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THE BANACH ALGEBRA $\mathcal{B}(X)$, WHERE X IS A BK SPACE AND APPLICATIONS

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Abstract. In this paper we give some properties of Banach algebras of bounded operators $\mathcal{B}(X)$, when X is a BK space. We then study the solvability of the equation $Ax = b$ for $b \in \{s_\alpha, s_\alpha^\circ, s_\alpha^{(c)}, l_p(\alpha)\}$ with $\alpha \in U^+$ and $1 \leqslant p < \infty$. We then deal with the equation $T_a x = b$, where $b \in \chi(\Delta^k)$ for $k \geqslant 1$ integer, $\chi \in \{s_\alpha, s_\alpha^\circ, s_\alpha^{(c)}, l_p(\alpha)\}$, $1 \leqslant p < \infty$ and T_a is a Toeplitz triangle matrix. Finally we apply the previous results to infinite tridiagonal matrices and explicitly calculate the inverse of an infinite tridiagonal matrix. These results generalize those given in [4, 9].

1. Preliminary results

Let $A = (a_{nm})_{n,m\geqslant 1}$ be an infinite matrix and consider the sequence $x = (x_n)_{n\geqslant 1}$. We will define the product $Ax = (A_n(x))_{n\geqslant 1}$ with $A_n(x) = \sum_{m=1}^{\infty} a_{nm}x_m$ whenever the series are convergent for all $n \geqslant 1$. Throughout this paper we use the convention that any term with subscript less than 1 is equal to naught. Let s denote the set of all complex sequences. We write φ, c_0, c and l_∞ for the sets of finite, null, convergent and bounded sequences respectively. For any given subsets X, Y of s, we shall say that the operator represented by the infinite matrix $A = (a_{nm})_{n,m\geqslant 1}$ maps X into Y, that is $A \in (X, Y)$, see [5], if

i) the series defined by $A_n(x) = \sum_{m=1}^{\infty} a_{nm}x_m$ are convergent for all $n \geqslant 1$ and for all $x \in X$;

ii) $Ax \in Y$ for all $x \in X$.

For any subset X of s, we shall write

$$AX = \{\, y \in s : \text{there is } x \in X, \quad y = Ax\,\}.$$

If Y is a subset of s, we shall denote the so-called matrix domain by

$$Y(A) = \{\, x \in s : y = Ax \in Y\}.$$

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Let $X \subset s$ be a Banach space, with norm $\|\cdot\|_X$. By $\mathcal{B}(X)$ we will denote the set of all bounded linear operators, mapping X into itself. We shall say that $A \in \mathcal{B}(X)$ if and only if $A\colon X \to X$ is a linear operator and

$$\|A\|^*_{\mathcal{B}(X)} = \sup_{x \neq 0}(\|Ax\|_X / \|x\|_X) < \infty.$$

It is well known that $\mathcal{B}(X)$ is a Banach algebra with the norm $\|A\|^*_{\mathcal{B}(X)}$, see [1]. A Banach space $X \subset s$ is a BK space if the projection $P_n\colon x \mapsto x_n$ from X into $\mathbb{C}$ is continuous for all n. A BK space $X \supset \varphi$ is said to have AK if for every $x \in X$, $x = \lim_{p\to\infty} \sum_{k=1}^{p} x_k e_k$, where $e_k = (0, \dots, 1, \dots)$, 1 being in the k-th position. It is well known that if X has AK then $\mathcal{B}(X) = (X, X)$, see [19]. In the following we shall explicitly give some new properties of particular algebras.

2. The set $\mathcal{B}(l_p(\alpha))$ for $1 \leqslant p \leqslant \infty$

2.1. The set $\mathcal{B}(l_p(\alpha))$ for $p \geqslant 1$ real

Put now

$$U^+ = \{\, x = (x_n)_{n\geqslant 1} \in s \ : \ x_n > 0 \text{ for all } n \,\}.$$

Recall that l_p, for $p > 0$ is the set of sequences $x = (x_n)_{n\geqslant 1}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Using Wilansky's notations [21], for any given $\alpha = (\alpha_n)_{n\geqslant 1} \in U^+$ and $p \geqslant 1$ real we have

$$l_p(\alpha) = \left(\frac{1}{\alpha}\right)^{-1} * l_p = \left\{ x \in s \ : \ \sum_{n=1}^{\infty} \left(\frac{|x_n|}{\alpha_n}\right)^p < \infty \right\}.$$

Define the diagonal matrix $D_\xi = (\xi_n \delta_{nm})_{n,m\geqslant 1}$, (where $\delta_{nm} = 0$ for all $n \neq m$ and $\delta_{nm} = 1$ otherwise), we then have $D_\alpha l_p = l_p(\alpha)$. It is easy to see that $l_p(\alpha)$ is a Banach space with the norm

$$\|x\|_{l_p(\alpha)} = \|D_{\frac{1}{\alpha}} x\|_{l_p} = \left[\sum_{n=1}^{\infty} \left(\frac{|x_n|}{\alpha_n}\right)^p\right]^{\frac{1}{p}}.$$

We have the following lemma.

Lemma 1. *Let $\alpha = (\alpha_n)_{n\geqslant 1}$, $\beta = (\beta_n)_{n\geqslant 1} \in U^+$ and $p \geqslant 1$ a real. The condition $\alpha/\beta \in l_\infty$ implies that $l_p(\alpha) \subset l_p(\beta)$.*

Proof. Since $p \geqslant 1$, we get $\alpha/\beta \in l_\infty$ if and only if $(\alpha/\beta)^p \in l_\infty$ and for all $x \in l_p(\alpha)$

$$\|x\|^p_{l_p(\beta)} = \sum_{n=1}^{\infty} \left(|x_n| \frac{1}{\alpha_n} \frac{\alpha_n}{\beta_n}\right)^p \leqslant \sup_{n\geqslant 1} \left[\left(\frac{\alpha_n}{\beta_n}\right)^p\right] \|x\|^p_{l_p(\alpha)}.$$

This gives the conclusion. ∎

We also have $l_p(\alpha) \subset l_{p'}(\alpha)$ for $1 \leqslant p \leqslant p'$.

As we will see later, $l_p(\alpha)$ has AK, so $\mathcal{B}(l_p(\alpha)) = (l_p(\alpha), l_p(\alpha))$ and $\mathcal{B}(l_p(\alpha))$ is a Banach algebra with identity. So we get

$$\|Ax\|_{l_p(\alpha)} \leqslant \|A\|^*_{\mathcal{B}(l_p(\alpha))} \|x\|_{l_p(\alpha)} \text{ for all } x \in l_p(\alpha).$$

We have $l_p = l_p(e)$, where $e = (1, \dots, 1, \dots)$ and

$$\|D_{\frac{1}{\alpha}} A D_\alpha\|^*_{\mathcal{B}(l_p)} = \|A\|^*_{\mathcal{B}(l_p(\alpha))} \quad \text{for all } A \in \mathcal{B}(l_p(\alpha)).$$

Indeed, writing $D_\alpha x = y$, we get

$$\sup_{x \neq 0} \left(\frac{\|(D_{\frac{1}{\alpha}} A D_\alpha) x\|_{l_p}}{\|x\|_{l_p}} \right) = \sup_{y \neq 0} \left(\frac{\|D_{\frac{1}{\alpha}} A y\|_{l_p}}{\|D_{\frac{1}{\alpha}} y\|_{l_p}} \right) = \|A\|^*_{\mathcal{B}(l_p(\alpha))}.$$

So we can say that $A \in \mathcal{B}(l_p(\alpha))$ if and only if $D_{1/\alpha} A D_\alpha \in \mathcal{B}(l_p)$. When $\alpha = (r^n)_{n \geqslant 1}$, for a given real $r > 0$, $l_p(\alpha)$ is denoted $l_p(r)$. When $p = \infty$, we obtain the next results.

2.2. The case $p = \infty$ and Banach algebra S_α

Let $\alpha = (\alpha_n)_{n \geqslant 1} \in U^+$. Using Wilansky's notation [20], we have $l_\infty(\alpha) = (1/\alpha)^{-1} * l_\infty = D_\alpha l_\infty$. We will write

$$s_\alpha = l_\infty(\alpha) = \{\, x \in s \ : \ x_n/\alpha_n = O(1) \ (n \to \infty) \,\},$$

see [6–15]. The set s_α is a Banach space with the norm $\|x\|_{s_\alpha} = \sup_{n \geqslant 1}(|x_n|/\alpha_n)$. The set

$$S_\alpha = \left\{ A = (a_{nm})_{n,m \geqslant 1} \ : \ \sup_{n \geqslant 1} \Big(\sum_{m=1}^{\infty} |a_{nm}| \tfrac{\alpha_m}{\alpha_n} \Big) < \infty \right\}, \tag{1}$$

is a Banach algebra with identity normed by

$$\|A\|_{S_\alpha} = \sup_{n \geqslant 1} \Big(\sum_{m=1}^{\infty} |a_{nm}| \tfrac{\alpha_m}{\alpha_n} \Big). \tag{2}$$

Recall that if $A \in (s_\alpha, s_\alpha)$, then $\|Ax\|_{s_\alpha} \leqslant \|A\|_{S_\alpha} \|x\|_{s_\alpha}$ for all $x \in s_\alpha$. Thus we obtain the following result where we put $B(s_\alpha) = \mathcal{B}(s_\alpha) \cap (s_\alpha, s_\alpha)$.

LEMMA 2. *For any given $\alpha \in U^+$ we have $B(s_\alpha) = S_\alpha = (s_\alpha, s_\alpha)$.*

As we have seen above when $\alpha = (r^n)_{n \geqslant 1}$, $r > 0$, S_α and s_α are denoted S_r and s_r. When $r = 1$, $s_1 = l_\infty$ is the set of all bounded sequences.

