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ON SEQUENCE-COVERING $msss$-MAPS

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Abstract. This paper gives characterizations of metric spaces under some sequence-covering $msss$-maps by means of certain kind of σ-locally countable networks.

1. Introduction and definitions

A study of some images of metric spaces under certain maps is an important task on general topology. The paper [1] introduced the concept of $msss$-maps, and established the relationships between spaces with σ-locally countable k-networks (bases) and metric spaces by means of $msss$-maps. In this paper, we study some spaces with σ-locally countable networks, and give characterizations of some sequence-covering $msss$-images of metric spaces.

In this paper all spaces are regular and T_1, all maps are continuous and onto. N denotes the set of all natural numbers, ω denotes $N \cup \{0\}$. For two family $\mathcal{A}$ and $\mathcal{B}$ of subsets of a space X, Denote $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}$ and $B \in \mathcal{B}\}$. For the usual product space $\prod_{i \in N} X_i$, p_i denotes the projection from $\prod_{i \in N} X_i$ onto X_i For a space X and each $x_n \in X$, (x_n) denotes a point of the usual product space X^ω whose n-th coordinate is x_n.

Definition 1.1. [10] Let X be a space, and $P \subset X$. Then,

(1) A sequence $\{x_n\}$ in X is eventually in P if $\{x_n\}$ converges to x, and there exists $m \in N$ such that $\{x\} \cup \{x_n : n \geq m\} \subset P$;

(2) P is a sequential neighborhood of x in X if whenever a sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is eventually in P;

(3) P is sequentially open in X if P is a sequential neighborhood at each of its points;

(4) X is a sequential space if any sequentially open subset of X is open in X.

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Definition 1.2. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of a space X satisfying for each $x \in X$,

(1) $\mathcal{P}_x$ is a network of x in X. i.e., $x \in \cap\mathcal{P}_x$ and for $x \in U$ with U open in X, $P \subset U$ for some $P \in \mathcal{P}_x$,

(2) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is a weak-base for X [8] if $G \subset X$ such that for each $x \in G$, there exists $P \in \mathcal{P}_x$ satisfying $P \subset G$, then G is open in X. $\mathcal{P}$ is an *sn*-network (i.e., sequential neighborhood network) for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X. $\mathcal{P}$ is an *so*-network (i.e., sequential open network) for X if each element of $\mathcal{P}_x$ is sequentially open in X. The above $\mathcal{P}_x$ respectively is a weak-base, an *sn*-network and an *so*-network of x in X.

Definition 1.3. [4] For a space X, let $\mathcal{P}$ be a family of subsets of X, there exists $\mathcal{P}_x \subset \mathcal{P}^\omega$ holding the following property: if $(P_n) \in \mathcal{P}_x$, then $\{P_n : n \in N\}$ is a decrease network of x in X. Denote $\mathcal{P} \approx \bigcup\{\mathcal{P}_x : x \in X\}$.

(1) $\mathcal{P}$ is an *s*-network (i.e., sequential network) for X if, whenever $P \subset X$, and for $x \in P$ and each $(P_n) \in \mathcal{P}_x$, $P_m \subset P$ for some $m \in M$, then P is sequentially open in X;

(2) $\mathcal{P}$ is a sequential quasi-bases for X if, whenever $P \subset X$, and for $x \in P$ and each $(P_n) \in \mathcal{P}_x$, $P_m \subset P$ for some $m \in M$, then P is open in X;

(3) $\mathcal{P}$ is a Fréchet quasi-bases for X if, whenever $P \subset X$, and for $x \in P$ and each $(P_n) \in \mathcal{P}_x$, $P_m \subset P$ for some $m \in M$, then P is a neighborhood of x in X.

Definition 1.4. Let $\mathcal{P}$ be a family of subsets of a space X.

(1) $\mathcal{P}$ is a *cs*-network [9] for X if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, then there are $P \in \mathcal{P}$ and $m \in N$ such that $\{x_n : n \geq m\} \cup \{x\} \subset P \subset U$;

(2) $\mathcal{P}$ is a cs^*-network for X if whenever $\{x_n\}$ is a sequence converging to a point $x \in U$ with U open in X, then there are a subsequence $\{x_{n_i}\}$ and $P \in \mathcal{P}$ such that $\{x_{n_i} : i \in N\} \cup \{x\} \subset P \subset U$.

Definition 1.5. Let $f\colon X \to Y$ be a map.

