V))^\star \subset (U \cap int(V))^\star = (int(U \cap V))^\star = (int(A))^\star \subset A^\star$ which implies that $int(A) \subset A \subset (int(A))^\star \subset A^\star$. Therefore, A and $int(A)$ are $\star$-dense in itself.

(b) By Lemma 1.2, we have $A^\star = cl(A) = cl^\star(A)$ and $(int(A))^\star = cl(int(A))$.

(c) From (a), $A \subset (int(A))^\star$ and so A is an $f_\mathcal{I}$-set.

(d) From (a), we have $int(A) \subset A \subset (int(A))^\star \subset A^\star$ and so $(int(A))^\star \subset A^\star \subset ((int(A))^\star)^\star \subset (int(A))^\star \subset A^\star$ and so $A^\star = (int(A))^\star = ((int(A))^\star)^\star = (A^\star)^\star$.

(e) From (d), it follows that $A^\star$ and $(int(A))^\star$ are $\star$-perfect and hence are $\mathcal{I}$-locally closed.

(f) From (d), $A^\star = (int(A))^\star$ and so by (b), $A^\star = cl(int(A))$. Since A is $\star$-dense in itself, $A^\star \subset cl(int(A^\star))$. Since $cl(int(A^\star)) = A^\star(\widetilde{\mathcal{I}}) \subset A^\star(\mathcal{I})$, we have $A^\star(\mathcal{I}) = cl(int(A)) = A^\star(\widetilde{\mathcal{I}})$ and each is regular closed, since $A^\star(\widetilde{\mathcal{I}})$ is regular closed [10, Theorem 3.2].

(g) Since $A^\star = cl^\star(A) = A \cup A^\star = A \cup cl(int(A))$ by (f), $A^\star = pcl(A)$.

(h) From (d), $A^\star = (int(A))^\star$. Let $B = (int(A))^\star$. Then $(int(B))^\star = (int(int(A))^\star)^\star = (int(A^\star))^\star \supset (int(A))^\star = B$, since A is $\star$-dense in itself. Therefore, $B \subset (int(B))^\star$. Also, $int(B) \subset B$ implies that $(int(B))^\star \subset B^\star = ((int(A))^\star)^\star \subset (int(A))^\star = B$ and so $(int(B))^\star \subset B$. Therefore, $B = (int(B))^\star$ which implies that B is regular $\mathcal{I}$-closed. Therefore, $A^\star$ is regular $\mathcal{I}$-closed.

(ii) The proof follows from Lemma 1.4. ∎

The following Theorems 2.2 and 2.4 give characterizations of codense ideals in terms of $\mathcal{A}_{\mathcal{I}}$-sets.

THEOREM 2.2. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space. Then* $\mathcal{I}$ *is codense if and only if* $\tau \subset \mathcal{A}_{\mathcal{I}}(X,\tau)$.

Proof. If $\mathcal{I}$ is codense, by Proposition 4(a) of [11], $\tau \subset \mathcal{A}_{\mathcal{I}}(X,\tau)$. Conversely, suppose the condition holds. By Theorem 2.1(ii), $\mathcal{A}_{\mathcal{I}}(X,\tau) \cap \mathcal{I} = \{\emptyset\}$ and so $\tau \cap \mathcal{I} = \{\emptyset\}$. Therefore, $\mathcal{I}$ is codense. ∎

COROLLARY 2.3. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space. Then the following are equivalent.*

(a) $\mathcal{I}$ *is codense.*
(b) $\tau = P_{\mathcal{I}}O(X,\tau) \cap \mathcal{A}_{\mathcal{I}}(X,\tau)$.
(c) $\tau = \alpha_{\mathcal{I}}O(X,\tau) \cap \mathcal{A}_{\mathcal{I}}(X,\tau)$.
(d) $\tau \subset \mathcal{A}_{\mathcal{I}}(X,\tau)$.

Proof. (a) implies (b) and (a) implies (c) follow from Proposition 6 of [11]. It is clear that (b) implies (d) and (c) implies (d). (d) implies (a) by Theorem 2.2. ∎

Every $\mathcal{A}_{\mathcal{I}}$-set is an $\mathcal{A}$-set [11, Proposition 5(b)] but not the converse [11, Example 5(3)]. Theorem 2.5 below shows that these two collection of sets are equal, if the ideal is codense and also it gives another characterization of codense ideals in terms of $\mathcal{A}_{\mathcal{I}}$-sets. Before that, we prove the following Theorem 2.4 which gives a characterization of codense ideals in terms of regular $\mathcal{I}$-closed sets.

