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FINITE GROUPS ADMITTING SOME COPRIME OPERATOR GROUPS

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Abstract. Let G be a finite group, with a finite operator group A, satisfying the following conditions: (1) $(|G|,|A|)=1$; (2) there exists a natural number m such that for any $\alpha,\beta\in A^\sharp$ we have: $[\,C_G(\alpha),\underbrace{C_G(\beta),\dots,C_G(\beta)}_{m}\,]=\{1\}$; (3) A is not cyclic. We prove the following: (1) If the exponent n of A is square-free, then G is nilpotent and its class is bounded by a function depending only on m and $\lambda(n)$ $(=n)$. (2) If $Z(A)=\{1\}$ and A has exponent n, then G is nilpotent and its class is bounded by a function depending only on m and $\lambda(n)$.

1. Introduction

Let G be a finite group and A a group acting on G. Since we are not supposing that the action is faithful, A will be referred to as an *operator group* on G. Giving some conditions on A and on G, deep information about the structure of G can be obtained. The most famous result is probably due to Thompson, who proved that if $|A|$ is a prime number and $C_G(A)=\{1\}$ then the finite group G is nilpotent (see Theorem 10.2.1 of [5]).

The aim of this paper is to extend some results obtained in [11] and [12]. We consider the following hypotheses:

($\star$) Let G be a finite group, with a finite operator group A, satisfying the following conditions:

(1) $(|G|,|A|)=1$;

(2) there exists a natural number m such that for any $\alpha,\beta\in A^\sharp$ we have:

$$[\,C_G(\alpha),\underbrace{C_G(\beta),\dots,C_G(\beta)}_{m}\,]=\{1\};$$

(3) A is not cyclic.

Here, if X is a group, $X^\sharp$ denotes the set $X\setminus\{1\}$.

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In [11] it is proved that, under the hypotheses ($\star$), if $|A| = p^2$ (p a prime number) then G is nilpotent with nilpotency class bounded by a function depending only on m and p. In [12] the case in which A is cyclic has been studied, proving that if $A = \langle\alpha\rangle$ has square-free exponent n, $C_G(A) = \{1\}$ and the hypotheses ($\star_1$) and ($\star_2$) are satisfied, then G is nilpotent with class bounded by a function depending only by m and n. In the cases considered, the exponent n of A is square free, that is $n = \lambda(n)$ where $\lambda(n)$ is the square free part of n defined as follows: given a natural number n and its factorization, $n = p_1^{h_1} \cdot p_2^{h_2} \cdot \ldots \cdot p_r^{h_r}$ ($p_i \neq p_j$ if $i \neq j$) we put $\lambda(n) = p_1 \cdot p_2 \cdot \ldots \cdot p_r$.

Using Lie methods, we prove:

Theorem 1. *Let G and A be two groups satisfying the hypotheses ($\star$). If the exponent n of A is square-free, then G is nilpotent and its class is bounded by a function depending only on m and $\lambda(n)$ ($= n$).*

Theorem 2. *Let G and A be two groups satisfying the hypotheses ($\star$). If $Z(A) = \{1\}$ and A has exponent n, then G is nilpotent and its class is bounded by a function depending only on m and $\lambda(n)$.*

Theorems 1 and 2 make us conjecture that if G and A satisfy the hypotheses ($\star$) then only *exceptionally* G is not nilpotent. In fact

Theorem 3. *If G and A satisfy the hypotheses ($\star_1$) and ($\star_2$) and if G is not nilpotent, then A has the structure of the complement of some finite Frobenius group.*

Using the classification of finite simple groups, we can prove that any group satisfying ($\star_1$) and ($\star_2$) is soluble (these hypotheses can be further weakened, see for example [13] and [3]; if $C_G(A) = \{1\}$ a nice proof can be found in [10]).

Using a result of Kurzweil ([8]), it can be proved in general that if G and A satisfy the hypotheses ($\star_1$) and ($\star_2$) then G has Fitting length at most 3 and this bound is reached. There exists in fact a finite group G with an automorphism σ of order a prime number p with $(|G|, p) = 1$, $[G, \langle\sigma\rangle] = G$ and such that $C_G(\sigma)$ is nilpotent and G has Fitting length 3 (see for example [2] and [6]).