Recall [14] that $s_\alpha = s_\beta$ if and only if there are K_1, $K_2 > 0$ such that $K_1 \leqslant \alpha_n/\beta_n \leqslant K_2$ for all n.

In the same way we will define the sets

$$\overset{\circ}{s}_\alpha = \left\{ x \in s \ : \ \frac{x_n}{\alpha_n} = o(1) \ \ (n \to \infty) \right\}$$

and

$$s_\alpha^{(c)} = \left\{ x \in s \ : \ \frac{x_n}{\alpha_n} \to l \quad (n \to \infty) \text{ for some } l \right\}.$$

The sets $\overset{\circ}{s}_\alpha$ and $s_\alpha^{(c)}$ are Banach spaces with the norm $\|\cdot\|_{s_\alpha}$.

3. New Banach algebras

3.1. The Banach algebra $\mathcal{B}(\chi)$ for $\chi \in \{s_\alpha, \overset{\circ}{s}_\alpha, s_\alpha^{(c)}, l_p(\alpha)\}$ with $1 \leqslant p < \infty$

In this section we will give an explicit expression for the norm $\|A\|^*_{\mathcal{B}(l_p(\alpha))}$ for $1 \leqslant p \leqslant \infty$ and give some properties of the equation $Ax = b$ for $A \in \mathcal{B}(\chi)$, $b \in \chi$ with $\chi \in \{s_\alpha, \overset{\circ}{s}_\alpha, s_\alpha^{(c)}, l_p(\alpha)\}$ and $1 \leqslant p < \infty$.

We have the next result where U is the set of all sequences $u = (u_n)_{n \geqslant 1}$ with $u_n \neq 0$ for all n and $\mathcal{L}$ is the set of lower triangular infinite matrices, (that is $A \in \mathcal{L}$ if $a_{nm} = 0$ for $m > n$).

Lemma 3. i) *Let $T \in \mathcal{L}$ be a triangle, that is $t_{nn} \neq 0$ for all n, and X a BK space. Then $X(T)$ is a BK space with the norm*

$$\|x\|_{X(T)} = \|Tx\|_X. \tag{3}$$

ii) *Let $T = D_a = (a_n \delta_{nm})_{n,m \geqslant 1}$ be a diagonal matrix with $a \in U$ and X a BK space with AK. Then $X(T)$ has AK with the norm given by (3).*

Proof. i) was shown in [17] and ii) comes from [20, Theorem 4.3.6 pp. 52]. ■

We then have the following result.

Lemma 4. i) *Let $\alpha \in U^+$. Then s_α, $\overset{\circ}{s}_\alpha$ and $s_\alpha^{(c)}$ are BK spaces with the norm $\|\cdot\|_{s_\alpha}$ and $\overset{\circ}{s}_\alpha$ has AK. The set $l_p(\alpha)$ for $1 \leqslant p < \infty$ is a BK space and has AK with the norm $\|\cdot\|_{l_p(\alpha)}$.*

ii) *Let χ be any of the spaces s_α, $\overset{\circ}{s}_\alpha$, or $s_\alpha^{(c)}$. Then*

$$|P_n(x)| = |x_n| \leqslant \alpha_n \|x\|_{s_\alpha} \text{ for all } n \geqslant 1 \text{ and for all } x \in \chi.$$

Proof. i) First we have $s_\alpha = s_1(D_{1/\alpha})$, since $x \in s_\alpha$ if and only if $D_{1/\alpha} x = x/\alpha \in s_1$. It is well known that s_1 is a BK space with respect to the norm $\|\cdot\|_{s_1}$, so by Lemma 3, the set $s_1(D_{1/\alpha})$ is also a BK space with

$$\|x\|_{s_1(D_{1/\alpha})} = \|D_{1/\alpha} x\|_{s_1} = \|x\|_{s_\alpha}.$$

We also have $\overset{\circ}{s}_\alpha = c_0(D_{1/\alpha})$ and $s_\alpha^{(c)} = c(D_{1/\alpha})$. We conclude since c_0 and c are BK spaces with respect to the norm $\|\|_{s_1}$, and c_0 has AK. Finally l_p for $1 \leqslant p < \infty$ being a BK space with AK, it follows that $l_p(\alpha) = l_p(D_{1/\alpha})$ also has AK.

ii) is a direct consequence of the definition of the sets s_α, $\overset{\circ}{s}_\alpha$, and $s_\alpha^{(c)}$. ■

Remark 1. Note that if X is a BK space with the norm $\|\cdot\|_X$, then

$$(X, X) \subset \mathcal{B}(X).$$

Indeed, by [17 Theorem 4.2.8 p. 57], since X is a BK space, the matrix map $A \in (X, X)$ is continuous and there is $M > 0$ such that

$$\|Ax\|_X \leqslant M \|x\|_X \text{ for all } x \in X.$$

To obtain other results we require some definitions and lemmas. We will write $B_X(0,1) = \{ x \in X \ : \ \|x\|_X \leqslant 1 \}$ for the unit ball, where X is any given BK space. Thus we get

$$\|A\|^*_{\mathcal{B}(X)} = \sup_{x \neq 0}(\|Ax\|_X / \|x\|_X) = \sup_{x \in B_X(0,1)} (\|Ax\|_X) \text{ for all } A \in \mathcal{B}(X).$$

Recall that for all $a = (a_n)_{n \geqslant 1} \in X$ such that the series $\sum_{n=1}^\infty a_n x_n$ is convergent for all $x \in X$ the identity

$$\|a\|^*_X = \sup_{x \in B_X(0,1)} \Big(\Big| \sum_{n=1}^\infty a_n x_n \Big| \Big)$$

is defined and finite. In Malkowsky [17], we have

Lemma 5. *Let X be a BK space. Then $A \in (X, l_\infty)$ if and only if*

$$\sup_{n \geqslant 1}(\|A_n\|^*_X) = \sup_{n \geqslant 1}\Big(\sup_{x \in B_X(0,1)} \Big| \sum_{n=1}^\infty a_{nk} x_k \Big| \Big) < \infty.$$

We also have, (cf. [17, Theorem 1.2.3, p. 155]),

Lemma 6. *For every $a \in l_1$, $\|a\|^*_{c_0} = \|a\|^*_c = \|a\|^*_{l_\infty} = \|a\|_{l_1} = \sum_{n=1}^\infty |a_n|$.*

It can be easily deduced that if X is a BK space, then $A \in (X, s_\beta)$ if and only if

$$\sup_{n \geqslant 1} \Big\| \frac{1}{\beta_n} A_n \Big\|^*_X < \infty.$$

Since there is no characterization of the set $(l_p(\alpha), l_p(\alpha))$ for $1 < p < \infty$ and $p \neq 2$, we need to define a subset $\widehat{\mathcal{B}}_p(\alpha)$ of $(l_p(\alpha), l_p(\alpha))$ permitting us to obtain the inverse of some well chosen matrix map $A \in (l_p(\alpha), l_p(\alpha))$. In this way we are led to define the number

$$N_{p,\alpha}(A) = \Big[\sum_{n=1}^\infty \Big(\sum_{m=1}^\infty \Big(|a_{nm}| \tfrac{\alpha_m}{\alpha_n} \Big)^q \Big)^{p-1} \Big]^{\frac{1}{p}},$$

for $1 < p < \infty$ and $q = p/(p-1)$. Thus we can state the following

Proposition 7. *Let $\alpha \in U^+$. Then*

i) *for every $A \in S_\alpha$*

$$\|A\|^*_{\mathcal{B}(s_\alpha)} = \|A\|_{S_\alpha} = \sup_{n \geqslant 1} \Big(\sum_{m=1}^\infty |a_{nm}| \tfrac{\alpha_m}{\alpha_n} \Big);$$

ii) a) $\mathcal{B}(l_1(\alpha)) = (l_1(\alpha), l_1(\alpha))$ *and $A \in \mathcal{B}(l_1(\alpha))$ if and only if $A^t \in S_{1/\alpha}$,*

b) $\|A\|^*_{\mathcal{B}(l_1(\alpha))} = \|A^t\|_{S_{1/\alpha}}$ *for all $A \in \mathcal{B}(l_1(\alpha))$.*

iii) *For $1 < p < \infty$ we have $\widehat{\mathcal{B}}_p(\alpha) \subset \mathcal{B}(l_p(\alpha))$, where*

$$\widehat{\mathcal{B}}_p(\alpha) = \{ A = (a_{nm})_{n,m \geqslant 1} \ : \ N_{p,\alpha}(A) < \infty \},$$

and for every $A \in \widehat{\mathcal{B}}_p(\alpha)$, $\|A\|^*_{\mathcal{B}(l_p(\alpha))} \leqslant N_{p,\alpha}(A)$.

