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SOME SUBCLASSES OF CLOSE-TO-CONVEX AND QUASI-CONVEX FUNCTIONS

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Abstract. In the present paper, the author introduce two new subclasses $\mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$ of close-to-convex functions and $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$ of quasi-convex functions with respect to $2k$-symmetric conjugate points. The coefficient inequalities and integral representations for functions belonging to these classes are provided, the inclusion relationships and convolution conditions for these classes are also provided.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U}=\{z\in\mathbf{C}:|z|<1\}$. Let $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{K}$, $\mathcal{C}$ and $\mathcal{C}^*$ denote the familiar subclasses of $\mathcal{A}$ consisting of functions which are univalent, starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$, respectively (see, for details, [2,4,6,7,8].

Al-Amiri, Coman and Mocanu [1] once introduced and investigated a class $\mathcal{S}_{sc}^{(k)}$ of functions starlike with respect to $2k$-symmetric conjugate points, which satisfy the following inequality

$$\Re\left\{\frac{zf'(z)}{f_{2k}(z)}\right\}>0 \quad (z\in\mathcal{U}),$$

where $k\geq 2$ is a fixed positive integer and $f_{2k}(z)$ is defined by the following equality

$$f_{2k}(z)=\frac{1}{2k}\sum_{\nu=0}^{k-1}\left[\varepsilon^{-\nu}f(\varepsilon^{\nu}z)+\varepsilon^{\nu}\overline{f(\varepsilon^{\nu}\overline{z})}\right] \quad (\varepsilon=\exp(2\pi i/k);\ z\in\mathcal{U}). \tag{1.2}$$

In the present paper, we introduce and investigate the following two more generalized subclasses $\mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$ and $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$ of $\mathcal{A}$ with respect to $2k$-symmetric conjugate points, and obtain some interesting results.

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Definition 1. Let $\mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$ denote the class of functions $f(z)$ in $\mathcal{A}$ satisfying the following inequality

$$\left|\frac{\frac{zf'(z)}{f_{2k}(z)}-1}{\beta\frac{zf'(z)}{f_{2k}(z)}+(1-\gamma)}\right|<1-\alpha, \tag{1.3}$$

where $0\leq\alpha<1,\ 0\leq\beta\leq1,\ 0\leq\gamma<1$ and $f_{2k}(z)$ is defined by equality (1.2). And a function $f(z)\in\mathcal{A}$ is in the class $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$ if and only if $zf'(z)\in\mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$.

Note that $\mathcal{S}_{sc}^{(k)}(0,1,0)=\mathcal{S}_{sc}^{(k)}$, so the class $\mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$ is a generalization of the class $\mathcal{S}_{sc}^{(k)}$.

In our proposed investigation of the classes $\mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$ and $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$, we shall also make use of the following lemmas.

Lemma 1. [3] *Let* $H(z)=1+\sum_{n=1}^{\infty}h_nz^n$ *be analytic in* $\mathcal{U}$, $0\leq\alpha<1,\ 0\leq\beta\leq1$ *and* $0\leq\gamma<1$, *then the inequality*

$$\left|\frac{H(z)-1}{\beta H(z)+(1-\gamma)}\right|<1-\alpha\quad(z\in\mathcal{U})$$

can be written as

$$H(z)\prec\frac{1+(1-\alpha)(1-\gamma)z}{1-(1-\alpha)\beta z}\quad(z\in\mathcal{U}),$$

where "$\prec$" stands for the usual subordination.

Lemma 2. *Let* $0\leq\alpha<1,\ 0\leq\beta\leq1$ *and* $0\leq\gamma<1$, *then we have*

$$\mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)\subset\mathcal{C}\subset\mathcal{S}.$$

Proof. Suppose that $f(z)\in\mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$, by Lemma 1, we know that the condition (1.3) can be written as

$$\frac{zf'(z)}{f_{2k}(z)}\prec\frac{1+(1-\alpha)(1-\gamma)z}{1-(1-\alpha)\beta z}\quad(z\in\mathcal{U}). \tag{1.4}$$

Thus we have

$$\Re\left\{\frac{zf'(z)}{f_{2k}(z)}\right\}>0\quad(z\in\mathcal{U}) \tag{1.5}$$

since

$$\Re\left\{\frac{1+(1-\alpha)(1-\gamma)z}{1-(1-\alpha)\beta z}\right\}>0\quad(z\in\mathcal{U}).$$