(1) f is a *msss*-map [1] (i.e., metrizably stratified strong s-map) if X is a subspace of the product space $\prod_{i\in N} X_i$ of a family $\{X_i : i \in N\}$ of metric spaces and for each $y \in Y$, there is a sequence $\{V_i\}$ of open neighborhoods of y such that each $p_i f^{-1}(V_i)$ is separable in X_i;

(2) f is a 1-sequence-covering map [2] if for each $y \in Y$, there exists $x \in f^{-1}(y)$ satisfying the following condition $(*)$: whenever $y_n \to y$, then there exists $x_n \in f^{-1}(y_n)$ such that $x_n \to x$;

(3) f is a 2-sequence-covering map [2] if for each $y \in Y$ and each $x \in f^{-1}(y)$ satisfying the above condition $(*)$;

(4) f is a sequence-covering map [13] (resp. compact-covering map) if each convergent sequence (including its limit point) of Y (resp. each compact subset of Y) is the image of some compact subset of X;

(5) f is a strong sequence-covering map [5] if each convergent sequence (including its limit point) in Y is the image of some convergent sequence (including its limit point) in X;

(6) f is a strong compact-covering map [5] if it is both strong sequence-covering and compact-covering;

(7) f is a sequentially quotient map [7] if whenever $R \subset Y$ and $f^{-1}(R)$ is sequentially open in X, then R is sequentially open in Y.

2. On 1-sequence-covering *msss*-images

Theorem 2.1. *A space X is a 1-sequence-covering msss-image of a metric space if and only if X has a σ-locally countable sn-network.*

Proof. Sufficiency. Suppose $\mathcal{P}$ is a σ-locally countable sn-network for X. Let $\mathcal{P} = \bigcup\{\mathcal{P}_i : i \in N\}$, where each $\mathcal{P}_i = \{P_\alpha : \alpha \in A_i\}$ is locally countable in X. We can assume that $\mathcal{P}_i$ is closed under finite intersections and $X \in \mathcal{P}_i \subset \mathcal{P}_{i+1}$. For each $i \in N$, endow A_i with discrete topology; then A_i is a metric space. Put

$$M = \left\{ \alpha = (\alpha_i) \in \prod_{i \in N} A_i : \{P_{\alpha_i} : i \in N\} \text{ is a network of some point } x_\alpha \text{ in } X \right\},$$

and endow M with the subspace topology induced from the product topology of a family $\{A_i : i \in N\}$ of metric spaces, then M is a metric space. Since X is Hausdorff, x_α is unique in X for each $\alpha \in M$. We define $f : M \to X$ by $f(\alpha) = x_\alpha$ for each $\alpha \in M$. Since $\mathcal{P}$ is a σ-locally countable sn-network for X, f is onto. For each $\alpha = (\alpha_i) \in M$, $f(\alpha) = x_\alpha$. Suppose V is an open neighborhood of x_α in X, there exists $n \in N$ such that $x_\alpha \in P_{\alpha_n} \subset V$, set $W = \{c \in M : \text{the n-the coordinate of } c \text{ is } \alpha_n\}$, then W is an open neighborhood of α in M, and $f(W) \subset P_{\alpha_n} \subset V$. Hence f is continuous. We will show that f is a 1-sequence-covering $msss$-map.

(i) f is an $msss$-map.

For each $x \in X$ and each $i \in N$, there exists an open neighborhood V_i of x in X such that $\{\alpha_i \in A_i : P_\alpha \cap V_i \neq \emptyset\}$ is countable. Put

$$B_i = \{\alpha_i \in A_i : P_\alpha \cap V_i \neq \emptyset\},$$

then $p_i f^{-1}(V_i) \subset B_i$. Thus $p_i f^{-1}(V_i)$ is separable in A_i. Hence f is a $msss$-map.

(ii) f is a 1-sequence-covering map.

For each $x \in X$, by the definition of $\mathcal{P}$, there exists $(\alpha_i) \in \prod_{i \in N} A_i$ such that $\{P_{\alpha_i} : i \in N\} \subset \mathcal{P}$ is an sn-network of x in X. Denote $\beta = (\alpha_i)$, then $\beta \in f^{-1}(x)$. For each $n \in N$, let $R_n = \{(\gamma_i) \in M : \text{if } i \leq n, \text{ then } \gamma_i = \alpha_i\}$. Then $\{R_n : n \in N\}$ is a decreasing neighborhood base of β in M. For each $n \in N$, it is easy to check that $f(R_n) = \bigcap_{i \leq n} P_{\alpha_i}$. Now suppose $x_j \to x$ in X. For each $n \in N$, since $f(B_n)$ is a sequential neighborhood of x in X, there exists $i(n) \in N$ such that if $i \geq i(n)$, then $x_i \in f(R_n)$. Thus $f^{-1}(x_i) \cap R_n \neq \phi$. We may assume $1 < i(n) < i(n+1)$. For each $j \in N$, let

$$\beta_j \in \begin{cases} f^{-1}(x_j), & \text{if } j < i(1), \\ f^{-1}(x_j) \cap R_n, & \text{if } i(n) \leq j < i(n+1), n \in N. \end{cases}$$

Then it is easy to show that sequence $\{\beta_j\}$ converges to β in M. Hence f is a 1-sequence-covering map.