Proof. i) First we have

$$\|Ax\|_{s_\alpha} = \sup_{n\geqslant 1}\Big(\frac{1}{\alpha_n}\Big|\sum_{k=1}^{\infty} a_{nk}x_k\Big|\Big) = \sup_{n\geqslant 1}\left(\frac{1}{\alpha_n}|A_n(x)|\right) \text{ for all } x \in s_\alpha \tag{4}$$

then

$$\|A\|^*_{\mathcal{B}(s_\alpha)} = \sup_{x\in B_{s_\alpha}(0,1)}\left(\sup_{n\geqslant 1}\left(\frac{1}{\alpha_n}|A_n(x)|\right)\right) = \sup_{n\geqslant 1}\left(\frac{1}{\alpha_n}\sup_{x\in B_{s_\alpha}(0,1)}(|A_n(x)|)\right). \tag{5}$$

Writing $x = \alpha y$ in (5) we obtain

$$\|A\|^*_{\mathcal{B}(s_\alpha)} = \sup_{n\geqslant 1}\left(\frac{1}{\alpha_n}\sup_{y\in B_{s_1}(0,1)}(|A_n(\alpha y)|)\right) = \sup_{n\geqslant 1}\left(\frac{1}{\alpha_n}\|A_n D_\alpha\|^*_{l_\infty}\right).$$

Now by Lemma 6 we have

$$\|A_n D_\alpha\|^*_{l_\infty} = \|(a_{nk}\alpha_k)_{k\geqslant 1}\|_{l_1} = \sum_{k=1}^{\infty}|a_{nk}|\alpha_k \text{ for all } n.$$

We conclude that $\|A\|^*_{\mathcal{B}(s_\alpha)} = \sup_{n\geqslant 1}(\sum_{m=1}^{\infty}|a_{nm}|\alpha_m/\alpha_n)$.

ii) By [17, Theorem 2.27, p. 175] we have $\mathcal{B}(l_1) = (l_1, l_1)$ and the condition $A \in (l_1(\alpha), l_1(\alpha))$ is equivalent to $D_{1/\alpha}AD_\alpha \in (l_1, l_1)$. It is well known that $D_{1/\alpha}AD_\alpha \in (l_1, l_1)$ if and only if $(D_{1/\alpha}AD_\alpha)^t \in S_1$. This means that $D_\alpha A^t D_{1/\alpha} \in S_1$ and $A^t \in S_{1/\alpha}$. Furthermore, since $\|A\|^*_{\mathcal{B}(l_1)} = \|A^t\|_{S_1}$, so $\|A\|^*_{\mathcal{B}(l_1(\alpha))} = \|A^t D_{1/\alpha}\|_{S_1} = \|A^t\|_{S_\alpha}$. This permits us to conclude for ii).

iii) Let $A \in \widehat{\mathcal{B}}_p(\alpha)$ be a given infinite matrix and take any $x \in l_p$. We have

$$\|Ax\|^p_{l_p} = \Big\|\Big(\sum_{m=1}^{\infty} a_{nm}x_m\Big)_{n\geqslant 1}\Big\|^p_{l_p} = \sum_{n=1}^{\infty}\Big|\sum_{m=1}^{\infty} a_{nm}x_m\Big|^p \leqslant \sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty}|a_{nm}x_m|\Big)^p,$$

and from the Hölder's inequality, we get for every n

$$\sum_{m=1}^{\infty}|a_{nm}x_m| \leqslant \Big(\sum_{m=1}^{\infty}|a_{nm}|^q\Big)^{\frac{1}{q}}\Big(\sum_{m=1}^{\infty}|x_m|^p\Big)^{\frac{1}{p}} = \Big(\sum_{m=1}^{\infty}|a_{nm}|^q\Big)^{\frac{1}{q}}\|x\|_{l_p},$$

with $q = p/(1-p)$. We deduce that

$$\|Ax\|^p_{l_p} \leqslant \sum_{n=1}^{\infty}\Big[\Big(\sum_{m=1}^{\infty}|a_{nm}|^q\Big)^{\frac{1}{q}}\|x\|_{l_p}\Big]^p = \sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty}|a_{nm}|^q\Big)^{\frac{p}{q}}\|x\|^p_{l_p};$$

and since $p/q = p - 1$, we have

$$\|Ax\|_{l_p} \leqslant \Big[\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty}|a_{nm}|^q\Big)^{p-1}\Big]^{\frac{1}{p}}\|x\|_{l_p}$$

and

$$\|A\|^*_{\mathcal{B}(l_p)} = \sup_{x\neq 0}\Big(\frac{\|Ax\|_{l_p}}{\|x\|_{l_p}}\Big) \leqslant \Big[\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty}|a_{nm}|^q\Big)^{p-1}\Big]^{\frac{1}{p}}. \tag{6}$$

We have proved that if $A \in \widehat{\mathcal{B}}_p(e)$, then $A \in \mathcal{B}(l_p)$. So if $A \in \widehat{\mathcal{B}}_p(\alpha)$ and $D_{1/\alpha} A D_\alpha \in \widehat{\mathcal{B}}_p(e)$, then $D_{1/\alpha} A D_\alpha \in \mathcal{B}(l_p)$ and A belongs to $\mathcal{B}(l_p(\alpha))$. This concludes the proof. ■

3.2. Application to the solvability of the equation $Ax = b$

For $a = (a_{nn})_{n\geqslant 1} \in U$ we get the following elementary but very usefull result

Proposition 8. *Let $X \subset s$ be a BK space. Assume that $D_{1/a} A \in (X, X)$ and*

$$\|I - D_{1/a} A\|^*_{\mathcal{B}(X)} < 1. \tag{7}$$

Then the equation $Ax = b$ with $D_{1/a} b = (b_n / a_{nn})_{n\geqslant 1} \in X$ admits a unique solution in X given by $x = (D_{1/a} A)^{-1} D_{1/a} b$.

Proof. First we see that since $D_{1/a} A \in (X, X)$ and condition (7) holds, then $D_{1/a} A$ is invertible in $\mathcal{B}(X)$. Since $\mathcal{B}(X)$ is a Banach algebra of operators with identity, then

$$(D_{1/a} A)^{-1} (D_{1/a} Ax) = [(D_{1/a} A)^{-1} o (D_{1/a} A)](x) = I(x) = x \quad \text{ for all } x \in X.$$

Thus the equation $Ax = b$ with $D_{1/a} b \in X$ is equivalent to $D_{1/a} Ax = D_{1/a} b$ which in turn is $x = (D_{1/a} A)^{-1} (D_{1/a} b) \in X$, this concludes the proof. ■

We can express a similar result in a more general case.

In the following we will write

$$\Gamma_\alpha = \{\, A = (a_{nm})_{n,m\geqslant 1} \in S_\alpha \,:\, \|I - A\|_{S_\alpha} < 1 \,\},$$

and for $1 < p < \infty$,

$$\Gamma'_{p,\alpha} = \{\, A = (a_{nm})_{n,m\geqslant 1} \in (l_p(\alpha), l_p(\alpha)) \,:\, N_{p,\alpha}(I - A) < 1 \,\}, \text{ for } 1 < p < \infty.$$

Note that since S_α is a Banach algebra, the condition $A \in \Gamma_\alpha$ means that A is invertible and $A^{-1} \in S_\alpha$.

In the following we will put $|a| = (|a_{nn}|)_{n\geqslant 1}$ for any given $a = (a_{nn})_{n\geqslant 1}$.

Corollary 9. *Let $\alpha \in U^+$ and A be an infinite matrix with $a \in U$. Assume that $D_{1/a} A \in \Gamma_\alpha$. Then*

i) a) *for any given $b \in s_{|a|\alpha}$, the equation $Ax = b$ admits in s_α a unique solution given by*

$$x^0 = (D_{1/a} A)^{-1} (D_{1/a} b) = A^{-1} b \tag{8}$$

with $A^{-1} \in (s_{|a|\alpha}, s_\alpha)$;

b) *if $\lim_{n\to\infty} (a_{nm} / a_{nn} \alpha_n) = 0$ for all $m \geqslant 1$, for any given $b \in \overset{\circ}{s}_{|a|\alpha}$, the equation $Ax = b$ admits a unique solution in $\overset{\circ}{s}_\alpha$ given by (8) and $A^{-1} \in (\overset{\circ}{s}_{|a|\alpha}, \overset{\circ}{s}_\alpha)$;*

c) *if $\lim_{n\to\infty} (a_{nm} / a_{nn} \alpha_n) = l_m$ for some l_m, $m \geqslant 1$ and*

$$\lim_{n\to\infty} \left(\frac{1}{\alpha_n a_{nn}} \sum_{m=1}^{\infty} a_{nm} \alpha_m \right) = l \quad \textit{for some } l \tag{9}$$

then for any given $b \in s^{(c)}_{|a|\alpha}$*, the equation* $Ax = b$ *admits a unique solution in* $s^{(c)}_{\alpha}$ *given by (8) with* $A^{-1} \in (s^{(c)}_{|a|\alpha}, s^{(c)}_{\alpha})$*.*

ii) a) *If* $A^t D_{1/a} \in \Gamma_{1/\alpha}$ *then for given* $b \in l_1(|a|\alpha)$ *the equation* $Ax = b$ *admits a unique solution in* $l_1(\alpha)$ *given by (8) with* $A^{-1} \in (l_1(|a|\alpha), l_1(\alpha))$*.*

b) *Let* $p > 1$ *real. If* $D_{1/a}A \in \Gamma'_{p,\alpha}$*, then for any given* $b \in l_p(|a|\alpha)$ *the equation* $Ax = b$ *admits a unique solution in* $l_p(\alpha)$ *given by (8) with* $A^{-1} \in (l_p(|a|\alpha), l_p(\alpha))$*.*

Proof. i) By Proposition 7 i), if $D_{1/a}A \in \Gamma_\alpha$, then

$$\|I - D_{1/a}A\|_{S_\alpha} = \|I - D_{1/a}A\|^*_{\mathcal{B}(s_\alpha)} < 1.$$

Thus $D_{1/a}A$ is invertible in $\mathcal{B}(s_\alpha) \cap S_\alpha$. Then $(D_{1/a}A)^{-1} \in S_\alpha$, that is $A^{-1} \in (s_{|a|\alpha}, s_\alpha)$ and we conclude by Proposition 8.