THEOREM 2.4. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space. Then* $\mathcal{I}$ *is codense if and only if* $\mathcal{R}_{\mathcal{I}}C(X,\tau) = RC(X,\tau)$ *where* $\mathcal{R}_{\mathcal{I}}C(X,\tau)$ *is the collection of all regular* $\mathcal{I}$*-closed sets in* $(X,\tau,\mathcal{I})$.

Proof. Suppose $\mathcal{I}$ is codense. Then $A \in \mathcal{R}_{\mathcal{I}}C(X,\tau)$ if and only if $A = (int(A))^\star$ if and only if $A = cl(int(A))$, by Lemma 1.1(c) and Lemma 1.2, if and only if $A \in RC(X,\tau)$. Conversely, suppose $\mathcal{R}_{\mathcal{I}}C(X,\tau) = RC(X,\tau)$. Since X is regular closed, X is regular $\mathcal{I}$-closed and so $X = (int(X))^\star = X^\star$ which implies that $\mathcal{I}$ is codense, by Lemma 1.1(b). ∎

THEOREM 2.5. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space. Then* $\mathcal{I}$ *is codense if and only if* $\mathcal{A}_{\mathcal{I}}(X,\tau) = \mathcal{A}(X,\tau)$ *where* $\mathcal{A}(X,\tau)$ *is the collection of all* $\mathcal{A}$*-sets in* (X,τ).

Proof. Suppose $\mathcal{I}$ is codense. $\mathcal{A}_{\mathcal{I}}(X,\tau) \subset \mathcal{A}(X,\tau)$ by [11, Proposition 5(b)]. On the other hand, $A \in \mathcal{A}(X,\tau)$ implies that $A = U \cap V$ where $U \in \tau$ and $V \in RC(X,\tau)$ and so $A = U \cap V$ where $U \in \tau$ and $V \in \mathcal{R}_{\mathcal{I}}C(X,\tau)$, by Theorem 2.4. So, $A \in \mathcal{A}_{\mathcal{I}}(X,\tau)$. Hence $\mathcal{A}_{\mathcal{I}}(X,\tau) = \mathcal{A}(X,\tau)$. Conversely, suppose $\mathcal{A}_{\mathcal{I}}(X,\tau) = \mathcal{A}(X,\tau)$. Since X is an $\mathcal{A}$-set, X is an $\mathcal{A}_{\mathcal{I}}$-set and so $X \subset X^\star$, by Theorem 2.1(a). Therefore $X = X^\star$ which implies that $\mathcal{I}$ is codense. ∎

A function $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is said to be $\mathcal{A}_{\mathcal{I}} -$ *continuous* [11] (resp. $\mathcal{A}$-*continuous* [18]) if $f^{-1}(V)$ is an $\mathcal{A}_{\mathcal{I}}$-set(resp. $\mathcal{A}$-set) in X for every open set V in Y. Every $\mathcal{A}_{\mathcal{I}}$-continuous function is $\mathcal{A}$-continuous [11, Proposition 7(c)] but not the converse [11, Example 6(3)]. The following Theorem 2.6 shows that the two concepts are equivalent, if the ideal $\mathcal{I}$ is codense.

THEOREM 2.6. *Let* $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ *be a mapping and* $\mathcal{I}$ *be codense. Then* f *is* $\mathcal{A}_{\mathcal{I}}$*-continuous if and only if* f *is* $\mathcal{A}$*-continuous.*

Proof. The proof follows from Theorem 2.5. ∎

Every $\mathcal{A}_{\mathcal{I}}$-set is $\mathcal{I}$-locally closed [11, Proposition 5(a)] but not the converse [11, Example 5(2)]. The following Theorem 2.7 shows that every $\mathcal{A}_{\mathcal{I}}$-set is an $f_{\mathcal{I}}$-set and characterizes $\mathcal{A}_{\mathcal{I}}$-set in terms of $f_{\mathcal{I}}$-set and $\mathcal{I}$-locally closed set. Example 2.8 below shows that $f_{\mathcal{I}}$-sets need not be $\mathcal{A}_{\mathcal{I}}$-sets.

