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WEAK FORMS OF OPEN MAPPINGS AND STRONG FORMS OF SEQUENCE-COVERING MAPPINGS

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Abstract. In this paper, we discuss some weak forms of open mappings and some strong forms of sequence-covering mappings, and establish some relations among these mappings. As some applications of these results, we obtain that images of metric spaces under certain weak forms of open mappings can be characterized as images of metric spaces under certain strong forms of sequence-covering mappings.

1. Introduction

Strong forms of sequence-covering mappings form an important class of mappings. Many interesting characterizations of images of metric spaces under these mappings have been obtained [7, 8, 10, 5, 4]. Recently, some weak forms of open mappings have attracted considerable attention, and some characterizations of images of metric spaces under these mappings have been obtained [13, 6, 7]. Note that these images are equivalent to images of metric spaces under certain strong forms of sequence-covering mappings. Naturally, we can ask: are these weak forms of open mappings and these strong forms of sequence-covering mappings, in case they are defined on metric spaces, equivalent? This arouses our interest in relations between weak forms of open mappings and strong forms of sequence-covering mappings.

In this paper, we investigate these mappings to establish some relations among them, and prove that a mapping f defined on a first countable space is open (resp. almost open) if and only if it is weak-open (resp. almost weak-open), if and only if it is sn-open (resp. almost sn-open) and quotient, if and only if it is 2-sequence-covering (resp. 1-sequence-covering) and quotient. As some applications of these results, we obtain that images of metric spaces under these mappings are equivalent.

Throughout this paper, all spaces are assumed to be regular T_1 and all mappings are continuous and onto. $\mathbf{N}$ denotes the set of all natural numbers. $\{x_n\}$ denotes a sequence, where the n-th term is x_n. Let X be a space and $P \subset X$. A

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sequence $\{x_n\}$ converging to x in X is eventually in P if $\{x_n : n > k\} \cup \{x\} \subset P$ for some $k \in \mathbf{N}$; it is frequently in P if $\{x_{n_k}\}$ is eventually in P for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}$ be a family of subsets of X. Then $\bigcup \mathcal{P}$ and $\bigcap \mathcal{P}$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$ and the intersection $\bigcap\{P : P \in \mathcal{P}\}$ respectively. For terms which are not defined here, we refer to [2].

2. The Main results

Definition 2.1. Let X be a space.

(1) Let $x \in P \subset X$. P is called a sequential neighborhood of x in X if whenever $\{x_n\}$ is a sequence converging to the point x, then $\{x_n\}$ is eventually in P;

(2) Let $P \subset X$. P is called a sequentially open subset in X if P is a sequential neighborhood of x in X for each $x \in P$;

(3) X is called a sequential space if each sequentially open subset in X is open;

(4) X is called a Fréchet space if for each $P \subset X$ and for each $x \in \overline{P}$, there exists a sequence $\{x_n\}$ in P converging to the point x.

Remark 2.2. (1) P is a sequential neighborhood of x if and only if each sequence $\{x_n\}$ converging to x is frequently in P.

(2) The intersection of finitely many sequential neighborhoods of x is a sequential neighborhood of x.

(3) It is well known that first countable $\Longrightarrow$ Fréchet $\Longrightarrow$ sequential.

Definition 2.3. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X such that for each $x \in X$, the following conditions (a) and (b) are satisfied:

(a) $\mathcal{P}_x$ is a network at x in X, i.e., $x \in \bigcap \mathcal{P}_x$ and for each neighborhood U of x in X, $P \subset U$ for some $P \in \mathcal{P}_x$;

(b) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is called a weak base [1] of X if whenever $G \subset X$, G is open in X if and only if for each $x \in G$ there exists $P \in \mathcal{P}_x$ with $P \subset G$, where $\mathcal{P}_x$ is called a *wn*-network (i.e., weak neighborhood network) at x in X. Furthermore, X is called g-first countable [11] if $\mathcal{P}_x$ is countable for each $x \in X$;

(2) $\mathcal{P}$ is called an *sn*-network [3] of X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X for each $x \in X$, where $\mathcal{P}_x$ is called an *sn*-network at x in X. Furthermore, X is called *sn*-first countable [3] if $\mathcal{P}_x$ is countable for each $x \in X$.