Necessity. Suppose $f\colon M \to X$ is a 1-sequence-covering $msss$-map, where M is a metric space. Since f is a $msss$-map, then there exists a base $\mathcal{B}$ for M such that $\mathcal{P}^* = \{f(B) : B \in \mathcal{B}\}$ is a σ-locally countable network for X by Lemma 1.2 of [1]. For each $x \in X$, $\beta \in f^{-1}(x)$ satisfies the condition $(*)$ of Definition 1.5 (2). Put

$$\mathcal{P}_x = \{f(B) : \beta_x \in B \in \mathcal{B}\}, \quad \mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}.$$

It is easy to show that each $\mathcal{P}_x$ is an sn-network of x in X, and $\mathcal{P}$ is an sn-network for X. Obviously, $\mathcal{P} \subset \mathcal{P}^*$. Hence X has a σ-locally countable sn-network. ■

Corollary 2.2. *A space X is a 1-sequence-covering and quotient msss-image of a metric space if and only if X has a σ-locally countable weak-base.*

Proof. Sufficiency. Suppose X has a σ-locally countable weak-base, then X is a sequential space with a σ-locally countable sn-network by [3, Proposition 1.6.15, Corollary 1.6.18]. Thus X is a 1-sequence-covering $msss$-image of a metric space by Theorem 2.1. This 1-sequence-covering $msss$-map is quotient by Lemma 2.1 of [14].

Necessity. Suppose X is a 1-sequence-covering and quotient $msss$-image of a metric space. Then X is a sequential space with a σ-locally countable sn-network $\mathcal{P}$. It is easy to prove that $\mathcal{P}$ is a σ-locally countable weak-base for X. ■

3. On 2-sequence-covering *msss*-images

The following Theorem 3.1 can be proved by Lemma 3.1 of [2] according to the proof of Theorem 2.1.

Theorem 3.1. *A space X is a 2-sequence-covering msss-image of a metric space if X has a σ-locally countable so-network.*

Corollary 3.2. *The following are equivalent for a space X:*

(1) *X has a σ-locally countable base;*

(2) *X is a 2-sequence-covering and quotient msss-image of a metric space;*

(3) *X is an open msss-image of a space having a σ-locally countable base;*

(4) *X is a countably-bi-quotient msss-image of a space having a σ-locally countable base.*

Proof. (1) $\Longrightarrow$ (2) follows from Theorem 3.1 of [1] and Corollary 3.2 of [2].

(2) $\Longrightarrow$ (1). By Theorem 3.1, X is a sequential space with a σ-locally countable so-network $\mathcal{P}$. It is easy to show that $\mathcal{P}$ is a σ-locally countable base for X.

(1) $\Longrightarrow$ (3) follows from Theorem 3.1 of [1].

(3) $\Longrightarrow$ (4) is obvious.

(4) $\Longrightarrow$ (1). Suppose X is the image of M under a countably-bi-quotient $msss$-map f, where M is a space with a σ-locally countable base. Because f is a $msss$-map, then there exists a base $\mathcal{B}$ for M such that $\mathcal{P} = \{f(B) : B \in \mathcal{B}\}$ is a σ-locally countable network for X by Lemma 1.2 of [1]. Thus $\mathcal{P}$ is a σ-locally countable k-network for X by Lemma 2.5 of [1]. By Proposition 2.3.1 of [3], countably-bi-quotient maps preserve strong Fréchet property, thus X is a strong Fréchet space with a σ-locally countable k-network. Hence X has a σ-locally countable base by Theorem 3.9 of [6] and Proposition 3.2 of [13]. ∎

Corollary 3.3. *A space with a σ-locally countable base is preserved by a countably-bi-quotient msss-map.*

4. On sequence-covering *msss*-images

Theorem 4.1. *The following are equivalent for a space X:*

(1) *X is a sequence-covering msss-image of a metric space;*

(2) *X is a sequentially quotient msss-image of a metric space;*

(3) *X has a σ-locally countable s-network.*

Proof. (1) $\Longrightarrow$ (2) follows from Proposition 2.1.17 of [3].

(2) $\Longrightarrow$ (3). Suppose $f : M \to X$ is a sequentially quotient $msss$-map, where M is a metric space. Since f is a $msss$-map, there exists a base $\mathcal{B}$ for M such that $\mathcal{P} = \{f(B) : B \in \mathcal{B}\}$ is a σ-locally countable network for X by Lemma 1.2 of [1]. Because s-networks are preserved by sequentially quotient maps by Lemma 3.1 of [4], X has a σ-locally countable s-network $\mathcal{P}$.