THEOREM 2.7. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space. Then* A *is an* $\mathcal{A}_{\mathcal{I}}$*-set if and only if* A *is both an* $f_{\mathcal{I}}$*-set and an* $\mathcal{I}$*-locally closed set.*

Proof. Suppose A is an $\mathcal{A}_{\mathcal{I}}$-set. Then A is $\mathcal{I}$-locally closed by [11, Proposition 5(a)]. Also, $A = U \cap V$ where $U \in \tau$ and $V \in \mathcal{R}_{\mathcal{I}}C(X,\tau)$ and so $int(A) = int(U \cap V) = U \cap int(V)$. Now $A = U \cap V$ implies that $A = U \cap (int(V))^\star \subset (U \cap int(V))^\star = (int(A))^\star$. Therefore, A is an $f_{\mathcal{I}}$-set. Conversely, suppose A is both an $f_{\mathcal{I}}$-set and an $\mathcal{I}$-locally closed set. Since A is an $f_{\mathcal{I}}$-set, $A \subset (int(A))^\star$ implies that $A^\star \subset ((int(A))^\star)^\star \subset (int(A))^\star \subset A^\star$ and so $A^\star = (int(A))^\star$. As in the proof of Theorem 2.1(h), we can prove that $A^\star$ is regular $\mathcal{I}$-closed. A is $\mathcal{I}$-locally closed implies that $A = U \cap A^\star$ for some $U \in \tau$, by Lemma 1.3. Since $A^\star$ is regular $\mathcal{I}$-closed, it follows that A is an $\mathcal{A}_{\mathcal{I}}$-set. ∎

EXAMPLE 2.8. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. If $A = \{a,c\}$, then $(int(A))^\star = \{a,c,d\}$ and so A is an $f_{\mathcal{I}}$-set. Since X is the only open containing A and A is not regular $\mathcal{I}$-closed, A is not an $\mathcal{A}_{\mathcal{I}}$-set.

A function $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is said to be *$f_{\mathcal{I}}$-continuous* [12] (resp. *$\mathcal{I}LC$-continuous* [3], *semicontinuous* [14], *LC-continuous* [5]) if $f^{-1}(V)$ is an $f_{\mathcal{I}}$-set (resp. $\mathcal{I}$-locally closed set, semiopen set, locally closed set) in X for every open set V in Y. Every $\mathcal{A}_{\mathcal{I}}$-continuous function is $\mathcal{I}LC$-continuous [11, Proposition 7(b)] but not the converse [11, Example 6(2)]. The following Theorem 2.9 shows that the converse is true, if f is $f_{\mathcal{I}}$-continuous and hence we have a decomposition of $\mathcal{A}_{\mathcal{I}}$-continuous functions.

THEOREM 2.9. *A function* $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ *is* $\mathcal{A}_{\mathcal{I}}$*-continuous if and only if* f *is both* $f_{\mathcal{I}}$*-continuous and* $\mathcal{I}LC$*-continuous.*

Proof. The proof follows from Theorem 2.7. ∎

THEOREM 2.10. *Let* $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ *be a mapping and* $\mathcal{I}$ *be codense. Then the following are equivalent.*

(a) f *is* $\mathcal{A}$*-continuous.*
(b) f *is* $\mathcal{A}_{\mathcal{I}}$*-continuous.*
(c) f *is both* $f_{\mathcal{I}}$*-continuous and* $\mathcal{I}LC$*-continuous.*
(d) f *is both semicontinuous and* LC*-continuous.*

Proof. (a) and (b) are equivalent, by Theorem 2.5. (b) and (c) are equivalent, by Theorem 2.9. (d) and (a) are equivalent by [5, Theorem 4(i)]. The proof will be over, if we prove (c) implies (d). From Lemma 1.3, it follows that every $\mathcal{I}$-locally closed set in $(X, \tau, \mathcal{I})$ is locally closed. Suppose A is an $f_{\mathcal{I}}$-set. Then $A \subset (int(A))^\star = cl(int(A))$ and so $A \in SO(X, \tau)$. This completes the proof. ■

The following Theorem 2.11 shows that the three collection of sets namely, $f_{\mathcal{I}}$-sets, $\mathcal{A}_{\mathcal{I}}$-sets and $\mathcal{I}$-locally closed sets coincide for the collection of open sets. The Example 2.12 below show that the condition *open* cannot be dropped.