Remark 2.4 [10]. For a space, weak base $\Longrightarrow$ *sn*-network. An *sn*-network for a sequential space is a weak base. So g-first countable $\Longrightarrow$ *sn*-first countable and for a sequential space, g-first countable $\Longleftrightarrow$ *sn*-first countable.

Definition 2.5. Let $f\colon X \longrightarrow Y$ be a mapping.

(1) f is called an open (resp. closed) mapping [2] if $f(U)$ is open (resp. closed) in Y for each open (resp. closed) subset U in X;

(2) f is called a weak-open mapping if there exists a weak base $\mathcal{P} = \{\mathcal{P}_y : y \in Y\}$ of Y such that for each $y \in Y$ and for each $x \in f^{-1}(y)$, whenever U is a neighborhood of x, then $P \subset f(U)$ for some $P \in \mathcal{P}_y$;

(3) f is called an sn-open mapping if there exists an sn-network $\mathcal{P} = \{\mathcal{P}_y : y \in Y\}$ of Y such that for each $y \in Y$ and for each $x \in f^{-1}(y)$, whenever U is a neighborhood of x, then $P \subset f(U)$ for some $P \in \mathcal{P}_y$;

(4) f is called an almost open mapping [8] if for each $y \in Y$ there exists $x \in f^{-1}(y)$ such that $f(U)$ is a neighborhood of y for each neighborhood U of x;

(5) f is called an almost weak-open mapping [13] if there exists a weak base $\mathcal{P} = \{\mathcal{P}_y : y \in Y\}$ of Y satisfying the condition: for every $y \in Y$, there exists $x \in f^{-1}(y)$ such that whenever U is a neighborhood of x, $P \subset f(U)$ for some $P \in \mathcal{P}_y$;

(6) f is called an almost sn-open mapping if there exists an sn-network $\mathcal{P} = \{\mathcal{P}_y : y \in Y\}$ of Y satisfying the condition: for every $y \in Y$, there exists $x \in f^{-1}(y)$ such that whenever U is a neighborhood of x, $P \subset f(U)$ for some $P \in \mathcal{P}_y$;

(7) f is called a 1-sequence-covering mapping [10] if for every $y \in Y$ there exists $x \in f^{-1}(y)$, such that whenever $\{y_n\}$ is a sequence converging to y in Y, there exists a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$;

(8) f is called a 2-sequence-covering mapping [7] if for each $y \in Y$ and for each $x \in f^{-1}(y)$, then whenever $\{y_n\}$ is a sequence converging to y in Y, there exists a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$;

(9) f is called a quotient mapping [2] if U is open in Y if and only if $f^{-1}(U)$ is open in X.

Remark 2.6. S. Xia introduced "weak-open mapping" in [13]. Since "weak-open mapping" in [13] is not only weak to "open mapping" but also weak to "almost open mapping", "weak-open mapping" in [13] is called "almost weak-open mapping" in this paper by Definition 2.5(5), and we define another mapping in this paper as "weak-open mappings" by Definition 2.5(2).

Remark 2.7. The following implications except for $(*)$ and $(**)$ hold from Definition 2.5. Implications $(*)$ and $(**)$ hold from Proposition 2.9(2) and Proposition 2.13(2), respectively. For every mapping

$$\begin{array}{ccccccc}
\text{open} & \Longrightarrow & \text{weak-open} & \Longrightarrow & sn\text{-open} & \stackrel{(*)}{\Longleftarrow} & \text{2-sequence-covering} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{almost open} & \Longrightarrow & \text{almost weak-open} & \Longrightarrow & \text{almost } sn\text{-open} & \stackrel{(**)}{\Longleftarrow} & \text{1-sequence-covering}
\end{array}$$

The following Lemma is due to the proof of [10, Theorem 4.4].