b) The set $\overset{\circ}{s}_\alpha$ being a BK space with AK, we have $\mathcal{B}(\overset{\circ}{s}_\alpha) = (\overset{\circ}{s}_\alpha, \overset{\circ}{s}_\alpha)$. Since $D_{1/a}A \in \Gamma_\alpha$ and $\lim_{n\to\infty}(a_{nm}/a_{nn}\alpha_n) = 0$ for all $m \geqslant 1$, we deduce that $D_{1/\alpha}(D_{1/a}A)D_\alpha \in (c_0, c_0)$. So $D_{1/a}A \in (\overset{\circ}{s}_\alpha, \overset{\circ}{s}_\alpha)$. Now by Lemma 6 we get

$$\|I - D_{1/a}A\|^*_{\mathcal{B}(\overset{\circ}{s}_\alpha)} = \sup_{x \in B_{s_\alpha}} (\|(I - D_{1/a}A)x\|_{s_\alpha}) = \|I - D_{1/a}A\|_{S_\alpha} < 1$$

and we conclude by Proposition 8. c) Here we have

$$\|I - D_{1/a}A\|^*_{\mathcal{B}(s^{(c)}_\alpha)} = \|I - D_{1/a}A\|_{S_\alpha} < 1.$$

Then $(D_{1/a}A)^{-1} \in S_\alpha \cap \mathcal{B}(s^{(c)}_\alpha)$, so $(D_{1/a}A)^{-1}$ is an operator represented by an infinite matrix, since $(D_{1/a}A)^{-1} \in S_\alpha$ and the condition $(D_{1/a}A)^{-1} \in \mathcal{B}(s^{(c)}_\alpha)$ implies $(D_{1/a}A)^{-1} \in (s^{(c)}_\alpha, s^{(c)}_\alpha)$. We conclude again by Proposition 8.

ii) a) By Proposition 7 ii) b) the condition $A^t D_{1/a} \in \Gamma_{1/\alpha}$ implies

$$\|I - D_{1/a}A\|^*_{\mathcal{B}(l_1(\alpha))} = \|I - (D_{1/a}A)^t\|_{S_{1/\alpha}} < 1$$

and $(D_{1/a}A)^{-1} \in \mathcal{B}(l_1(\alpha)) = (l_1(\alpha), l_1(\alpha))$. Thus $A^{-1} \in (l_1(|a|\alpha), l_1(\alpha))$ and we conclude by Proposition 8. ii) b) By Proposition 7 iii) we have $(D_{1/a}A)^{-1} \in \mathcal{B}(l_p(\alpha)) = (l_p(\alpha), l_p(\alpha))$, so $A^{-1} \in (l_p(|a|\alpha), l_p(\alpha))$ and again we conclude by Proposition 8. ∎

4. Matrix transformations mapping in the set $\chi(\Delta^k)$ for $k \geqslant 1$ integer and $\chi \in \{s_\alpha, \overset{\circ}{s}_\alpha, s^{(c)}_\alpha, l_p(\alpha)\}$ with $1 \leqslant p < \infty$.

In this section we will give some properties of the set $\chi(\Delta)$ for $\chi = s_\alpha$, $\overset{\circ}{s}_\alpha$, $s^{(c)}_\alpha$, or $l_p(\alpha)$. The characterization of the set $(\chi(\Delta^k), \chi(\Delta^k))$ given in [15] being complicated, we will deal with the subset $(\chi(\Delta^k), \chi(\Delta^k))_{\mathcal{L}'}$ of infinite Toeplitz triangles that map $\chi(\Delta^k)$ into itself.

4.1. The sets $X = \chi(\Delta^k)$ for $\chi = s_\alpha$, $\overset{\circ}{s}_\alpha$, $s_\alpha^{(c)}$, or $l_p(\alpha)$, with $1 \leqslant p < \infty$

Recall that the operator of first difference [1, 3, 6, 7, 8], [10–13] and [14–18], is defined by $\Delta = (\eta_{nm})_{n,m\geqslant 1} \in \mathcal{L}$, with $\eta_{nn} = 1$ for all $n \geqslant 1$, $\eta_{n,n-1} = -1$ for all $n \geqslant 2$ and $\eta_{nm} = 0$ otherwise. It is well known that $\Sigma = \Delta^{-1} = (\eta'_{nm})_{n,m\geqslant 1}$, with $\eta'_{nm} = 1$ for $m \leqslant n$ and $\eta'_{nm} = 0$ otherwise. We need to use the following sets

$$\begin{aligned}
\widehat{C_1} &= \{\, \alpha \in U^+ \,:\, (\alpha_1 + \cdots + \alpha_n)/\alpha_n = O(1)\ (n \to \infty) \,\},\\
\widehat{C} &= \{\, \alpha \in U^+ \,:\, \lim_{n\to\infty} (\alpha_1 + \cdots + \alpha_n)/\alpha_n = l \text{ for some } l \in \mathbb{C} \,\},\\
\Gamma &= \left\{ \alpha \in U^+ \,:\, \varlimsup_{n\to\infty} \left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1 \right\},\\
\widehat{\Gamma} &= \left\{ \alpha \in U^+ \,:\, \lim_{n\to\infty} \left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1 \right\}.
\end{aligned}$$

It can be easily seen that $\widehat{\Gamma} \subset \Gamma$ and it was shown in [12] that $\Gamma \subset \widehat{C_1}$ and by [15] we have $\widehat{C} = \widehat{\Gamma}$. So we have

Lemma 10. $\widehat{C} = \widehat{\Gamma} \subset \Gamma \subset \widehat{C_1}$.

We also have

Lemma 11. *Let $\alpha \in U^+$ and $k \geqslant 1$ be an integer. Then*

i) $s_\alpha^{(c)}(\Delta^k) = s_\alpha^{(c)}$ *if and only if* $\alpha \in \widehat{\Gamma}$.

ii) *If $\alpha \in \Gamma$, then $s_\alpha(\Delta^k) = s_\alpha$, $\overset{\circ}{s}_\alpha(\Delta^k) = \overset{\circ}{s}_\alpha$ and $l_p(\alpha)(\Delta^k) = l_p(\alpha)$ for $1 \leqslant p < \infty$.*

Proof. i) comes from [12, Theorem 2.6, p. 1789]. ii) By [12, Proposition 2.1, p. 1786], the condition $\alpha \in \Gamma$ implies $\alpha \in \widehat{C_1}$, so Δ and Δ^k are bijective from s_α to itself and from $\overset{\circ}{s}_\alpha$ to itself then $s_\alpha(\Delta^k) = s_\alpha$ and $\overset{\circ}{s}_\alpha(\Delta^k) = \overset{\circ}{s}_\alpha$. It remains to show that if $\alpha \in \Gamma$, then $l_p(\alpha)(\Delta) = l_p(\alpha)$ for $1 \leqslant p < \infty$. If we put $l = \overline{\lim}_{narrow\infty}(\alpha_{n-1}/\alpha_n) < 1$, for given ε_0, such that $0 < \varepsilon_0 < 1 - l$, there exists N_0 such that $\sup_{n\geqslant N_0+1}(\alpha_{n-1}/\alpha_n) \leqslant l + \varepsilon_0 < 1$. Consider now the infinite matrix

$$\Sigma_\alpha^{(N_0)} = \begin{pmatrix} [\Delta_\alpha^{(N_0)}]^{-1} & & O \\ & 1 & \\ O & & 1 \\ & & & \cdot \end{pmatrix},$$

$\Delta_\alpha^{(N_0)}$ being the finite matrix whose entries are those of $\Delta_\alpha = D_{1/\alpha}\Delta D_\alpha$ for all n, $m \leqslant N_0$. We get

$$Q = \Sigma_\alpha^{(N_0)}\Delta_\alpha = (q_{nm})_{n,m\geqslant 1},$$

with

$$q_{nm} = \begin{cases} 1, & \text{for } m = n,\\ -\frac{\alpha_m}{\alpha_{m+1}}, & \text{for } m = n-1 \geqslant N_0,\\ 0, & \text{otherwise.} \end{cases}$$

For every $x \in l_p$ we get $(I - Q)x =$ $(0, \dots, 0, (\alpha_{N_0}/\alpha_{N_0+1})x_{N_0}, \dots, (\alpha_{n-1}/\alpha_n)x_{n-1}, \dots)^t$, where $\alpha_{N_0}x_{N_0}/\alpha_{N_0+1}$ is in the (N_0+1) position. So we get

$$\|(I-Q)x\|_{l_p}^p = \sum_{n=N_0+1}^{\infty} \left(\frac{\alpha_{n-1}}{\alpha_n}\right)^p |x_{n-1}|^p \leqslant \sup_{n \geqslant N_0+1} \left[\left(\frac{\alpha_{n-1}}{\alpha_n}\right)^p\right] \Big(\sum_{n=N_0}^{\infty} |x_n|^p\Big),$$

and

$$\|I-Q\|^*_{\mathcal{B}(l_p)} = \sup_{x \neq 0} \left(\frac{\|(I-Q)x\|_{l_p}}{\|x\|_{l_p}}\right) \leqslant \Big[\sup_{n \geqslant N_0+1} \left(\frac{\alpha_{n-1}}{\alpha_n}\right)^p\Big]^{\frac{1}{p}}.$$

Since $\alpha_{n-1}/\alpha_n \leqslant l + \varepsilon_0 < 1$ for all $n \geqslant N_0 + 1$, we deduce

$$\sup_{n \geqslant N_0+1} \left(\frac{\alpha_{n-1}}{\alpha_n}\right)^p < 1.$$

Hence

$$\|I-Q\|_{\mathcal{B}(l_p)} \leqslant \Big[\sup_{n \geqslant N_0+1} \left(\frac{\alpha_{n-1}}{\alpha_n}\right)^p\Big]^{\frac{1}{p}} < 1.$$