THEOREM 2.11. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$ be open. Then the following hold.*

(a) *A is an $f_{\mathcal{I}}$-set if and only if A is an $\mathcal{A}_{\mathcal{I}}$-set.*

(b) *A is an $\mathcal{I}$-locally closed set if and only if A is an $\mathcal{A}_{\mathcal{I}}$-set.*

Proof. (a) Suppose A is an open, $f_{\mathcal{I}}$-set. Then $A \subset (int(A))^\star \subset A^\star$ and so $A^\star = (int(A))^\star$ which implies that $A^\star$ is regular $\mathcal{I}$-closed. Since $A = A \cap A^\star$, it follows that A is an $\mathcal{A}_{\mathcal{I}}$-set. Conversely, if A is an $\mathcal{A}_{\mathcal{I}}$-set, by Theorem 2.7, A is an $f_{\mathcal{I}}$-set.

(b) Suppose A is an open, $\mathcal{I}$-locally closed set. Then $A = G \cap A^\star$ for some $G \in \tau$, by Lemma 1.3 and so $A \subset A^\star$. We prove that $A^\star$ is regular $\mathcal{I}$-closed. Since $int(A^\star) \subset A^\star$, we have $(int(A^\star))^\star \subset (A^\star)^\star \subset A^\star$. Therefore, $(int(A^\star))^\star \subset A^\star$. On the other hand, since A is open and $\star$-dense in itself, $A^\star = (int(A))^\star \subset (int(A^\star))^\star$ and so $A^\star = (int(A^\star))^\star$ which implies that $A^\star$ is regular $\mathcal{I}$-closed. Therefore, A is an $\mathcal{A}_{\mathcal{I}}$-set. Conversely, if A is an $\mathcal{A}_{\mathcal{I}}$-set, by Theorem 2.7, A is an $\mathcal{I}$-locally closed set. ■

EXAMPLE 2.12. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. If $A = \{a, c\}$, then $int(A) = \{a\}$ and $(int(A))^\star = \{a, c, d\}$ and so A is an $f_{\mathcal{I}}$-set which is not open. Since X is the only open set containing A and $\mathcal{R}_{\mathcal{I}}C(X, \tau) = \{\emptyset, \{a, c, d\}, \{b, c, d\}, X\}$, A is not an $\mathcal{A}_{\mathcal{I}}$-set. If $B = \{d\}$, then $B^\star = \{c, d\}$ and $B = \{a, b, d\} \cap B^\star$ and so B is $\mathcal{I}$-locally closed which is not open. Since X and $\{a, b, d\}$ are the only open sets containing B, it follows that B is not an $\mathcal{A}_{\mathcal{I}}$-set.

3. $\mathcal{I}$-locally closed sets

In this section, we characterize codense ideals in terms of $\mathcal{I}$-locally closed sets. Theorem 3.3 gives a decomposition of continuity. We deduce some results established in [5] as corollaries to Theorem 3.3.

THEOREM 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{I}LC(X, \tau) \cap \mathcal{I} = \{\emptyset\}$.*

Proof. Let $A \in \mathcal{I}LC(X, \tau)$. Then by Lemma 1.3, $A \subset A^\star$ and so by Lemma 1.4, $\mathcal{I}LC(X, \tau) \cap \mathcal{I} = \{\emptyset\}$. ■

THEOREM 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.*

(a) $\mathcal{I}$ *is codense.*
(b) $\tau = P_{\mathcal{I}}O(X,\tau) \cap \mathcal{I}LC(X,\tau)$.
(c) $\tau = \alpha_{\mathcal{I}}O(X,\tau) \cap \mathcal{I}LC(X,\tau)$.
(d) $\tau \subset \mathcal{I}LC(X,\tau)$.

Proof. (a) implies (b) follows from [3, Proposition 4.1]. (b) implies (d) and (c) implies (d) are clear. (d) implies (a) follows from Theorem 3.1. Therefore, the proof will be over, if we prove (a) implies (c). Suppose $\mathcal{I}$ is codense. If A is open, then A is $\alpha - \mathcal{I}$-open and $A \subset A^\star$. By Lemma 1.3, it follows that A is $\mathcal{I}$-locally closed. Conversely, suppose A is both $\alpha - \mathcal{I}$-open and $\mathcal{I}$-locally closed. A is $\mathcal{I}$-locally closed implies $A = U \cap A^\star$ for some open set U. A is $\alpha - \mathcal{I}$-open implies $A \subset int(cl^\star(int(A))) \subset int(cl^\star(A)) = int(cl^\star(U \cap A^\star)) \subset int(cl^\star(A^\star)) = int(A^\star)$. Since $A \subset U$, $A \subset U \cap int(A^\star) = int(U \cap A^\star) = int(A)$ and so A is open. This completes the proof of the theorem. ∎