2. Notations and preliminary results

All the groups considered in this paper are finite, even if the results of Theorems 1, 2 and 3 holds also under the weaker hypothesis that G is locally finite. We denote by G and A a pair of groups satisfying the hypothesis ($\star_1$) and by p, q, r and s prime numbers; the function λ has already been defined. If Γ is a set of parameters we say that a number is Γ-bounded if it is bounded by a function depending only on elements of Γ.

We use standard notation for groups and Lie rings (see [5], [9], [4], [7], [1]). In particular if G is a nilpotent group and A is an operator group of G with

$(|G|,|A|)=1$, then we can associate to G a nilpotent Lie ring $\mathcal{L}$ of the same class of G on which A acts (essentially) as on G.

If X and Y are subgroups of a group H (or subrings of a Lie ring $\mathcal{L}$) the subgroup (the subring) $[X, {}_mY]$ is defined inductively by $[X, {}_1Y] = [X,Y]$ and $[X, {}_mY] = [[X,Y], {}_{m-1}Y]$.

The following result will be used without further reference in many proofs by induction on $|G|$. Its proof is immediate.

Lemma 1. *The hypotheses ($\star$) are inductive on G, that is they are inherited by A-invariant subgroups and A-invariant quotients of G.* ∎

Lemma 2. *Let V be a vector space of finite dimension over a field of characteristic $p>0$. Let B be a non abelian group of order rs with $r>s$ distinct prime numbers and $(p,rs)=1$. If B acts on V then:*

$$V=\sum_{\beta\in B^\sharp} C_V(\beta).$$

Proof. Let $1\neq\langle\alpha\rangle$ be the normal subgroup of order r of B; then $C_V(\alpha)$ is a B-invariant subspace of V. By Maske Theorem (Theorem 3.3.1 of [5]) V admits a B-invariant complement W of $C_V(\alpha)$. We put $X=B\backslash\langle\alpha\rangle$ and $W_0=\sum_{\kappa\in X} C_W(\kappa)$. If we suppose $W_0\neq W$, then α and any $\kappa\in X$ act on $\overline{W}=W/W_0$ fixed points free. In this case Theorem 5.3.14.iii of [5] shows that the group $\langle\alpha\rangle\langle\kappa\rangle$ is cyclic, against the hypothesis. ∎

Lemma 3. *Let X be a group and assume that either*

(a) X is non abelian of square-free exponent of X or

(b) $Z(X)=\{1\}$.

If X does not contain elementary abelian subgroups of order p^2 for any $p\in\pi(X)$, then X is metacyclic and it contains a non abelian subgroup of order rs with $r>s$ distinct prime numbers.

Proof. By hypothesis all the Sylow p-subgroups of X, p odd, are cyclic, while the Sylow 2-subgroups are cyclic or generalized quaternion (Theorem 5.4.10.ii of [5]).

The Sylow 2-subgroups of X cannot be generalized quaternion. This is clear in case (a); in case (b) either

(i) $O_{2'}(X)=\{1\}$, then for Brauer-Suzuki Theorem (see chapter 12 of [5]) $Z(X)$ contains an involution: a contradiction;

or

(ii) $O_{2'}(X)\neq\{1\}$; since all the Sylow subgroups of $O_{2'}(X)$ are cyclic, $O_{2'}(X)$ is (cyclic or) metacyclic (Theorem 10.1.10 of [9]). A generalized quaternion group cannot act faithfully on a (cyclic or) metacyclic group and therefore we should have $Z(X)\neq\{1\}$ against the hypothesis.