We have shown that Q is invertible in $\mathcal{B}(l_p)$. Now let $b \in l_p$. The equations

$$\Delta_\alpha x = b \quad \text{and} \quad Qx = \Sigma^{(N_0)} b$$

are equivalent in l_p. Since $Q^{-1} \in \mathcal{B}(l_p)$, reasoning as in Proposition 8 that $Q^{-1}(Qx) = (Q^{-1}Q)x = x = (\Delta_\alpha)^{-1}b$ for all $x \in l_p$. This shows the map Δ_α is bijective from l_p to l_p and Δ is bijective from $l_p(\alpha)$ to $l_p(\alpha)$. ■

4.2. The set $(\boldsymbol{\chi}(\boldsymbol{\Delta}), \boldsymbol{\chi}(\boldsymbol{\Delta}))$ for $\boldsymbol{\chi} \in \{s_\alpha, \overset{\circ}{s}_\alpha, s_\alpha^{(c)}\}$

First recall the characterizations of the set $(\chi(\Delta), \chi(\Delta))$ for $\chi = s_\alpha$, $\overset{\circ}{s}_\alpha$, or $s_\alpha^{(c)}$. For this we will consider the following properties

$$\lim_{l \to \infty} \Big(\sum_{j=1}^{l} \alpha_j \Big| \sum_{k=l}^{\infty} \left(\frac{a_{nk} - a_{n-1,k}}{\alpha_n}\right)\Big|\Big) = 0 \text{ for all } n; \tag{10}$$

$$\sup_{n \geqslant 1} \Big| \sum_{j=1}^{\infty} \alpha_j \Big(\sum_{k=j}^{\infty} \left(\frac{a_{nk} - a_{n-1,k}}{\alpha_n}\right)\Big)\Big| < \infty; \tag{11}$$

$$\lim_{n \to \infty} \Big[\alpha_j \Big(\sum_{k=j}^{\infty} \left(\frac{a_{nk} - a_{n-1,k}}{\alpha_n}\right)\Big)\Big] = 0 \text{ for all } j; \tag{12}$$

$$\sup_{l \geqslant 1} \Big(\sum_{j=1}^{l} \alpha_j \Big| \sum_{k=l}^{\infty} \left(\frac{a_{nk} - a_{n-1,k}}{\alpha_n}\right)\Big|\Big) < \infty \text{ for all } n; \tag{13}$$

$$\lim_{n \to \infty} \Big(\alpha_j \Big(\sum_{k=j}^{\infty} \left(\frac{a_{nk} - a_{n-1,k}}{\alpha_n}\right)\Big)\Big) = l_j \text{ for all } j; \tag{14}$$

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} \Big[\left(\frac{a_{nk} - a_{n-1,k}}{\alpha_n}\right) \Big(\sum_{j=1}^{k} \alpha_j\Big)\Big] = l \quad \text{exists.} \tag{15}$$

The following Proposition was proved in [15]

PROPOSITION 12. *Let $\alpha \in U^+$. Then*

i) $A \in (s_\alpha(\Delta), s_\alpha(\Delta))$ *if and only if (10) and (11) hold;*

ii) $A \in (s^\circ_\alpha(\Delta), s^\circ_\alpha(\Delta))$ *if and only if (11), (12) and (13) hold;*

iii) $A \in (s^{(c)}_\alpha(\Delta), s^{(c)}_\alpha(\Delta))$ *if and only if (11), (13), (14) and (15) hold.*

Note that $A \in (\chi(\Delta^k), \chi(\Delta^k))$ if and only if

$$D_{1/\alpha}\Delta^k A \in (\chi(\Delta^k), Y) \text{ with } Y \in \{l_\infty, c_0, c\},$$

see [16]. Now we will give necessary conditions for a matrix map to be bijective from $\chi(\Delta^k)$ into itself when $\chi \in \{s_\alpha, s^\circ_\alpha, s^{(c)}_\alpha, l_p(\alpha)\}$ and $p \geqslant 1$ real. We have

PROPOSITION 13. *Let $\alpha \in U^+$ and $k \geqslant 1$ an integer.*

i) *Let $b \in \chi(\Delta^k)$, $\chi \in \{s_\alpha, s^\circ_\alpha, s^{(c)}_\alpha\}$ and assume that there is $\tau \in]0,1[$ such that*

$$\|\Delta^k[(I-A)x]\|_{s_\alpha} \leqslant \tau\|\Delta^k x\|_{s_\alpha} \text{ for all } x \in \chi(\Delta^k). \tag{16}$$

Then

a) the map $A\colon x \mapsto Ax$ is bijective from $\chi(\Delta^k)$ into itself;

b) if (16) holds and $\alpha \in \widehat{\Gamma}$ then the matrix map $A\colon x \mapsto Ax$ is bijective from χ into itself.

ii) *a) If $\alpha \in \Gamma$ and $A^t \in \Gamma_{1/\alpha}$, then $A\colon x \mapsto Ax$ is bijective from $l_1(\alpha)(\Delta^k)$ into itself.*

b) Let $1 < p < \infty$. If $\alpha \in \Gamma$ and $A \in \Gamma'_{p,\alpha}$, then the map $A\colon x \mapsto Ax$ is bijective from $l_p(\alpha)(\Delta^k)$ into itself.

Proof. i) a) It is enough to show the proposition in the case when $X = s_\alpha$. Using Lemma 3, condition (16) means that

$$\|I-A\|^*_{\mathcal{B}(s_\alpha(\Delta^k))} = \sup_{x\neq 0}\left(\frac{\|\Delta^k[(I-A)x]\|_{s_\alpha}}{\|\Delta^k x\|_{s_\alpha}}\right) \leqslant \tau < 1,$$

so A is invertible in $\mathcal{B}(s_\alpha(\Delta^k))$ and by Proposition 8 the equation $Ax = b$ for $b \in s_\alpha(\Delta^k)$ admits a unique solution in $s_\alpha(\Delta^k)$. i) b) Since $\widehat{\Gamma} \subset \Gamma$, by Lemma 10 and Lemma 11 ii), we easily deduce that $\alpha \in \widehat{\Gamma}$ implies $\chi(\Delta^k) = \chi$, $\|x\|_\chi = \|x\|_{s_\alpha}$ for $\chi \in \{s_\alpha, s^\circ_\alpha, s^{(c)}_\alpha\}$. We conclude since condition (16) means that

$$\|I-A\|_{B(s_\alpha)} = \|I-A\|_{S_\alpha} \leqslant \tau < 1.$$

ii) By Lemma 11, the condition $\alpha \in \Gamma$ implies Δ is bijective from $l_p(\alpha)$ to itself for any given p with $1 \leqslant p < \infty$. Then Δ^k is also bijective and $l_p(\alpha)(\Delta^k) = l_p(\alpha)$. We conclude applying Corollary 9 ii) with $a = e$. ∎

EXAMPLE 14. Let $\gamma, \eta \in \mathbb{C}$ and put

$$M(\gamma,\eta) = \begin{pmatrix} 1 & \eta & & & \\ \gamma & 1 & \eta & \mathbf{0} & \\ & \gamma & 1 & \eta & \\ & \mathbf{0} & . & . & . \\ & & & & \end{pmatrix}. \tag{17}$$

As a direct consequence of Proposition 13 we can state the following. *For* $\alpha \in U^+$ *and*

$$\xi_\alpha = |\gamma| \sup_{n \geqslant 2} \left(\frac{\alpha_{n-1}}{\alpha_n} \right) + |\eta| \sup_{n \geqslant 1} \left(\frac{\alpha_{n+1}}{\alpha_n} \right) < 1, \tag{18}$$

the equation $M(\gamma, \eta)x = b$ *with* $b \in s_\alpha(\Delta)$ *admits a unique solution* $\overset{\circ}{x} = [M(\gamma, \eta)]^{-1} b \in s_\alpha(\Delta)$. *Furthermore if* $\alpha \in \Gamma$ *the equation* $M(\gamma, \eta)x = b$ *with* $b \in s_\alpha$ *admits a unique solution* $\overset{\circ}{x} \in s_\alpha$.

Proof. First we have $[M(\gamma, \eta) - I]x = (\gamma x_{n-1} + \eta x_{n+1})_{n \geqslant 1}$, with $x_0 = 0$. Then we easily get $\Delta[(M(\gamma, \eta) - I)x] = (y_n)_{n \geqslant 1}$with

$$y_n = \eta(x_{n+1} - x_n) + \gamma(x_{n-1} - x_{n-2}) = 0 \text{ and } x_{-1} = x_0 = 0.$$

Hence

$$\begin{aligned} \frac{1}{\alpha_n} |y_n| &\leqslant |\eta| |x_{n+1} - x_n| \frac{1}{\alpha_{n+1}} \frac{\alpha_{n+1}}{\alpha_n} + |\gamma| |x_{n-1} - x_{n-2}| \frac{1}{\alpha_{n-1}} \frac{\alpha_{n-1}}{\alpha_n} \\ &\leqslant |\eta| \frac{\alpha_{n+1}}{\alpha_n} \|\Delta x\|_{s_\alpha} + |\gamma| \frac{\alpha_{n-1}}{\alpha_n} \|\Delta x\|_{s_\alpha}. \end{aligned}$$

Thus

$$\sup_{n \geqslant 1} \left(\frac{1}{\alpha_n} |y_n| \right) = \|\Delta[(I - M(\gamma, \eta))x]\|_{s_\alpha} \leqslant \xi_\alpha \|\Delta x\|_{s_\alpha} < \|\Delta x\|_{s_\alpha}$$

and

$$\|I - M(\gamma, \eta)\|^*_{\mathcal{B}(s_\alpha(\Delta))} < 1.$$

As above we conclude that if (18) holds the equation $M(\gamma, \eta)x = b$ with $b \in s_\alpha(\Delta)$ admits in $s_\alpha(\Delta)$ the unique solution $\overset{\circ}{x} = [M(\gamma, \eta)]^{-1} b$. If $\alpha \in \Gamma$, then $s_\alpha(\Delta) = s_\alpha$ by Lemma 11 and we conclude from the preceding that the equation $M(\gamma, \eta)x = b$ with $b \in s_\alpha$ admits a unique solution in s_α. ∎

4.3. The equation $T_a x = b$, where T_a is a Toeplitz triangle matrix

We will denote by $\mathcal{L}' \subset \mathcal{L}$ the set of all infinite Toeplitz triangles. We will say that $T = (t_{nm})_{n,m \geqslant 1} \in \mathcal{L}'$ if there is a sequence $a = (a_n)_{n \geqslant 1} \in s$, such that $t_{nm} = a_{n-m+1}$ for $m \leqslant n$ and $t_{nm} = 0$ otherwise; then we will write $T = T_a$; $\mathcal{L}'$ is a subset of the set of Toeplitz matrices, see [1]. It can easily be seen that $T_a T_{a'} = T_{a'} T_a$ for all a, $a' \in s$. We then have the next result.