A function $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is said to be *$\alpha - \mathcal{I}$-continuous* [6] (resp. *pre-$\mathcal{I}$-continuous* [3], *α-continuous* [16], *pre-continuous* [15]) if $f^{-1}(V)$ is an $\alpha - \mathcal{I}$-open (resp. *pre-$\mathcal{I}$*-open, α-open, preopen) set in X for every open set V in Y. The following Theorem 3.3, which is a decomposition of continuous function in ideal topological spaces, follows from Theorem 3.2.

THEOREM 3.3. *Let $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ be a mapping and $\mathcal{I}$ be codense. Then the following are equivalent.*
(a) *f is continuous.*
(b) *f is α-$\mathcal{I}$-continuous and $\mathcal{I}LC$-continuous.*
(c) *f is pre-$\mathcal{I}$-continuous and $\mathcal{I}LC$-continuous [3, Theorem 4.3].*

COROLLARY 3.4. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a mapping. Then the following are equivalent.*
(a) *f is continuous.*
(b) *f is α-continuous and LC-continuous [5, Theorem 4(ii)].*
(c) *f is pre-continuous and LC-continuous [5, Theorem 4(iv)].*

Proof. Suppose $\mathcal{I} = \{\emptyset\}$ in Theorem 3.3. If $\mathcal{I} = \{\emptyset\}$, then α-open sets coincide with $\alpha - \mathcal{I}$-open sets, preopen sets coincide with pre-$\mathcal{I}$-open sets and locally closed sets coincide with $\mathcal{I}$-locally closed sets. Hence the proof follows from Theorem 3.3. ∎

COROLLARY 3.5. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a mapping. Then the following are equivalent.*
(a) *f is continuous.*
(b) *f is α-continuous and $\mathcal{A}$-continuous.*
(c) *f is pre-continuous and $\mathcal{A}$-continuous [5, Theorem 4(v)].*

Proof. Suppose $\mathcal{I} = \mathcal{N}$ in Theorem 3.3. If $\mathcal{I} = \mathcal{N}$, then α-open sets coincide with α-$\mathcal{I}$-open sets, preopen sets coincide with pre-$\mathcal{I}$-open sets and $\mathcal{A}$-sets coincide with $\mathcal{I}$-locally closed sets [3]. Hence the proof follows from Theorem 3.3. ∎

4. Almost strong $\mathcal{I}$-open sets

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be *almost strong $\mathcal{I}$-open* [7] if $A \subset cl^\star(int(A^\star))$. Every $\mathcal{I}$-open set is an almost strong $\mathcal{I}$-open set but not the converse [7]. We will denote the family of all almost strong $\mathcal{I}$-open sets by $as\mathcal{I}O(X,\tau)$. The following Theorem 4.1 gives some properties of almost strong $\mathcal{I}$-open sets.

THEOREM 4.1. (i) *If A is an almost strong $\mathcal{I}$-open set of an ideal space $(X,\tau,\mathcal{I})$, then the following hold:*
(a) *A is $\star$-dense in itself.*
(b) $A^\star = cl(A) = cl^\star(A)$.
(c) $A^\star(\mathcal{I}) = (cl^\star(int(A^\star)))^\star = (cl(int(A^\star)))^\star = (A^\star)^\star = (A^\star(\widetilde{\mathcal{I}}))^\star(\mathcal{I})$.
(d) *$A^\star$ is $\star$-perfect, regular closed and $\mathcal{I}$-locally closed.*
(e) $A^\star = A^\star(\widetilde{\mathcal{I}})$.
(f) *$(cl^\star(int(A^\star)))^\star$ is $\star$-perfect and $\mathcal{I}$-locally closed.*
(ii) *In any ideal space $(X,\tau,\mathcal{I})$, $as\mathcal{I}O(X,\tau) \cap \mathcal{I} = \{\emptyset\}$.*

Proof. (i)(a) Since $A \subset cl^\star(int(A^\star)) \subset cl(int(A^\star)) \subset cl(A^\star) = A^\star$, A is $\star$-dense in itself.

(b) Follows from Lemma 1.2 and (a).