Lemma 2.8. *Let $f\colon X \longrightarrow Y$ be a mapping. If $\{B_n\}$ is a decreasing network at some x in X, and each $f(B_n)$ is a sequential neighborhood of $y = f(x)$ in Y, then whenever $\{y_n\}$ is a sequence converging to y in Y, there exists a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.*

Proof. Let $\{B_n\}$ be a decreasing network at some $x \in X$, and each $f(B_n)$ be a sequential neighborhood of $y = f(x)$ in Y. If $\{y_n\}$ is a sequence converging to y in Y, then for each $k \in \mathbf{N}$, there exists $n_k \in \mathbf{N}$ such that $y_n \in f(B_k)$ for all $n \geq n_k$. Thus $f^{-1}(y_n) \cap B_k \neq \emptyset$ for each $n > n_k$. Without loss of generality, we can assume $1 < n_k < n_{k+1}$ for each $k \in \mathbf{N}$. For each $n \in \mathbf{N}$, pick

$$x_n \in \begin{cases} f^{-1}(y_n), & n < n_1 \\ f^{-1}(y_n) \cap B_k, & n_k \leq n < n_{k+1} \end{cases}$$

Then $x_n \in f^{-1}(y_n)$ for each $n \in \mathbf{N}$. It is easy to show that $\{x_n\}$ converges to x. In fact, let U be a neighborhood of x; then there exists $k \in \mathbf{N}$ such that $x \in B_k \subset U$. Obviously, $x_n \in B_k \subset U$ for each $n > n_k$, so $\{x_n\}$ converges to x. ■

Proposition 2.9. *Let $f\colon X \longrightarrow Y$ be a mapping. Then the following hold:*

(1) If f is sn-open and X is first countable, then f is 2-sequence-covering;

(2) If f is 2-sequence-covering, then f is sn-open;

(3) If f is almost weak-open, then f is quotient.

Proof. (1) Let f be *sn*-open and X be first countable, $\mathcal{P} = \bigcup\{\mathcal{P}_y : y \in Y\}$ be a *sn*-network of Y as stated in Definition 2.5(3). Since X is first countable for each $y \in Y$ and each $x \in f^{-1}(y)$ there exists a countable decreasing neighborhood base $\{U_n\}$ at x in X. For each $n \in \mathbf{N}$, there exists $P_n \in \mathcal{P}_y$ such that $P_n \subset f(U_n)$. P_n is a sequential neighborhood of y in Y, so is $f(U_n)$. By Lemma 2.8, whenever $\{y_n\}$ is a sequence converging to y in Y, there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$. So f is 2-sequence-covering.

(2) Let f be 2-sequence-covering. For each $y \in Y$, we construct an *sn*-network $\mathcal{P}_y$ at y in Y as follows. For each $x \in f^{-1}(y)$, let $\mathcal{B}_x = \{B : B \text{ is a neighborhood of } x\}$. Put $\mathcal{P}'_y = \{f(B) : B \in \mathcal{B}_x \text{ and } x \in f^{-1}(y)\}$; then each element of $\mathcal{P}'_y$ is a sequential neighborhood of y in Y. In fact, Let $P' = f(B) \in \mathcal{P}'_y$, where $B \in \mathcal{B}_x$ for some $x \in f^{-1}(y)$. Since f is 2-sequence-covering, whenever $\{y_n\}$ is a sequence converging to y in Y there exists a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$. Note that B is a neighborhood of x. $\{x_n\}$ is eventually in B, so $\{y_n\} = \{f(x_n)\}$ is eventually in $f(B) = P'$. Thus P' is a sequential neighborhood of y in Y. Put $\mathcal{P}_y = \{\bigcap \mathcal{F}_y : \mathcal{F}_y \text{ is a finite subfamily of } \mathcal{P}'_y\}$; then each element of $\mathcal{P}_y$ is a sequential neighborhood of y in Y from Remark 2.2(2). Also, it is not difficult to check that $\mathcal{P}_y$ satisfies the conditions (a) and (b) in Definition 2.3. Thus we obtain that $\mathcal{P}_y$ is an *sn*-network at y in Y. Put $\mathcal{P} = \bigcup\{\mathcal{P}_y : y \in Y\}$; then $\mathcal{P}$ is an *sn*-network of Y. For each $y \in Y$ and for each $x \in f^{-1}(y)$, whenever U is a neighborhood of x, there exists $B \in \mathcal{B}_x$ such that $x \in B \subset U$, so $f(B) \in \mathcal{P}_y$ and $f(B) \subset f(U)$. This proves that f is *sn*-open.