Now it suffices to show that $f_{2k}(z)\in\mathcal{S}^*\subset\mathcal{S}$. Substituting z by $\varepsilon^\mu z$ ($\mu=0,1,2,\ldots,k-1$) in (1.5), then (1.5) is also true, that is,

$$\Re\left\{\frac{\varepsilon^\mu zf'(\varepsilon^\mu z)}{f_{2k}(\varepsilon^\mu z)}\right\}>0\quad(z\in\mathcal{U}). \tag{1.6}$$

From inequality (1.6), we have

$$\Re\left\{\frac{\overline{\varepsilon^{\mu}\overline{z}}\ \overline{f'(\varepsilon^{\mu}\overline{z})}}{\overline{f_{2k}(\varepsilon^{\mu}\overline{z})}}\right\} > 0 \quad (z \in \mathcal{U}). \tag{1.7}$$

Note that $f_{2k}(\varepsilon^{\mu}z) = \varepsilon^{\mu} f_{2k}(z)$ and $\overline{f_{2k}(\varepsilon^{\mu}\overline{z})} = \varepsilon^{-\mu} f_{2k}(z)$, then inequalities (1.6) and (1.7) can be written as

$$\Re\left\{\frac{zf'(\varepsilon^{\mu}z)}{f_{2k}(z)}\right\} > 0 \quad (z \in \mathcal{U}), \tag{1.8}$$

and

$$\Re\left\{\frac{z\overline{f'(\varepsilon^{\mu}\overline{z})}}{f_{2k}(z)}\right\} > 0 \quad (z \in \mathcal{U}). \tag{1.9}$$

Summing inequalities (1.8) and (1.9), we can get

$$\Re\left\{\frac{z\left(f'(\varepsilon^{\mu}z) + \overline{f'(\varepsilon^{\mu}\overline{z})}\right)}{f_{2k}(z)}\right\} > 0 \quad (z \in \mathcal{U}). \tag{1.10}$$

Letting $\mu = 0, 1, 2, \ldots, k-1$ in (1.10), respectively, and summing them we can get

$$\Re\left\{\frac{z\left[\frac{1}{2k}\sum_{\mu=0}^{k-1}\left(f'(\varepsilon^{\mu}z) + \overline{f'(\varepsilon^{\mu}\overline{z})}\right)\right]}{f_{2k}(z)}\right\} > 0 \quad (z \in \mathcal{U}),$$

or equivalently,

$$\Re\left\{\frac{zf'_{2k}(z)}{f_{2k}(z)}\right\} > 0 \quad (z \in \mathcal{U}),$$

that is $f_{2k}(z) \in \mathcal{S}^* \subset \mathcal{S}$. This means that $\mathcal{S}_{sc}^{(k)}(\alpha, \beta, \gamma) \subset \mathcal{C} \subset \mathcal{S}$, hence the proof of Lemma 2 is complete. ■

Similarly, for the class $\mathcal{C}_{sc}^{(k)}(\alpha, \beta, \gamma)$, we have

Lemma 3. *Let $0 \leq \alpha < 1$, $0 \leq \beta \leq 1$ and $0 \leq \gamma < 1$, then we have*

$$\mathcal{C}_{sc}^{(k)}(\alpha, \beta, \gamma) \subset \mathcal{C}^* \subset \mathcal{C}.$$

Lemma 4. [5] *Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, then we have*

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z}.$$

In the present paper, we shall provide the coefficient inequalities and integral representations for functions belonging to the classes $\mathcal{S}_{sc}^{(k)}(\alpha, \beta, \gamma)$ and $\mathcal{C}_{sc}^{(k)}(\alpha, \beta, \gamma)$, we shall also provide the inclusion relationships and convolution conditions for these classes.

2. Inclusion relationships

We first give some inclusion relationships for the classes $\mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$ and $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$.