Therefore X is metacyclic (Theorem 10.1.10 of [9]) and $X = \langle x\rangle\langle y\rangle$ with $\langle x\rangle$ normal in X. If $Z(X) = \{1\}$ we have $\langle x\rangle \cap \langle y\rangle = \{1\}$ because $\langle x\rangle \cap \langle y\rangle \leq Z(X)$ and therefore $(|\langle x\rangle|, |\langle y\rangle|) = 1$ by hypothesis. In any case there exists an element x_0 of $\langle x\rangle$ of prime order r. Since $C_X(x_0) \neq X$ there exists $y_0 \in \langle y\rangle$ of order a prime number $s \neq r$ that does not centralize x_0. Then $\langle x_0\rangle\langle y_0\rangle$ is the subgroup we are looking for. ■

Lemma 4. *If G and A are groups satisfying the hypothesis $(\star_1)$ then:*
(a) if N is a normal A-invariant subgroup of G, then

$$C_{G/N}(A) = C_G(A)N/N\ ;$$

(b) for any $p \in \pi(G)$, there exists an A-invariant Sylow p-subgroup of G;
(c) if A is not abelian and its exponent is square-free or if $Z(A) = \{1\}$ we have

$$G = \langle\ C_G(\alpha) \mid \alpha \in A^\sharp\ \rangle.$$

Proof. (a) and (b) are a direct consequence of Schur-Zassenhaus Theorem (see Theorem 6.2.2 of [5]).

If A contains an elementary abelian subgroup of order p^2 (p a prime number) then the conclusion is given by Theorem 6.2.4 of [5]. Hence, by Lemma 3, A contains a non abelian subgroup of order rs with r and s distinct prime numbers. The conclusion can be obtained using Lemma 2 and point (a). ■

Lemma 5. *Let B be a group of operators of the Lie ring $\mathcal{L}$. If B has exponent n and if $n\mathcal{L} = \mathcal{L}$ we have:*
(a) if $\mathcal{I}$ is a B-invariant ideal of $\mathcal{L}$ then $C_{\mathcal{L}/\mathcal{I}}(B) = (C_{\mathcal{L}}(B) + \mathcal{I})/\mathcal{I}$;
(b) if B is not cyclic and n is square-free or if $Z(B) = \{1\}$ then

$$\mathcal{L} = \sum_{\beta \in B^\sharp} C_{\mathcal{L}}(\beta).$$

Proof. (a) is a simple extension of Lemma 2.2.1 of [11]. If $C \leq B$ is elementary abelian of order p^2 (p a prime number) then clearly $p\mathcal{L} = \mathcal{L}$, Lemma 2.2.2 of [11] gives

$$\mathcal{L} = \sum_{\gamma \in C^\sharp} C_{\mathcal{L}}(\gamma)$$

and the conclusion. If B do not contains elementary abelian subgroups of order p^2 then by, Lemma 3, B contains a non abelian subgroup of order rs. The conclusion follows from Lemma 2 and point (a). ■

Lemma 6. *Let $\mathcal{L}$ be a metabelian Lie ring and B an operator group of $\mathcal{L}$ of exponent n. Suppose that $n\mathcal{L} = \mathcal{L}$ and that $[C_{\mathcal{L}}(\alpha), {}_m C_{\mathcal{L}}(\beta)] = \{0\}$ for any $\alpha, \beta \in B^\sharp$. If B is not cyclic and n is square-free or if $Z(B) = \{1\}$ then $\mathcal{L}$ is nilpotent of class $\{m,\ \lambda(n)\}$-bounded.*

Proof. Let $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$; by Lemma 5 (b) we have:

$$\mathcal{L} = \sum_{\kappa \in K^\sharp} C_{\mathcal{L}}(\kappa)\,, \qquad \mathcal{L}' = \sum_{\kappa \in K^\sharp} C_{\mathcal{L}'}(\kappa).$$

Since $\mathcal{L}$ is metabelian, for any $x \in \mathcal{L}'$ and any $a, b \in \mathcal{L}$ we have $[x, a, b] = [x, b, a]$. If $t = m(rs - 1) + 1$, by the bilinearity of $[\ ,\ ]$ we get $[\mathcal{L}',\, {}_t\mathcal{L}] = \{0\}$ and therefore $\mathcal{L}$ is nilpotent of class at most $t + 1$. It is now enough to observe that $rs \leq n = \lambda(n)$. ∎