(3) This namely is [13, Proposition 3.2]. ■

Proposition 2.10. *Let $f\colon X \longrightarrow Y$ be an sn-open mapping. If one of the following two conditions is satisfied, then f is open:*

(1) Y is a sequential space;

(2) X is a sequential space and f is a quotient mapping.

Proof. Quotient mappings preserve sequential spaces, so condition (2) implies condition (1). Thus we only need to prove that f is open if condition (1) is satisfied.

As f is an sn-open mapping, let $\mathcal{P} = \bigcup\{\mathcal{P}_y : y \in Y\}$ be an sn-network of Y as stated in Definition 2.5(3). Since Y is a sequential space, it suffices to prove that whenever U is an open subset of X, then $f(U)$ is a sequential neighborhood of y in Y for each $y \in f(U)$. Let $y \in f(U)$; pick $x \in f^{-1}(y) \cap U$. Then U is a neighborhood of x. So there exists $P \in \mathcal{P}_y$ such that $P \subset f(U)$. P is a sequential neighborhood of y in Y, so is $f(U)$. This completes the proof. ∎

The following theorem is obtained from Remark 2.7, Proposition 2.9 and Proposition 2.10.

THEOREM 2.11. *Let $f\colon X \longrightarrow Y$ be a mapping. If X is first countable (especially, if X is metric), then the following are equivalent:*

(1) f is an open mapping;

(2) f is a weak-open mapping;

(3) f is an sn-open, quotient mapping;

(4) f is a 2-sequence-covering, quotient mapping.

We denote some mapping property by P.

COROLLARY 2.12. *The following are equivalent for a space X:*

(1) X is an open, P-image of a metric space;

(2) X is a weak-open, P-image of a metric space;

(3) X is an sn-open, quotient, P-image of a metric space;

(4) X is a 2-sequence-covering, quotient, P-image of a metric space.

In a similar way as in the above proofs, we can obtain the following results. For example, let $f\colon X \longrightarrow Y$ be a 1-sequence-covering mapping. For every $y \in Y$, there exists $x_y \in f^{-1}(y)$ satisfies Definition 2.5(7). In the proof of Proposition 2.9(2), if $\mathcal{P}_y$ is replaced by $\{f(B) : B \text{ is a neighborhood of } x_y\}$, then it is obtained easily that f is almost open.

PROPOSITION 2.13. *Let $f\colon X \longrightarrow Y$ be a mapping. Then the following hold:*

(1) If f is almost sn-open and X is first countable, then f is 1-sequence-covering;

(2) If f is 1-sequence-covering, then f is almost sn-open.

PROPOSITION 2.14. *Let $f\colon X \longrightarrow Y$ be an almost sn-open mapping. If one of the following two conditions is satisfied, then f is almost open:*

(1) Y is a sequential space;

(2) X is a sequential space and f is a quotient mapping.

Theorem 2.15. *Let $f\colon X \longrightarrow Y$ be a mapping. If X is first countable (especially, if X is metric), then the following are equivalent:*

(1) f is an almost open mapping;

(2) f is an almost weak-open mapping;

(3) f is an almost sn-open, quotient mapping;

(4) f is a 1-sequence-covering, quotient mapping.

Corollary 2.16. *The following are equivalent for a space X.*

(1) X is an almost open, P-image of a metric space;

(2) X is an almost weak-open, P-image of a metric space;

(3) X is an almost sn-open, quotient, P-image of a metric space;

(4) X is a 1-sequence-covering, quotient, P-image of a metric space.

3. Some Examples

S. Lin gave an example: there exists a 1-sequence-covering, closed mapping $f\colon X \longrightarrow Y$ such that f is not almost weak-open [9, Example 1]. We can prove f is also 2-sequence-covering by revising the proof of this example. So we have the following example.

Example 3.1. 2-sequence-covering, closed mapping $\not\Longrightarrow$ almost weak-open mapping.

Proof. Let $X_1 = \{x : x \leq \omega_1\}$, where ω_1 denotes the first uncountable ordinal. We define the topology on X_1 as follows.

If $x < \omega_1$, then $\{x\}$ is open in X_1. If $x = \omega_1$, then the neighborhood base at x is the neighborhood base at x in ordinal space $\omega_1 + 1$.