Proposition 15. *Let* $k \geqslant 1$ *be an integer and let* $T_a \in \mathcal{L}'$ *for* $a \in U$. *Then*

i) $T_a \in \mathcal{B}(s_\alpha(\Delta^k))$ *if and only if* $T_a \in S_\alpha$;

ii) $T_a \in \mathcal{B}(\overset{\circ}{s}_\alpha(\Delta^k))$ *if and only if* $T_a \in S_\alpha$ *and*

$$\lim_{n \to \infty} \frac{a_{n-m+1}}{\alpha_n} = 0 \quad \textit{for all } m; \tag{19}$$

iii) $T_a \in \mathcal{B}(s^{(c)}_\alpha(\Delta^k))$ *if and only if* $T_a \in S_\alpha$,

$$\lim_{n \to \infty} \frac{a_{n-m+1}}{\alpha_n} = l_m \quad \textit{for all } m \tag{20}$$

and

$$\lim_{n\to\infty}\left(\sum_{m=1}^{n} a_{n-m+1}\frac{\alpha_m}{\alpha_n}\right) = l. \tag{21}$$

iv) $\|T_a\|^*_{\mathcal{B}(s^\circ_\alpha(\Delta^k))} = \|T_a\|^*_{\mathcal{B}(s_\alpha(\Delta^k))} \leqslant \|T_a\|_{S_\alpha}$ *and* $\|T_a\|^*_{\mathcal{B}(s_\alpha(\Delta^k))} = \|T_a\|_{S_\alpha}$ *for* $\alpha \in \Gamma$.

v) *Let* $1 \leqslant p < \infty$. *Then* $T_a \in \mathcal{B}(l_p(\alpha)(\Delta^k))$ *if and only if* $T_a \in (l_p(\alpha), l_p(\alpha))$.

Proof. i) The set $s_\alpha(\Delta^k)$ is a BK space by Lemma 3 in which $T = \Delta^k$. So by Remark 1, $T_a \in \mathcal{B}(s_\alpha(\Delta^k))$ if and only if $T_a \in (s_\alpha(\Delta^k), s_\alpha(\Delta^k))$. Thus the condition $T_a \in \mathcal{B}(s_\alpha(\Delta^k))$ is equivalent to $T_a(\Sigma^k x) \in s_\alpha(\Delta^k)$ for all $x \in s_\alpha$. So $T_a \in \mathcal{B}(s_\alpha(\Delta^k))$ if and only if

$$\Delta^k[T_a(\Sigma^k x)] \in s_\alpha \quad \text{for all } x \in s_\alpha.$$

Since Δ^k, $\Sigma^k \in \mathcal{L}'$, we have

$$\Delta^k[T_a(\Sigma^k x)] = (\Delta^k T_a \Sigma^k)x = (T_a(\Delta\Sigma)^k)x = T_a x \text{ for all } x \in s_\alpha.$$

We conclude that $T_a \in \mathcal{B}(s_\alpha(\Delta^k))$ if and only if $T_a \in (s_\alpha, s_\alpha)$, that is $T_a \in S_\alpha$. ii) As above $s^\circ_\alpha(\Delta^k)$ is a BK space and $T_a \in \mathcal{B}(s^\circ_\alpha(\Delta^k))$ if and only if $T_a \in (s^\circ_\alpha, s^\circ_\alpha)$. Now we have $D_{1/\alpha} T_a D_\alpha = (\xi_{nm})_{n,m\geqslant 1}$, with $\xi_{nm} = a_{n-m+1}\alpha_m/\alpha_n$ for $m \leqslant n$ and $\xi_{nm} = 0$ otherwise. Thus we get $D_{1/\alpha} T_a D_\alpha \in (c_0, c_0)$ if and only if $T_a \in S_\alpha$ and (19) holds. We get iii) by a similar argument. So $T_a \in \mathcal{B}(s^\circ_\alpha(\Delta^k))$ if and only if $T_a \in (s^\circ_\alpha, s^\circ_\alpha)$ and $T_a \in \mathcal{B}(s^{(c)}_\alpha(\Delta^k))$ if and only if $T_a \in (s^{(c)}_\alpha, s^{(c)}_\alpha)$.

iv) Let us show that $\|T_a\|^*_{\mathcal{B}(s_\alpha(\Delta^k))} \leqslant \|T_a\|_{S_\alpha}$. We have

$$\|T_a\|^*_{\mathcal{B}(s_\alpha(\Delta^k))} = \sup_{x\neq 0}\left(\frac{\|T_a x\|_{s_\alpha(\Delta^k)}}{\|x\|_{s_\alpha(\Delta^k)}}\right) < \infty.$$

Since $\Delta^k \in \mathcal{L}'$, we have $\Delta^k T_a = T_a \Delta^k$ and for every $x \in s_\alpha(\Delta^k)$

$$\|T_a x\|_{s_\alpha(\Delta^k)} = \|\Delta^k(T_a x)\|_{s_\alpha} = \|T_a(\Delta^k x)\|_{s_\alpha},$$

so

$$\|T_a x\|_{s_\alpha(\Delta^k)} \leqslant \|T_a\|_{S_\alpha}\|\Delta^k x\|_{s_\alpha} = \|T_a\|_{S_\alpha}\|x\|_{s_\alpha(\Delta^k)}$$

and

$$\|T_a\|^*_{\mathcal{B}(s_\alpha(\Delta^k))} \leqslant \|T_a\|_{S_\alpha}.$$

Now it can easily be seen that $\|T_a\|^*_{\mathcal{B}(s_\alpha(\Delta^k))} = \|T_a\|^*_{\mathcal{B}(s^\circ_\alpha(\Delta^k))}$. In the case when $\alpha \in \Gamma$ we have $s_\alpha(\Delta^k) = s_\alpha$ and by Proposition 7 i) we get

$$\|T_a\|^*_{\mathcal{B}(s_\alpha(\Delta^k))} = \|T_a\|_{S_\alpha}.$$

v) By Lemma 3 where $T = \Delta^k$, we see that $l_p(\alpha)(\Delta^k)$ is a BK space. We have $T_a \in \mathcal{B}(l_p(\alpha)(\Delta^k))$ if and only if $T_a \in (l_p(\alpha)(\Delta^k), l_p(\alpha)(\Delta^k))$. Now $T_a \in (l_p(\alpha)(\Delta^k), l_p(\alpha)(\Delta^k))$ if and only if $\Delta^k(T_a\Sigma^k x) = T_a x \in l_p(\alpha)$ for all $x \in l_p(\alpha)$. This means that $T_a \in (l_p(\alpha), l_p(\alpha))$. ∎

Proposition 16. *Let $T_a \in \mathcal{L}'$ with $a_1 \neq 0$. Then*

i) *If $(1/a_1)T_a \in \Gamma_\alpha$ then for any given $b \in s_\alpha(\Delta^k)$ the equation $T_a x = b$ admits in $s_\alpha(\Delta^k)$ a unique solution*

$$x = T_a^{-1} b. \tag{22}$$

ii) *If $(1/a_1)T_a \in \Gamma_\alpha$ and (19) holds, then for any given $b \in \overset{\circ}{s}_\alpha(\Delta^k)$ the equation $T_a x = b$ admits in $\overset{\circ}{s}_\alpha(\Delta^k)$ a unique solution given by (22).*

iii) *If $(1/a_1)T_a \in \Gamma_\alpha$, (20) and (21) hold, then for any given $b \in s_\alpha^{(c)}(\Delta^k)$ the equation $T_a x = b$ admits in $s_\alpha^{(c)}(\Delta^k)$ a unique solution given by (22).*

iv) a) *If $(1/a_1)T_a^t \in \Gamma_{1/\alpha}$, then for any given $b \in l_1(\alpha)(\Delta^k)$ the equation $T_a x = b$ admits in $l_1(\alpha)(\Delta^k)$ a unique solution given by (22).*

b) *Let $1 < p < \infty$ and assume that $(1/a_1)T_a \in \Gamma'_{p,\alpha}$. Then for any given $b \in l_p(\alpha)(\Delta^k)$ the equation $T_a x = b$ admits in $l_p(\alpha)(\Delta^k)$ a unique solution given by (22).*

Proof. i) Since $T_a \in \mathcal{L}$ the equation $T_a x = b$ is equivalent to $x = T_a^{-1} b$. Now the condition $(1/a_1)T_a \in \Gamma_\alpha$ implies $T_a^{-1} \in S_\alpha$ and by Proposition 15 i), this means that $T_a^{-1} \in \mathcal{B}(s_\alpha(\Delta^k))$. We conclude $T_a^{-1} b \in s_\alpha(\Delta^k)$.