LEMMA 7. *Let $\mathcal{L}$ be a soluble Lie ring and B be an operator group of $\mathcal{L}$ of exponent n. Suppose that $n\mathcal{L} = \mathcal{L}$ and that $[C_{\mathcal{L}}(\alpha),\, {}_mC_{\mathcal{L}}(\beta)] = \{0\}$ for any $\alpha, \beta \in B^\sharp$. If one of the following conditions is satisfied:*

(a) B is not cyclic and n is square-free,

(b) $Z(B) = \{1\}$,

then $\mathcal{L}$ is nilpotent of class $\{m, \lambda(n)\}$-bounded.

Proof. Let d be the derived length of $\mathcal{L}$.

It follows from Lemma 6, using Proposition 7.1.1 of [1] (or Theorem 2.4 of [11]), that $\mathcal{L}$ is nilpotent of class $\{m, \lambda(n), d\}$-bounded. Hence the statement is proved if we show that d is $\{m, \lambda(n)\}$-bounded. Let $\beta \in B^\sharp$, by Lemma 5 (b) we have:

$$[\mathcal{L},\, {}_mC_{\mathcal{L}}(\beta)] = [\sum_{\kappa \in B^\sharp} C_{\mathcal{L}}(\kappa),\, {}_mC_{\mathcal{L}}(\beta)] = \sum_{\kappa \in B^\sharp} [C_{\mathcal{L}}(\kappa),\, {}_mC_{\mathcal{L}}(\beta)] = 0.$$

Theorem 2.3 of [11] gives the conclusion. ∎

3. Proofs of the theorems

To prove Theorems 1 and 2 it is sufficient to show that if the group G satisfies their hypotheses, then G is nilpotent; the bound of the nilpotency class is then a consequence of Lemma 7 and of the correspondence between (finite) nilpotent groups and nilpotent Lie rings (described in [7] or in chapter VIII of [4]).

We suppose, by contradiction, that there exist counterexamples to Theorem 1 or Theorem 2. We choose among these counterexample, one such that $|G| + |A|$ is minimal. Let $p \in \pi(G)$ be an odd prime number and P an A-invariant Sylow p-subgroup of G. Let $N = N_G(Z(J(P)))$. If $N \neq P$ then $|N| + |A| < |G| + |A|$ and N is nilpotent. By Glauberman-Thompson Theorem (Theorem 8.3.1 of [5]) we have that G has a normal p-complement, which is A-invariant. If $N = G$ then $\{1\} \neq Z(J(P))$ is a normal, nilpotent and A-invariant subgroup of G. In both cases we have $F(G) \neq \{1\}$ and since $|G/F(G)| + |A| < |G| + |A|$, $G/F(G)$ is nilpotent. With a similar argument it can be proved that $G/F(G)$ is minimal A-invariant and therefore there exists $q \in \pi(G)$ such that $G/F(G)$ is an elementary abelian q-group (on which A acts irreducibly).

We suppose that G admits two minimal normal A-invariant subgroups N_1 and N_2, then G/N_1 and G/N_2 are nilpotent. Since $N_1 \cap N_2 = \{1\}$, the group G should be nilpotent: a contradiction.

We only have to examine the case in which G has a unique minimal normal A-invariant subgroup N; in this case N, and therefore $F(G)$, is a p-subgroup for some $p \in \pi(G)$, $p \neq q$. If $\Phi(F(G)) \neq \{1\}$ we get a contradiction, when we consider $G/\Phi(F(G))$. We can therefore suppose $F(G) = P$, an elementary abelian p-subgroup of G and $G = PQ$ with P and Q elementary abelian Sylow subgroups. As in the proof of Lemma 3 we obtain that A contains a non abelian subgroup B of order rs with r and s distinct prime numbers. Since the hypotheses $(\star)$ are still valid for the non-cyclic subgroups of A, if $B \neq A$ we have $|G| + |B| < |G| + |A|$ and G should be nilpotent. Therefore $A = B$ and $|A| = rs$. Lemma 4 (c) gives $P = \langle\, C_P(\alpha) \mid \alpha \in A^\sharp \,\rangle$ and $Q = \langle\, C_Q(\alpha) \mid \alpha \in A^\sharp \,\rangle$.