Put $X_2 = X_1$ and let X be the topological sum $X_1 \oplus X_2$ of X_1 and X_2. We write ω_1 in X_1 and ω_1 in X_2 by x_1 and x_2 respectively. Put $A = \{x_1, x_2\} \subset X$ and Y is the quotient space X/A obtained from X by shrinking the set A to a point. Let $f\colon X \longrightarrow Y$ be the natural mapping. Then:

(1) f is a closed mapping, and it is not almost weak-open [9, Example 1].

(2) Let $\{y_n\}$ be a sequence in Y converging to $y \in Y$. Then there exists $k \in \mathbb{N}$ such that $y_n = y$ for $n > k$.

If $y \notin f(A)$, then $\{y\}$ is open in Y. $\{y_n\}$ converges to y, so there exists $k \in \mathbf{N}$ such that $y_n \in \{y\}$ for $n > k$, that is, $y_n = y$ for $n > k$. If $y \in f(A)$, let $\alpha_n \in f^{-1}(y_n)$ for each $n \in \mathbf{N}$, and put $x_0 = \bigcup\{\alpha_n : n \in \mathbf{N} \text{ and } \alpha_n < \omega_1\}$; then $x_0 < \omega_1$. Put $U = (x_0, \omega_1]$; then U is a neighborhood of ω_1 in X, so $f(U)$ is neighborhood of y in Y. It is easy to see that $\{y_n : n \in \mathbf{N} \text{ and } y_n \neq y\} \cap f(U) = \emptyset$. Since $\{y_n\}$ converges to y, there exists $k \in \mathbf{N}$ such that $y_n \in f(U)$ for $n > k$. So $y_n = y$ for $n > k$.

(3) f is 2-sequence-covering.

Let $y \in Y$, and $x \in f^{-1}(y)$. If $\{y_n\}$ is a sequence in Y converging to y, then there exists $k \in \mathbf{N}$ such that $y_n = y$ for $n > k$ from the above (2). Pick $x_n \in f^{-1}(y_n)$

for $n \le k$, and put $x_n = x \in f^{-1}(y) = f^{-1}(y_n)$ for $n > k$. The sequence $\{x_n\}$ converges to x in X with each $x_n \in f^{-1}(y_n)$. So f is 2-sequence-covering.

By the above (1) and (3), we complete the proof of this example. ■

The following example is due to [7, Example 3.10].

EXAMPLE 3.2. Open mapping $\not\Longrightarrow$ 1-sequence-covering mapping.

Proof. Let $X = \{0\} \cup (\mathbf{N} \times \mathbf{N})$. Put $V(n,i) = \{(n,k) \in \mathbf{N} \times \mathbf{N} : k \ge i\}$ for each $n \in \mathbf{N}$ and each $i \in \mathbf{N}$. The topology on X is defined as follows.

If $(n,m) \in \mathbf{N} \times \mathbf{N}$, then (n,m) is open in X. For $0 \in X$, an open neighborhood base at 0 is $\{\{0\} \cup (\bigcup\{V(n,i_n) : n \ge m\}) : m, i_n \in \mathbf{N}\}$.

Let $Y = \{0\} \cup \{1/n : n \in \mathbf{N}\}$ be the subspace of the real line. Put $f \colon X \longrightarrow Y$ by $f(0) = 0$ and $f((n,i)) = 1/n$ for each $n \in \mathbf{N}$ and each $i \in \mathbf{N}$. Then f is an open mapping, and it is not 1-sequence-covering [7, Example 3.10]. ■

It is well known that open mappings preserve first countable spaces. However, do open mappings preserve g-first countable spaces? This question was raised by Y. Tanaka in [12, Question 2.19(2)], and it is still open. We have the following relative question.

QUESTION 3.3. Do open mappings preserve sn-first countable spaces?

If the answer to Question 3.3 is affirmative, then so is the answer to Y. Tanaka's question. Note that almost sn-open mappings defined on first countable spaces are 1-sequence-covering mappings from Proposition 2.13(1) and 1-sequence-covering mappings preserve sn-first countable spaces [8, Corollary 2.4.13(1)]. It is an interesting question whether first countability of X in 2.13(1) can be relaxed to sn-first countability or g-first countability. We have the following question.

QUESTION 3.4. Is each open mapping defined on an sn-first countable space or on a g-first countable space a 1-sequence-covering mapping?

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