THEOREM 1. *Let* $0 \le \beta_2 \le \beta_1 \le 1,\ 0 \le \alpha_1 \le \alpha_2 < 1$ *and* $0 \le \gamma_1 \le \gamma_2 < 1$, *then we have*

$$\mathcal{S}_{sc}^{(k)}(\alpha_2,\beta_2,\gamma_2) \subset \mathcal{S}_{sc}^{(k)}(\alpha_1,\beta_1,\gamma_1).$$

Proof. Suppose that $f(z) \in \mathcal{S}_{sc}^{(k)}(\alpha_2,\beta_2,\gamma_2)$, by (1.4), we have

$$\frac{zf'(z)}{f_{2k}(z)} \prec \frac{1+(1-\alpha_2)(1-\gamma_2)z}{1-(1-\alpha_2)\beta_2 z}.$$

Since $0 \le \alpha_1 \le \alpha_2 < 1,\ 0 \le \beta_2 \le \beta_1 \le 1$ and $0 \le \gamma_1 \le \gamma_2 < 1$, then we have

$$-1 \le -(1-\alpha_1)\beta_1 \le -(1-\alpha_2)\beta_2 < (1-\alpha_2)(1-\gamma_2) \le (1-\alpha_1)(1-\gamma_1) \le 1.$$

Thus, by Lemma 4, we have

$$\frac{zf'(z)}{f_{2k}(z)} \prec \frac{1+(1-\alpha_2)(1-\gamma_2)z}{1-(1-\alpha_2)\beta_2 z} \prec \frac{1+(1-\alpha_1)(1-\gamma_1)z}{1-(1-\alpha_1)\beta_1 z},$$

that is $f(z) \in \mathcal{S}_{sc}^{(k)}(\alpha_1,\beta_1,\gamma_1)$. This means that $\mathcal{S}_{sc}^{(k)}(\alpha_2,\beta_2,\gamma_2) \subset \mathcal{S}_{sc}^{(k)}(\alpha_1,\beta_1,\gamma_1)$. ∎

Similarly, for the class $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$, we have

COROLLARY 1. *Let* $0 \le \beta_2 \le \beta_1 \le 1,\ 0 \le \alpha_1 \le \alpha_2 < 1$ *and* $0 \le \gamma_1 \le \gamma_2 < 1$, *then we have*

$$\mathcal{C}_{sc}^{(k)}(\alpha_2,\beta_2,\gamma_2) \subset \mathcal{C}_{sc}^{(k)}(\alpha_1,\beta_1,\gamma_1).$$

3. Coefficient inequalities

In this section, we give some coefficient inequalities for functions belonging to the classes $\mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$ and $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$.

THEOREM 2. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* $\mathcal{U}$, *if for* $0 \le \alpha < 1,\ 0 \le \beta \le 1$ *and* $0 \le \gamma < 1$, *we have*

$$\sum_{n=2}^{\infty} n[1+(1-\alpha)\beta]\,|a_n| + \sum_{l=1}^{\infty} [(1-\alpha)(1-\gamma)+1]\,|\Re(a_{lk+1})| \le (1-\alpha)(1+\beta-\gamma), \quad (3.1)$$

then $f(z) \in \mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$.

Proof. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and $f_{2k}(z)$ is defined by equality (1.2). We now let M be denoted by

$$\begin{aligned} M &:= |zf'(z) - f_{2k}(z)| - (1-\alpha)\,|\beta z f'(z) + (1-\gamma) f_{2k}(z)| \\ &= \left| \sum_{n=2}^{\infty} n a_n z^n - \sum_{n=2}^{\infty} \Re(a_n) c_n z^n \right| \\ &\quad - (1-\alpha) \left| \beta \left(z + \sum_{n=2}^{\infty} n a_n z^n \right) + (1-\gamma) \left(z + \sum_{n=2}^{\infty} \Re(a_n) c_n z^n \right) \right|, \end{aligned}$$

where

$$c_n = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu} = \begin{cases} 1, & n = lk+1, \\ 0, & n \neq lk+1. \end{cases} \quad (\varepsilon = \exp(2\pi i/k);\ l \in \mathbf{N} = \{1, 2, \dots\}). \tag{3.2}$$