ii) By Proposition 15 ii) we have $T_a \in \mathcal{B}(\overset{\circ}{s}_\alpha(\Delta^k))$ since $T_a \in (\overset{\circ}{s}_\alpha, \overset{\circ}{s}_\alpha)$; and the condition $(1/a_1)T_a \in \Gamma_\alpha$ and Proposition 15 iv) imply

$$\left\| I - \frac{1}{t_{11}} T_a \right\|^*_{\mathcal{B}(s_\alpha(\Delta^k))} = \left\| I - \frac{1}{t_{11}} T_a \right\|^*_{\mathcal{B}(\overset{\circ}{s}_\alpha(\Delta^k))} \leqslant \left\| I - \frac{1}{t_{11}} T_a \right\|_{S_\alpha} < 1.$$

Then $T_a^{-1} \in \mathcal{B}(\overset{\circ}{s}_\alpha(\Delta^k))$ and the unique solution of the equation $T_a x = b$ for $b \in \overset{\circ}{s}_\alpha(\Delta^k)$ is $x = T_a^{-1} b \in \overset{\circ}{s}_\alpha(\Delta^k)$. We get iii) from the characterization of $(s_\alpha^{(c)}, s_\alpha^{(c)})$ and reasoning as above. iv) a) is a direct consequence of Corollary 9 ii) a) and Proposition 15 v). iv) b) Here the condition $(1/a_1)T_a \in \Gamma'_{p,\alpha}$ implies $T_a^{-1} \in (l_p(\alpha), l_p(\alpha))$, since $\mathcal{B}(l_p(\alpha)) = (l_p(\alpha), l_p(\alpha))$. Then from Proposition 15 v) $T_a^{-1} \in (l_p(\alpha)(\Delta^k), l_p(\alpha)(\Delta^k))$ and for all $b \in l_p(\alpha)(\Delta^k)$ the unique solution $x = T_a^{-1} b$ of the equation $T_a x = b$ belongs to $l_p(\alpha)(\Delta^k)$. ■

Remark 2. The conditions $(1/a_1)T_a \in \Gamma_\alpha$ and $(1/a_1)T_a^t \in \Gamma_{1/\alpha}$ are equivalent to

$$\sup_{n \geqslant 2} \Big(\sum_{m=1}^{n-1} |a_{n-m+1}| \frac{\alpha_m}{\alpha_n} \Big) < |a_1| \quad \text{and} \quad \sup_{n \geqslant 1} \Big(\sum_{m=n+1}^{\infty} |a_{m-n+1}| \frac{\alpha_n}{\alpha_m} \Big) < |a_1|;$$

respectively. Note that the condition $(1/a_1)T_a \in \Gamma'_{p,\alpha}$ means

$$\sum_{n=2}^{\infty} \Big(\sum_{m=1}^{n-1} |a_{n-m+1}|^{p/(p-1)} \Big)^{p-1} < |a_1|^{p/(p-1)}.$$

Furthermore, it can be verified from Proposition 15 that if $T_a \in \mathcal{L}'$, then

i) $T_a \in \mathcal{B}(c_0(\Delta^k))$ if and only if $T_a \in S_1$ and $a \in c_0$;

ii) $T_a \in \mathcal{B}(c(\Delta^k))$ if and only if $T_a \in S_1$ and $a \in c_s$.

5. Application to infinite tridiagonal matrices

In this section we will consider infinite tridiagonal matrices. These matrices are used in many applications, let us cite for instance the case of continued fractions [9], or the finite differences method, see [20]. We deal with some properties of the matrix map $M(\gamma, a, \eta)$ between particular sequence spaces. Then we will explicitly calculate the inverse of $M(\gamma, a, \eta)$.

5.1. Properties of infinite matrices considered as operators in certain BK spaces

Let $\gamma = (\gamma_n)_{n\geqslant 1}$, $\eta = (\eta_n)_{n\geqslant 1}$, $a' = (a_n)_{n\geqslant 1}$ be sequences with $a' \in U$. Consider the infinite tridiagonal matrix

$$M(\gamma, a, \eta) = \begin{pmatrix} a_1 & \eta_1 & & & \\ \gamma_2 & a_2 & \eta_2 & & \mathbf{0} \\ & . & . & . & \\ \mathbf{0} & & \gamma_n & a_n & \eta_n \\ & & & . & . \end{pmatrix}.$$

We then have

PROPOSITION 17. *Assume that $D_{1/a}M(\gamma, a, \eta) \in \Gamma_\alpha$, that is*

$$\sup_{n\geqslant 1}\left[\frac{1}{a_n}\left(|\gamma_n|\frac{\alpha_{n-1}}{\alpha_n} + |\eta_n|\frac{\alpha_{n+1}}{\alpha_n}\right)\right] < 1.$$

Then

i) *$M(\gamma, a, \eta)$ is bijective from s_α into $s_{|a|\alpha}$ and $M(\gamma, a, \eta)^{-1} \in (s_{|a|\alpha}, s_\alpha)$;*

ii) *$M(\gamma, a, \eta)$ is bijective from $\overset{\circ}{s}_\alpha$ into $\overset{\circ}{s}_{|a|\alpha}$ and $M(\gamma, a, \eta)^{-1} \in (\overset{\circ}{s}_{|a|\alpha}, \overset{\circ}{s}_\alpha)$;*

iii) *if*

$$\lim_{n\to\infty}\left[\frac{1}{a_n}\left(\gamma_n\frac{\alpha_{n-1}}{\alpha_n} + \eta_n\frac{\alpha_{n+1}}{\alpha_n}\right)\right] = l \neq 0,$$

then $M(\gamma, a, \eta)$ is bijective from $s_\alpha^{(c)}$ into $s_{|a|\alpha}^{(c)}$ and $M(\gamma, a, \eta)^{-1} \in (s_{|a|\alpha}^{(c)}, s_\alpha^{(c)})$.

iv) *Let $p \geqslant 1$ be a real. If $\widetilde{K}_{p,\alpha} = K_1 + K_2 < 1$ with*

$$K_1 = \sup_{n\geqslant 1}\left(\left|\frac{\gamma_n}{a_n}\right|\frac{\alpha_{n-1}}{\alpha_n}\right) \text{ and } K_2 = \sup_{n\geqslant 1}\left(\left|\frac{\eta_n}{a_n}\right|\frac{\alpha_{n+1}}{\alpha_n}\right),$$

then $M(\gamma, a, \eta)$ is bijective from $l_p(\alpha)$ into $l_p(|a|\alpha)$ and $M(\gamma, a, \eta)^{-1} \in (l_p(|a|\alpha), l_p(\alpha))$.

Proof. i), ii) and iii) are direct consequences of Corollary 9. iv) We get

$$\begin{aligned}\|(I - D_{1/a}M(\gamma, a, \eta))x\|_{l_p(\alpha)} &= \Big(\sum_{n=1}^{\infty}\frac{1}{\alpha_n^p}\Big|\frac{\gamma_n}{a_n}x_{n-1} + \frac{\eta_n}{a_n}x_{n+1}\Big|^p\Big)^{1/p} \\ &= \Big(\sum_{n=1}^{\infty}\Big|\frac{\gamma_n}{a_n}\frac{\alpha_{n-1}}{\alpha_n}\frac{x_{n-1}}{\alpha_{n-1}} + \frac{\eta_n}{a_n}\frac{\alpha_{n+1}}{\alpha_n}\frac{x_{n+1}}{\alpha_{n+1}}\Big|^p\Big)^{1/p} \\ &\leqslant \Big(\sum_{n=1}^{\infty}\Big(K_1\Big|\frac{x_{n-1}}{\alpha_{n-1}}\Big| + K_2\Big|\frac{x_{n+1}}{\alpha_{n+1}}\Big|\Big)^p\Big)^{1/p};\end{aligned}$$

Applying Minkowski's inequality we get

$$\Big(\sum_{n=1}^{\infty}\Big(K_1\Big|\frac{x_{n-1}}{\alpha_{n-1}}\Big|+K_2\Big|\frac{x_{n+1}}{\alpha_{n+1}}\Big|\Big)^p\Big)^{1/p}\leqslant K_1\Big(\sum_{n=1}^{\infty}\Big|\frac{x_{n-1}}{\alpha_{n-1}}\Big|^p\Big)^{1/p}+K_2\Big(\sum_{n=1}^{\infty}\Big|\frac{x_{n+1}}{\alpha_{n+1}}\Big|^p\Big)^{1/p}$$
$$\leqslant (K_1+K_2)\|x\|_{l_p(\alpha)}.$$

we conclude that

$$\|I-D_{1/a}M(\gamma,a,\eta)\|^*_{\mathcal{B}(l_p(\alpha))}\leqslant (K_1+K_2)<1.$$

So $D_{1/a}M(\gamma,a,\eta)$ is invertible in $\mathcal{B}(l_p(\alpha))$ and $A=D_a(D_{1/a}M(\gamma,a,\eta))$ is bijective from $l_p(\alpha)$ into $l_p(|a|\alpha)$. Since $\mathcal{B}(l_p(\alpha))=(l_p(\alpha),l_p(\alpha))$, we conclude

$$[D_{1/a}M(\gamma,a,\eta)]^{-1}\in (l_p(\alpha),l_p(\alpha))$$

and $M(\gamma,a,\eta)^{-1}\in (l_p(|a|\alpha),l_p(\alpha))$. ∎

We deduce the next corollary.