(c) Follows from the inequality in (a) and the fact that $cl(int(A^\star)) = A^\star(\widetilde{\mathcal{I}})$ [10, Theorem 3.2].

(d) $A^\star$ is $\star$-perfect by (c) and hence it is $\mathcal{I}$-locally closed. Since every almost strong $\mathcal{I}$-open set is β-open [7] and the closure of a β-open set is regular closed, by (b), $A^\star$ is regular closed.

(e) Since $A^\star$ is regular closed, by (d), we have $A^\star = cl(int(A^\star)) = A^\star(\widetilde{\mathcal{I}})$.

(f) Since $A \subset cl^\star(int(A^\star)) \subset A^\star$, it follows that $cl^\star(int(A^\star))$ is $\star$-perfect and hence it is $\mathcal{I}$-locally closed.

(ii) Follows from Lemma 1.4. ∎

COROLLARY 4.2. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $\mathcal{I}$ be codense. If $A \subset X$ is almost strong $\mathcal{I}$-open, then $A^\star$ is regular $\mathcal{I}$-closed and an $f_{\mathcal{I}}$-set. Moreover, if A is $\star$-closed, then A is regular $\mathcal{I}$-closed and an $f_{\mathcal{I}}$-set.*

Proof. By Theorem 4.1(d), $A^\star$ is regular closed. By Theorem 2.4, $A^\star$ is regular $\mathcal{I}$-closed and hence an $f_{\mathcal{I}}$-set. If A is $\star$-closed, then A is $\star$-perfect and so $A = A^\star$. Therefore, A is regular $\mathcal{I}$-closed and hence A is an $f_{\mathcal{I}}$-set. ∎

The following Theorem 4.3 gives a characterization of codense ideals in terms of almost strong $\mathcal{I}$-open sets. Theorem 4.4 below gives another property of almost strong $\mathcal{I}$-open sets.

THEOREM 4.3. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is codense if and only if $S\mathcal{I}O(X,\tau) \subset as\mathcal{I}O(X,\tau)$.*

Proof. Suppose $\mathcal{I}$ is codense. If $A \in S\mathcal{I}O(X,\tau)$, then $A \subset cl^\star(int(A)) \subset cl(int(A))$ and so A is semiopen. By Lemma 1.1(d), $A \subset A^\star$. Therefore, $A \subset$

$cl^\star(int(A^\star))$ and so $A \in as\mathcal{I}O(X,\tau)$. Conversely, suppose the condition holds. Since $\tau \subset S\mathcal{I}O(X,\tau) \subset as\mathcal{I}O(X,\tau)$, by Theorem 4.1(ii), $\tau \cap \mathcal{I} = \{\emptyset\}$ and so $\mathcal{I}$ is codense. ■

THEOREM 4.4. *Let $(X,\tau,\mathcal{I})$ be an ideal space and A and B be subsets of X such that $A \subset B \subset A^\star$. If A is almost strong $\mathcal{I}$-open, then B is almost strong $\mathcal{I}$-open and so $cl^\star(int(A^\star))$ is almost strong $\mathcal{I}$-open.*

Proof. If $A \subset B \subset A^\star$, then $A^\star \subset B^\star \subset (A^\star)^\star \subset A^\star$ and so $A^\star = B^\star$ which implies that B is $\star$-dense in itself and $B^\star$ is $\star$-perfect. If $A \in as\mathcal{I}O(X,\tau)$, then $A \subset cl^\star(int(A^\star)) = cl^\star(int(B^\star))$. Now $B \subset A^\star$ implies $B \subset (cl^\star(int(B^\star)))^\star \subset cl^\star(cl^\star(int(B^\star))) = cl^\star(int(B^\star))$ and so B is an almost strong $\mathcal{I}$-open set. Since $A \subset cl^\star(int(A^\star)) \subset A^\star$, $cl^\star(int(A^\star))$ is an almost strong $\mathcal{I}$-open set. ■

We define the *almost strong $\mathcal{I}$-interior* of any subset A of X as the largest almost strong $\mathcal{I}$-open set contained in A and denote it by $as\mathcal{I}int(A)$. The following Theorem 4.5 deals with the almost strong $\mathcal{I}$-interior of subsets of X. Moreover, Theorem 4.5(b) is a generalization of Theorem 4.1(4) of [10].