(3) $\Longrightarrow$ (1). Suppose $\mathcal{P}$ is a σ-locally countable s-network for X. then $\mathcal{P}$ is a σ-locally countable cs^*-network for X by Theorem 2.4 of [4]. Hence X is a sequence-covering $msss$-image of a metric space by Theorem 1 of [11]. ∎

The following corollaries can be proved by Theorem 4.1, Corollary 2.3 of [4], Lemma 3.1 of [4] and Proposition 2.1.16 of [3].

Corollary 4.2. *A space X has a σ-locally countable sequential quasi-base if and only if X is a quotient msss-image of a metric space.*

Corollary 4.3. *A space X has a σ-locally countable Fréchet quasi-base if and only if X is a pseudo-open msss-image of a metric space.*

5. On strong sequence-covering *msss*-images

Theorem 5.1. *The following are equivalent for a space X:*

(1) *X is a strong sequence-covering msss-image of a metric space;*

(2) *X is a strong compact-covering msss-image of a metric space;*

(3) *X has a σ-locally-countable cs-network.*

Proof. (1) $\Longrightarrow$ (3) follows from Lemma 1.2 of [1] and the fact: cs-networks are preserved by strong sequence-covering maps.

(3) $\Longrightarrow$ (2). Suppose $\mathcal{P}$ is a σ-locally-countable *cs*-network for X. Denote $\mathcal{P} = \bigcup\{\mathcal{P}_i : i \in N\}$, where each $\mathcal{P}_i = \{P_\alpha : \alpha \in A_i\}$ is locally-countable in X. We can assume that each $\mathcal{P}_i$ is closed under finite intersections and $X \in \mathcal{P}_i \subset \mathcal{P}_{i+1}$. By the proof of Theorem 2.1, there exist a metric space M and a *msss*-map $f : M \to X$. We will prove that f is a strong compact-covering map. For each sequence $\{x_n\}$ converging to x_0, we can assume that all $x_n's$ are distinct, and that $x_n \neq x_0$ for each $n \in N$. Let $K = \{x_m : m \in \omega\}$. Suppose V is an open neighborhood of K in X. A subfamily $\mathcal{A}$ of $\mathcal{P}$ is said to have the following property, which is denoted by $F(K, V)$, if:

(a) $\mathcal{A}$ is finite,

(b) for each $P \in \mathcal{A}$, $\phi \neq P \cap K \subset P \subset V$,

(c) for each $z \in K$, there exists a unique $P_z \in \mathcal{A}$ such that $z \in \mathcal{P}_z$,

(d) if $x_0 \in P \in \mathcal{A}$, then $K \setminus P$ is finite.

For each $i \in N$, put

$$\mathcal{P}_i(K) = \{\mathcal{A} \subset \mathcal{P}_i : \mathcal{A} \text{ has the property } F(K, X)\};$$

then $|\mathcal{P}_i(K)| < \aleph_0$. Denote $\mathcal{P}_i(K)$ by $\{\mathcal{P}_{ij} : j \in N\}$ (when $\mathcal{P}_i(K) = \{\mathcal{P}_{i1}, \cdots, \mathcal{P}_{is}\}$, denote $\mathcal{P}_{ij} = \mathcal{P}_{is}$ if $j > s$). For each $n \in N$, put

$$\mathcal{P}'_n = \bigwedge_{i,j \leq n} \mathcal{P}_{ij},$$

then $\mathcal{P}'_n \subset \mathcal{P}_n$ and $\mathcal{P}'_n$ also has the property $F(K, X)$.

For each $i \in N$ and each $m \in \omega$, there exists $\alpha_{im} \in A_i$ such that $x_m \in P_{\alpha_{im}} \in \mathcal{P}'_i$. Let $b_m = (\alpha_{im}) \in \prod_{i \in N} A_i$. It is easy to prove that $\{P_{\alpha_{im}} : i \in N\}$ is a network of x_m in X. Then $b_m \in M$ and $f(b_m) = x_m$ for each $m \in \omega$. For each $i \in N$, there exists $n(i) \in N$ such that $\alpha_{in} = \alpha_{i0}$ when $n \geq n(i)$. Hence the sequence $\{\alpha_{in}\}$ converges α_{i0} in A_i. Thus, there is a sequence $\{b_n\}$ converging to b_0 in X. This shows that f is sequence-covering.

Since X has a σ-locally-countable *cs*-network, each compact subset L of X has a countable *cs*-network. So L is metrizable. We can prove that f is compact-covering by the proof of Theorem 2 in [5].

(2) $\Longrightarrow$ (1) is obvious. ∎

Corollary 5.2. *The following are equivalent for a space X:*

(1) *X is a strong sequence-covering and quotient msss-image of a metric space;*

(2) *X is a strong compact-covering and msss-image of a metric space;*

(3) *X is a k-space with a σ-locally-countable cs-network.*

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