Let α be an element of A with $C_Q(\alpha) \neq \{1\}$ and let y be a non trivial element in $C_Q(\alpha)$. Take $x_1, x_2 \in P$, and recall that $[x_1, y], [x_2, y] \in P$. From the fact that P is abelian, we get $[x_1x_2, y] = [x_1, y]^{x_2}[x_2, y] = [x_1, y][x_2, y]$.

By hypothesis we have $[C_P(\beta), {}_m\langle y\rangle] = \{1\}$ for all $\beta \in A^\sharp$ and therefore $[C_P(\beta), \langle y\rangle] = \{1\}$ because $(|P|, |\langle y\rangle|) = 1$. Since $P = \langle C_P(\beta) \mid \beta \in A^\sharp\rangle$ we obtain $[P, \langle y\rangle] = \{1\}$. Then $1 \neq y \in C_G(P) \leq P$ is the contradiction that concludes the proof. ■

The proof of Theorem 3 holds even without the hypothesis that G is soluble and it is therefore independent from the classification of finite simple groups.

For any $p \in \pi(G)$, by Lemma 4 (b) there exists an A-invariant Sylow p-subgroup of G. We can consider two cases.

- ✓ For any odd prime number $p \in \pi(G)$ the group G has a normal p-complement. If P is a Sylow 2-subgroup of G, then P is normal (and therefore A-invariant) and we can choose $q \in \pi(G)$ and a Sylow q-subgroup Q of G such that PQ is not nilpotent.
- ✓ There exists an odd prime number $p \in \pi(G)$ such that G has not a normal p-complement. If T is an A-invariant Sylow p-subgroup of G then the subgroup $N = N_G(Z(J(T)))$ is not nilpotent. If we consider the A-invariant subgroup $P = O_p(N)$, we can conclude that there exists $q \in \pi(N)$ and an A-invariant Sylow q-subgroup Q of N such that PQ is not nilpotent.

In any case G admits a non-nilpotent A-invariant $\{p, q\}$-subgroup PQ with P normal in PQ. It is possible to choose a non-nilpotent A-invariant section $\overline{S}$ in PQ of minimal order. It is easy to prove that $\overline{S} = \overline{P}\,\overline{Q}$ where $\overline{P}$ and $\overline{Q}$ are respectively an A-invariant Sylow p-subgroup and q-subgroup of $\overline{S}$ and $F(\overline{S}) = \overline{P}$. In particular, by Theorem 6.1.3 of [5], $C_{\overline{S}}(\overline{P}) = \overline{P}$. We can consider two cases.

- ✓ Any element of $A^\sharp$ acts fixed point free on $\overline{P}$. Then the semidirect product $\overline{P}A$ is a Frobenius group with complement A and the statement is proved.
- ✓ There exists $\alpha \in A^\sharp$ with $C_{\overline{P}}(\alpha) \neq \{1\}$. Let x be a non trivial element of $C_{\overline{P}}(\alpha)$. By the minimality of $|\overline{S}|$ we have $\overline{P} = \langle x\rangle^{\overline{Q}A}$. If there exists $\beta \in A^\sharp$

with $C_{\overline{Q}}(\beta) \neq \{1\}$ then taken $y \in C_{\overline{Q}}(\beta)$, $y \neq 1$, by hypothesis we have $[x, {}_m y] = 1$ and therefore, since $\overline{P} = \langle x \rangle^{\overline{Q}A}$, $[\overline{P}, {}_m y] = \{1\}$ and $(|\overline{P}|, |\overline{Q}|) = 1$, $[\overline{P}, y] = \{1\}$, against the fact that $C_{\overline{S}}(\overline{P}) = \overline{P}$. Therefore any element of $A^\sharp$ acts fixed points free on $\overline{Q}$. Then the semidirect product $\overline{Q}A$ is a Frobenius group with complement A and the statement is proved. ■

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