THEOREM 4.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then the following hold.*
(a) *$as\mathcal{I}int(A) = A \cap cl^\star(int(A^\star))$.*
(b) *$as\mathcal{I}int(A) = \emptyset$ if and only if $A \in \widetilde{\mathcal{I}}$.*

Proof. (a) $A \cap cl^\star(int(A^\star)) \subset cl^\star(int(A^\star)) = cl^\star(int(int(A^\star))) = cl^\star(int(A^\star \cap int(A^\star))) \subset cl^\star(int(A \cap int(A^\star))^\star) \subset cl^\star(int(A \cap cl^\star(int(A^\star)))^\star)$. Therefore, $A \cap cl^\star(int(A^\star))$ is an almost strong $\mathcal{I}$-open set contained in A. Hence $A \cap cl^\star(int(A^\star)) \subset as\mathcal{I}int(A)$. Since $as\mathcal{I}int(A)$ is almost strong $\mathcal{I}$-open, $as\mathcal{I}int(A) \subset cl^\star(int(as\mathcal{I}int(A))^\star) \subset cl^\star(int(A^\star))$ and so $A \cap as\mathcal{I}int(A) \subset A \cap cl^\star(int(A^\star))$ which implies that $as\mathcal{I}int(A) \subset A \cap cl^\star(int(A^\star))$. Therefore, $as\mathcal{I}int(A) = A \cap cl^\star(int(A^\star))$.

(b) $as\mathcal{I}int(A) = \emptyset$ implies $A \cap cl^\star(int(A^\star)) = \emptyset$ implies $A \cap int(A^\star) = \emptyset$ implies $\mathcal{I}int(A) = \emptyset$ implies $A \in \widetilde{\mathcal{I}}$, by [10, Theorem 4.1(4)]. Conversely, $A \in \widetilde{\mathcal{I}}$ implies $int(A^\star) = \emptyset$ implies $cl^\star(int(A^\star)) = \emptyset$ implies $A \cap cl^\star(int(A^\star)) = \emptyset$ implies $as\mathcal{I}int(A) = \emptyset$. ■

In [7], it is established that the intersection of an almost strong $\mathcal{I}$-open set with an open set is always an almost strong $\mathcal{I}$-open set. The following Theorem 4.6 shows that, in the above result, open set can be replaced by α-$\mathcal{I}$-open set.

THEOREM 4.6. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If A is almost strong $\mathcal{I}$-open and B is α-$\mathcal{I}$-open, then $A \cap B$ is almost strong $\mathcal{I}$-open.*

Proof. A is almost strong $\mathcal{I}$-open implies $A \subset cl^\star(int(A^\star))$ and B is α-$\mathcal{I}$-open implies $B \subset int(cl^\star(int(B)))$. Now, $A \cap B \subset cl^\star(int(A^\star)) \cap int(cl^\star(int(B))) = (int(A^\star) \cup (int(A^\star))^\star) \cap int(cl^\star(int(B))) = (int(A^\star) \cap int(cl^\star(int(B)))) \cup ((int(A^\star))^\star \cap int(cl^\star(int(B)))) \subset int(int(A^\star) \cap cl^\star(int(B))) \cup (int(A^\star) \cap int(cl^\star(int(B))))^\star \subset int(cl^\star(int(A^\star) \cap int(B))) \cup (int(int(A^\star) \cap cl^\star(int(B))))^\star \subset int(cl^\star(int(A^\star \cap int(B)))) \cup (int(cl^\star(int(A^\star) \cap int(B))))^\star \subset int(cl^\star(int((A \cap int(B))^\star))) \cup (int(cl^\star(int(A^\star \cap

$int(B)))))^{\star} \subset int(cl^{\star}(int((A \cap int(B))^{\star}))) \cup (int(cl^{\star}(int((A \cap int(B))^{\star}))))^{\star} = cl^{\star}(int(cl^{\star}(int((A \cap int(B))^{\star})))) \subset cl^{\star}(int(cl^{\star}((A \cap int(B))^{\star}))) = cl^{\star}(int((A \cap int(B))^{\star})) \subset cl^{\star}(int((A \cap B)^{\star}))$. Therefore, $A \cap B$ is almost strong $\mathcal{I}$-open. ■

Acknowledgement. The authors sincerely thank Professor T. Noiri (Japan) for sending some of his reprints which were useful for the preparation of this manuscript and the referee for the valuable comments for a better presentation of the paper.

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(received 05.01.2005, in revised form 12.09.2007)

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