Thus, for $|z| = r < 1$, we have

$$\begin{aligned} M &\leq \sum_{n=2}^{\infty} (n\,|a_n| + |\Re(a_n)|\,c_n) r^n \\ &\quad - (1-\alpha)\left[(1+\beta-\gamma)r - \sum_{n=2}^{\infty}[n\beta\,|a_n| + (1-\gamma)\,|\Re(a_n)|\,c_n]r^n\right] \\ &< \Bigg(\sum_{n=2}^{\infty}\{n[1+(1-\alpha)\beta]\,|a_n| + [(1-\alpha)(1-\gamma)+1]\,|\Re(a_n)|\,c_n\} \\ &\quad - (1-\alpha)(1+\beta-\gamma)\Bigg) r \\ &< \sum_{n=2}^{\infty}\{n[1+(1-\alpha)\beta]\,|a_n| + [(1-\alpha)(1-\gamma)+1]\,|\Re(a_n)|\,c_n\} - (1-\alpha)(1+\beta-\gamma) \\ &= \sum_{n=2}^{\infty} n[1+(1-\alpha)\beta]\,|a_n| + \sum_{l=1}^{\infty}[(1-\alpha)(1-\gamma)+1]\,|\Re(a_{lk+1})| - (1-\alpha)(1+\beta-\gamma). \end{aligned}$$

From inequality (3.1), we know that $M < 0$, thus we can get inequality (1.3), that is $f(z) \in \mathcal{S}_{sc}^{(k)}(\alpha, \beta, \gamma)$. This completes the proof of Theorem 2. ■

Similarly, for the class $\mathcal{C}_{sc}^{(k)}(\alpha, \beta, \gamma)$, we have

COROLLARY 2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathcal{U}$, if for $0 \leq \alpha < 1$, $0 \leq \beta \leq 1$ and $0 \leq \gamma < 1$, we have*

$$\sum_{n=2}^{\infty} n^2[1+(1-\alpha)\beta]\,|a_n| + \sum_{l=1}^{\infty}[(1-\alpha)(1-\gamma)+1](lk+1)\,|\Re(a_{lk+1})| \leq (1-\alpha)(1+\beta-\gamma),$$

then $f(z) \in \mathcal{C}_{sc}^{(k)}(\alpha, \beta, \gamma)$.

4. Integral representations

In this section, we provide the integral representations for functions belonging to the classes $\mathcal{S}_{sc}^{(k)}(\alpha, \beta, \gamma)$ and $\mathcal{C}_{sc}^{(k)}(\alpha, \beta, \gamma)$.

THEOREM 3. *Let $f(z) \in \mathcal{S}_{sc}^{(k)}(\alpha, \beta, \gamma)$, then we have*

$$\begin{aligned} f_{2k}(z) = z \cdot \exp\Bigg\{ \frac{1}{2k}\sum_{\mu=0}^{k-1} \int_0^z \frac{(1-\alpha)(1+\beta-\gamma)}{\zeta} \times \\ \times \left[\frac{\omega(\varepsilon^{\mu}\zeta)}{1-(1-\alpha)\beta\omega(\varepsilon^{\mu}\zeta)} + \frac{\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}{1-(1-\alpha)\beta\overline{\omega(\varepsilon^{\mu}\overline{\zeta})}}\right] d\zeta \Bigg\}, \end{aligned} \tag{4.1}$$

where $f_{2k}(z)$ is defined by equality (1.2), $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.

Proof. Suppose that $f(z) \in \mathcal{S}_{sc}^{(k)}(\alpha, \beta, \gamma)$, by (1.4), we have

$$\frac{zf'(z)}{f_{2k}(z)} = \frac{1+(1-\alpha)(1-\gamma)\omega(z)}{1-(1-\alpha)\beta\omega(z)}, \tag{4.2}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$. Substituting z by $\varepsilon^\mu z$ ($\mu = 0, 1, 2, \ldots, k-1$) in (4.2), we have

$$\frac{\varepsilon^\mu z f'(\varepsilon^\mu z)}{f_{2k}(\varepsilon^\mu z)} = \frac{1+(1-\alpha)(1-\gamma)\omega(\varepsilon^\mu z)}{1-(1-\alpha)\beta\omega(\varepsilon^\mu z)}. \tag{4.3}$$

From equality (4.3), we have

$$\frac{\overline{\varepsilon^\mu \overline{z}}\ \overline{f'(\varepsilon^\mu \overline{z})}}{\overline{f_{2k}(\varepsilon^\mu \overline{z})}} = \frac{1+(1-\alpha)(1-\gamma)\overline{\omega(\varepsilon^\mu \overline{z})}}{1-(1-\alpha)\beta\overline{\omega(\varepsilon^\mu \overline{z})}}. \tag{4.4}$$