Corollary 18. *If $\widetilde{K}_{1,\alpha}<1$, then $M(\gamma,a,\eta)$ is bijective from $l_1(\alpha)$ to $l_1(|a|\alpha)$ and bijective from s_α to $s_{|a|\alpha}$.*

Proof. First taking $p=1$ in Proposition 17 iv), we deduce that A is bijective from $l_1(\alpha)$ to $l_1(|a|\alpha)$. Then from

$$\|I-D_{1/a}M(\gamma,a,\eta)\|_{S_\alpha}=\widetilde{K'_\alpha}\leqslant \widetilde{K}_{1,\alpha}<1,$$

we conclude that $M(\gamma,a,\eta)$ is bijective from s_α to $s_{|a|\alpha}$. ∎

Remark 3. Note that in the case when $p=1$, the condition

$$\|I-[D_{1/a}M(\gamma,a,\eta)]^t\|_{S_\alpha}=\sup_{n\geqslant 1}\left(\left|\frac{\gamma_{n+1}}{a_{n+1}}\right|\frac{\alpha_{n+1}}{\alpha_n}+\left|\frac{\eta_{n-1}}{a_{n-1}}\right|\frac{\alpha_{n-1}}{\alpha_n}\right)<1,$$

also implies $M(\gamma,a,\eta)$ is bijective from $l_1(\alpha)$ to $l_1(|a|\alpha)$.

5.2. The inverse of an infinite tridiagonal matrix.

In this subsection, among other things, we are interested in the calculation of the inverse of $M(\gamma,\eta)$ defined in Example 14.

For this we need to recall the next results. We can associate with any power series $f(z)=\sum_{k=0}^{\infty}a_kz^k$ defined in the open disk $|z|<R$ the upper triangular infinite matrix $A=\varphi(f)\in\bigcup_{0<r<R}S_r$ defined by

$$\varphi(f)=\begin{pmatrix} a_0 & a_1 & a_2 & . \\ & a_0 & a_1 & . \\ O & & a_0 & . \\ & & & . \end{pmatrix},$$

(see [6]). Practically we shall write $\varphi[f(z)]$ instead of $\varphi(f)$. We have

Lemma 19. i) *The map $\varphi\colon f\mapsto A$ is an isomorphism from the algebra of the power series defined in $|z|<R$, into the algebra of the corresponding matrices $\bar{A}$.*

ii) *Let* $f(z)=\sum_{k=0}^{\infty}a_k z^k$, *with* $a_0\neq 0$, *and assume that* $1/f(z)=\sum_{k=0}^{\infty}a'_k z^k$ *admits* $R'>0$ *as radius of convergence. We then have*

$$\varphi\left(\frac{1}{f}\right)=[\varphi(f)]^{-1}\in\bigcup_{0<r<R'}S_r.$$

From the previous results we get

Proposition 20. *Let* γ, η *be reals with* $0<\gamma+\eta<1$. *Then*

i) $M(\gamma,\eta):x\longmapsto M(\gamma,\eta)x$ *is bijective from* χ *into itself, for* $\chi\in\{s_1,c_0,c\}$.

ii) a) *Let* χ *be either one of the sets* s_1, *or* c_0, *or* c *and put*

$$u=(1-\sqrt{1-4\gamma\eta})/2\gamma\quad\text{and}\quad v=(1-\sqrt{1-4\gamma\eta})/2\eta.$$

Then for any given $b\in\chi$ *the equation* $M(\gamma,\eta)x=b$ *admits a unique solution* $\overset{\circ}{x}=(\overset{\circ}{x}_n)_{n\geqslant 1}$ *in* χ *given by*

$$\overset{\circ}{x}_n=\left(\frac{uv+1}{uv-1}\right)(-1)^n v^n\sum_{m=1}^{\infty}[1-(uv)^{-l}](-1)^m u^m b_m\text{ for all }n. \tag{23}$$

with $l=\min(n,m)$.

b) *The inverse* $[M(\gamma,\eta)]^{-1}=(a'_{nm})_{n,m\geqslant 1}$ *is given by*

$$a'_{nm}=\left(\frac{uv+1}{uv-1}\right)(-1)^{n+m}v^{n-m}[(uv)^l-1]\text{ for all }n,m\geqslant 1\text{ and }l=\min(n,m). \tag{24}$$

Proof. i) We have $\|I-M(\gamma,\eta)\|_{S_1}=\gamma+\eta<1$, so $M(\gamma,\eta)\in\Gamma_1$ and we conclude by Corollary 9 i) a) in which $\alpha=e$. ii) Let u, v be reals with $0<u<1$, $0<v<1$ and consider the matrices

$$\Delta_u^+=\begin{pmatrix}1 & u & & O\\ & 1 & u & \\ O & & . & .\end{pmatrix}\quad\text{and}\quad\Delta_v=\begin{pmatrix}1 & & & O\\ v & 1 & & \\ O & v & 1 & \\ & & . & .\end{pmatrix}.$$

By a short calculation we get

$$\frac{1}{1+uv}\Delta_u^+\Delta_v=\begin{pmatrix}1 & \frac{u}{1+uv} & & O\\ \frac{v}{1+uv} & 1 & \frac{u}{1+uv} & \\ O & \frac{v}{1+uv} & 1 & \frac{u}{1+uv}\\ & & . & .\end{pmatrix}.$$

Thus the identity $M(\gamma,\eta)=\frac{1}{1+uv}\Delta_u^+\Delta_v$ is equivalent to $\eta=u/(1+uv)$ and $\gamma=v/(1+uv)$. Putting $\xi=1+uv$, we get $\eta=u/\xi$ and $\gamma=v/\xi$, so $\xi=1+\gamma\eta\xi^2$. Since $0<\gamma+\eta<1$, we have $4\gamma\eta<1$ and $\xi=(1\pm\sqrt{1-4\gamma\eta})/2\gamma\eta$ and the condition $|u|$, $|v| < 1$ implies $u = (1-\sqrt{1-4\gamma\eta})/2\gamma$ and $v = (1-\sqrt{1-4\gamma\eta})/2\eta$. Then

$$\left\|I - \frac{1}{1+uv}\Delta_u^+\Delta_v\right\|_{S_1} = \gamma + \eta = \frac{u+v}{1+uv} < 1.$$

Furthermore we have $\|I-\Delta_u^+\|_{S_1} = u < 1$ and $\|I-\Delta_v\|_{S_1} = v < 1$. So Δ_u^+ and Δ_v are invertible in S_1 and by Lemma 19, we have

$$\begin{aligned}
(\Delta_u^+)^{-1} &= \varphi(1/(1+uz)) = \varphi\Big(\sum_{k=0}^{\infty}(-1)^k u^k z^k\Big),\\
(\Delta_v)^{-1} &= \Big[\varphi\Big(\sum_{k=0}^{\infty}(-1)^k v^k z^k\Big)\Big]^t,
\end{aligned}$$

with $|uz|$, $|vz| < 1$. This means that

$$(\Delta_u^+)^{-1} = \Sigma_u^+ = \begin{pmatrix} 1 & -u & u^2 & . \\ & 1 & -u & u^2 \\ O & & . & . \\ & & & . \end{pmatrix} \text{ and } (\Delta_v)^{-1} = \Sigma_v = \begin{pmatrix} 1 & & O & \\ -v & 1 & & \\ v^2 & -v & 1 & \\ . & . & . & . \end{pmatrix}.$$

Then $(\frac{1}{1+uv}\Delta_u^+\Delta_v)^{-1} = (1+uv)\Sigma_v\Sigma_u^+$. For any given $b \in s_1$, we successively get

$$\begin{aligned}
\Sigma_u^+ b &= \Big(\sum_{k=0}^{\infty}(-1)^k u^k b_{n+k}\Big)_{n\geqslant 1},\\
\Sigma_v(\Sigma_u^+ b) &= \Big(\sum_{s=1}^{n}(-1)^{n-s}v^{n-s}\Big(\sum_{k=0}^{\infty}(-1)^k u^k b_{s+k}\Big)\Big)_{n\geqslant 1}
\end{aligned}$$

and the unique solution is given by

$$\overset{\circ}{x}_n = (1+uv)\sum_{s=1}^{n}\sum_{k=0}^{\infty}(-1)^{n-s+k}v^{n-s}u^k b_{s+k}. \tag{25}$$

Hence writing

$$\overset{\circ}{x}_n = (1+uv)(-v)^n\sigma_n \text{ with } \sigma_n = \sum_{s=1}^{n}(uv)^{-s}\sum_{k=0}^{\infty}(-u)^{k+s}b_{s+k},$$

putting $l = \min(n,m)$, we get

$$\begin{aligned}
\sigma_n &= \sum_{s=1}^{n}(uv)^{-s}\sum_{m=0}^{\infty}(-u)^m b_m\\
&= \sum_{s=1}^{n}(uv)^{-s}\sum_{m=s}^{n}(-u)^m b_m + \sum_{s=1}^{n}(uv)^{-s}\sum_{m=n+1}^{\infty}(-u)^m b_m\\
&= \sum_{m=1}^{\infty}(-u)^m b_m\sum_{s=1}^{m}(uv)^{-s} + \sum_{m=n+1}^{\infty}(-u)^m b_m\sum_{s=1}^{n}(uv)^{-s}\\
&= \sum_{m=1}^{\infty}(-u)^m b_m\sum_{s=1}^{l}(uv)^{-s} = \sum_{m=1}^{\infty}(-u)^m b_m\tfrac{(uv)^{-1}-(uv)^{-l-1}}{1-(uv)^{-1}}\\
&= \frac{1}{uv-1}\sum_{m=1}^{\infty}(-u)^m b_m(1-(uv)^{-l}),
\end{aligned}$$

that is (23) and ii) a) holds. ii) b) is a direct consequence of the identity $\overset{\circ}{x}_n = \sum_{m=1}^{\infty} a'_{nm} b_m$ for all n, where $M^{-1} = (a'_{nm})_{n,m \geq 1}$. ■

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