Summing equalities (4.3) and (4.4), and making use of the same method as in Lemma 2, we have

$$\frac{zf'_{2k}(z)}{f_{2k}(z)} = \frac{1}{2k}\sum_{\mu=0}^{k-1}\left[\frac{1+(1-\alpha)(1-\gamma)\omega(\varepsilon^\mu z)}{1-(1-\alpha)\beta\omega(\varepsilon^\mu z)} + \frac{1+(1-\alpha)(1-\gamma)\overline{\omega(\varepsilon^\mu \overline{z})}}{1-(1-\alpha)\beta\overline{\omega(\varepsilon^\mu \overline{z})}}\right], \tag{4.5}$$

from equality (4.5), we can get

$$\frac{f'_{2k}(z)}{f_{2k}(z)} - \frac{1}{z} = \frac{1}{2k}\sum_{\mu=0}^{k-1}\frac{1}{z}\left\{\left[\frac{1+(1-\alpha)(1-\gamma)\omega(\varepsilon^\mu z)}{1-(1-\alpha)\beta\omega(\varepsilon^\mu z)} + \right.\right.$$
$$\left.\left. + \frac{1+(1-\alpha)(1-\gamma)\overline{\omega(\varepsilon^\mu \overline{z})}}{1-(1-\alpha)\beta\overline{\omega(\varepsilon^\mu \overline{z})}}\right] - 2\right\}. \tag{4.6}$$

Integrating equality (4.6), we have

$$\log\left\{\frac{f_{2k}(z)}{z}\right\} = \frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z \frac{(1-\alpha)(1+\beta-\gamma)}{\zeta}\times$$
$$\times\left[\frac{\omega(\varepsilon^\mu\zeta)}{1-(1-\alpha)\beta\omega(\varepsilon^\mu\zeta)} + \frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-(1-\alpha)\beta\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right]d\zeta. \tag{4.7}$$

From equality (4.7), we can get equality (4.1) easily. This completes the proof of Theorem 3. ∎

THEOREM 4. *Let $f(z) \in \mathcal{S}_{sc}^{(k)}(\alpha, \beta, \gamma)$, then we have*

$$f(z) = \int_0^z \exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^\xi \frac{(1-\alpha)(1+\beta-\gamma)}{\zeta}\left[\frac{\omega(\varepsilon^\mu\zeta)}{1-(1-\alpha)\beta\omega(\varepsilon^\mu\zeta)}\right.\right.$$
$$\left.\left. + \frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-(1-\alpha)\beta\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right]d\zeta\right\}\cdot\frac{1+(1-\alpha)(1-\gamma)\omega(\xi)}{1-(1-\alpha)\beta\omega(\xi)}d\xi, \tag{4.8}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.

Proof. Suppose that $f(z) \in \mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$, from equalities (4.1) and (4.2), we can get

$$\begin{aligned} f'(z) &= \frac{f_{2k}(z)}{z} \cdot \frac{1+(1-\alpha)(1-\gamma)\omega(z)}{1-(1-\alpha)\beta\omega(z)} \\ &= \exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^z \frac{(1-\alpha)(1+\beta-\gamma)}{\zeta}\left[\frac{\omega(\varepsilon^\mu\zeta)}{1-(1-\alpha)\beta\omega(\varepsilon^\mu\zeta)}\right.\right. \\ &\quad \left.\left. + \frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-(1-\alpha)\beta\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right]d\zeta\right\} \cdot \frac{1+(1-\alpha)(1-\gamma)\omega(z)}{1-(1-\alpha)\beta\omega(z)}. \end{aligned}$$

Integrating the above equality, we can get equality (4.8) easily. Hence the proof of Theorem 4 is complete. ■

Similarly, for the class $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$, we have

COROLLARY 3. *Let $f(z) \in \mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$, then we have*

$$\begin{aligned} f_{2k}(z) = \int_0^z \exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^\xi \frac{(1-\alpha)(1+\beta-\gamma)}{\zeta}\times\right. \\ \left. \times\left[\frac{\omega(\varepsilon^\mu\zeta)}{1-(1-\alpha)\beta\omega(\varepsilon^\mu\zeta)} + \frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-(1-\alpha)\beta\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right]d\zeta\right\}d\xi, \end{aligned}$$

where $f_{2k}(z)$ is defined by equality (1.2), $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.

COROLLARY 4. *Let $f(z) \in \mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$, then we have*

$$\begin{aligned} f(z) = \int_0^z \frac{1}{t}\int_0^t \exp\left\{\frac{1}{2k}\sum_{\mu=0}^{k-1}\int_0^\xi \frac{(1-\alpha)(1+\beta-\gamma)}{\zeta}\left[\frac{\omega(\varepsilon^\mu\zeta)}{1-(1-\alpha)\beta\omega(\varepsilon^\mu\zeta)}\right.\right. \\ \left.\left. + \frac{\overline{\omega(\varepsilon^\mu\overline{\zeta})}}{1-(1-\alpha)\beta\overline{\omega(\varepsilon^\mu\overline{\zeta})}}\right]d\zeta\right\} \cdot \frac{1+(1-\alpha)(1-\gamma)\omega(\xi)}{1-(1-\alpha)\beta\omega(\xi)}\,d\xi\,dt, \end{aligned}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$.

5. Convolution conditions

Finally, we provide the convolution conditions for the classes $\mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$ and $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$. Let $f,\ g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then the Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g*f)(z).$$

Theorem 5. *A function $f(z) \in \mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$ if and only if*

$$\frac{1}{z}\left\{f * \left\{\frac{z}{(1-z)^2}\left[1-(1-\alpha)\beta e^{i\theta}\right] - \frac{1+(1-\alpha)(1-\gamma)e^{i\theta}}{2}h\right\}(z)\right.$$

$$\left. - \frac{1+(1-\alpha)(1-\gamma)e^{i\theta}}{2} \cdot \overline{(f * h)(\overline{z})}\right\} \neq 0 \tag{5.1}$$

for all $z \in \mathcal{U}$ and $0 \leq \theta < 2\pi$, where $h(z)$ is given by (5.6).

Proof. Suppose that $f(z) \in \mathcal{S}_{sc}^{(k)}(\alpha,\beta,\gamma)$, we know that the condition (1.3) can be written as (1.4), since (1.4) is equivalent to

$$\frac{zf'(z)}{f_{2k}(z)} \neq \frac{1+(1-\alpha)(1-\gamma)e^{i\theta}}{1-(1-\alpha)\beta e^{i\theta}} \tag{5.2}$$

for all $z \in \mathcal{U}$ and $0 \leq \theta < 2\pi$. It is easy to know that the condition (5.2) can be written as

$$\frac{1}{z}\left\{\left[1-(1-\alpha)\beta e^{i\theta}\right]zf'(z) - \left[1+(1-\alpha)(1-\gamma)e^{i\theta}\right]f_{2k}(z)\right\} \neq 0. \tag{5.3}$$

On the other hand, it is well known that

$$zf'(z) = f(z) * \frac{z}{(1-z)^2}. \tag{5.4}$$

And from the definition of $f_{2k}(z)$, we know that

$$f_{2k}(z) = \frac{1}{2}\left[(f*h)(z) + \overline{(f*h)(\overline{z})}\right], \tag{5.5}$$

where

$$h(z) = \frac{1}{k}\sum_{\upsilon=0}^{k-1}\frac{z}{1-\varepsilon^{\upsilon}z}. \tag{5.6}$$

Substituting (5.4) and (5.5) into (5.3), we can get (5.1) easily. This completes the proof of Theorem 5. ∎

Similarly, for the class $\mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$, we have

Corollary 5. *A function $f(z) \in \mathcal{C}_{sc}^{(k)}(\alpha,\beta,\gamma)$ if and only if*

$$\frac{1}{z}\left\{f * \left\{z\left\{\frac{z}{(1-z)^2}\left[1-(1-\alpha)\beta e^{i\theta}\right] - \frac{1+(1-\alpha)(1-\gamma)e^{i\theta}}{2}h\right\}'\right\}(z)\right.$$

$$\left. - \frac{1+(1-\alpha)(1-\gamma)e^{i\theta}}{2} \cdot \overline{[f*(zh')](\overline{z})}\right\} \neq 0$$

for all $z \in \mathcal{U}$ and $0 \leq \theta < 2\pi$, where $h(z)$ is given by (5.6).

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