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ON A SYSTEM OF LINEAR THERMOELASTICITY WITH THE BESSEL OPERATOR

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Abstract. In this paper, we study an initial value problem for a one-dimensional system of thermoelasticity. Using an a priori bound and a density argument, we prove the existence and uniqueness of a generalized solution.

1. Introduction

In this paper we are concerned with a coupled system of thermoelasticity of the form

$$\mathcal{L}_1 u = u_{tt} - \frac{a}{r}(ru_r)_r + br\theta_r = f(r,t) \tag{1.1}$$

$$\mathcal{L}_2 \theta = \theta_t - \frac{\varkappa}{r}(r\theta_r)_r + bru_{rt} = g(r,t) \tag{1.2}$$

in the bounded domain

$$Q = \Omega \times (0,T) = \{(r,t) : 0 < r < 1,\ 0 < t < T\},$$

where u is the displacement, θ is the difference absolute temperature, f is an external force, g is a heat supply, and $a, b, \varkappa$ are positive constants. This system can be regarded as a model for the radially symmetric deformation and temperature distribution in the unit disk.

For $b = 0$, system (1.1), (1.2) decouples and we obtain two independent equations; namely the hyperbolic wave equation and the parabolic heat equation "with the Bessel operator". Both equations have been extensively investigated and several results concerning existence, uniqueness, and well-posedness have been established. For the parabolic case we mention here Cannon [3], Bouziani [2], Ionkin [5], Kamynin [6], P. Shi [15], Yurchuk [18], Mesloub [8], Mesloub and Bouziani [9], [10], [11], Kartynik [7]. For the hyperbolic case, we mention Mesloub and Bouziani [12], [13], Muravei and Philinovskii [14], Pulkina [16], [17] and Beilin [1].

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We supplement (1.1), (1.2) with the initial conditions

$$\ell_1 u = u(r,0) = u_0(r), \qquad 0 < r < 1 \tag{1.3}$$

$$\ell_2 u = u_t(r,0) = u_1(r), \qquad 0 < r < 1 \tag{1.4}$$

$$\ell_3 \theta = \theta(r,0) = \theta_0(r), \qquad 0 < r < 1 \tag{1.5}$$

and the one-point boundary conditions

$$u(1,t) = 0, \qquad 0 < t < T \tag{1.6}$$

$$\theta(1,t) = 0, \qquad 0 < t < T, \tag{1.7}$$

where the data functions satisfy, for compatibility,

$$u_0(1) = u_1(1) = \theta_0(1) = 0. \tag{1.8}$$

Based on an a priori bound and on the density of the range of the operator generated by the problem in consideration, we prove the existence, uniqueness and the continuous dependence on the data of the strong solution of problem (1.1)–(1.7). We should note here that for the best of our knowledge, system (1.1), (1.2) has never been treated using this approach.

2. Functions spaces

Let $L^2_\rho(Q)$ be the weighted $L^2(Q)$ Hilbert space of square integrable functions on Q with scalar product

$$(u,\theta)_{L^2_\rho(Q)} = \int_Q r u\theta \, dr\, dt,$$

and with the associated finite norm

$$\|u\|^2_{L^2_\rho(Q)} = \int_Q r u^2 \, dr\, dt,$$

and $W^{1,1}_{2,\rho}$ be the weighted Hilbert space consisting of the elements u of $L^2_\rho(Q)$ having first order generalized derivatives square summable on Q. $W^{1,1}_{2,\rho}$ is equipped with the scalar product

$$(u,\theta)_{W^{1,1}_{2,\rho}(Q)} = (u,\theta)_{L^2_\rho(Q)} + (u_r,\theta_r)_{L^2_\rho(Q)} + (u_t,\theta_t)_{L^2_\rho(Q)},$$

and the associated norm is

$$\|u\|^2_{W^{1,1}_{2,\rho}(Q)} = \|u\|^2_{L^2_\rho(Q)} + \|u_r\|^2_{L^2_\rho(Q)} + \|u_t\|^2_{L^2_\rho(Q)}.$$

We also use the weighted spaces on Ω, such as $L^2_\rho(\Omega)$ and $W^1_{2,\rho}(\Omega)$, whose definitions are analogous to the spaces on Q.

3. Reformulation of the problem

We reformulate problem (1.1)–(1.7) as the problem of solving the operator equation

$$AU = \mathcal{H}, \tag{3.1}$$

where U, AU and $\mathcal{H}$ are respectively the pairs:

$$U = (u, \theta), \tag{3.2}$$

$$AU = (L_1 u, L_2 \theta), \tag{3.3}$$

and

$$\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2). \tag{3.4}$$

The right-hand sides of (3.3) and (3.4) are respectively defined by

$$L_1 u = \{\mathcal{L}_1 u, \ell_1 u, \ell_2 u\}, \quad L_2\theta = \{\mathcal{L}_2\theta, \ell_3\theta\}, \tag{3.5}$$

and

$$\mathcal{H}_1 = \{f, u_0, u_1\}, \quad \mathcal{H}_2 = \{g, \theta_0\}. \tag{3.6}$$

The operator A is considered from a space $B = B_1 \times B_2$ into the space $H = H_1 \times H_2$, where B is a Banach space consisting of all functions $(u, \theta) \in (L^2_\rho(Q))^2$ satisfying conditions (1.6)–(1.7) and having the finite norm

$$\|U\|^2_B = \sup_{0\le\tau\le T}\left(\|u(.,\tau)\|^2_{W^{1,1}_{2,\rho}(\Omega)} + \|\theta(.,\tau)\|^2_{L^2_\rho(\Omega)}\right) + \|\theta_r\|^2_{L^2_\rho(Q)} \tag{3.7}$$

and $H = H_1 \times H_2$ is the completion of the Hilbert space $\left\{L^2_\rho(Q) \times W^1_{2,\rho}(\Omega) \times L^2_\rho(\Omega)\right\} \times \left\{L^2_\rho(Q) \times L^2_\rho(\Omega)\right\}$ with respect to the norm

$$\|\mathcal{H}\|^2_H = \|f\|^2_{L^2_\rho(Q)} + \|u_0\|^2_{W^1_{2,\rho}(\Omega)} + \|u_1\|^2_{L^2_\rho(\Omega)} + \|g\|^2_{L^2_\rho(Q)} + \|\theta_0\|^2_{L^2_\rho(\Omega)}. \tag{3.8}$$

Let $D(L)$, be the domain of definition of the operator A, defined by:

$$D(A) = \left\{(u,\theta) \in (L^2_\rho(Q))^2 /\ u_t,\ \theta_t,\ u_{tt},\ u_r,\ \theta_r,\ u_{rr}, \theta_{rr}, u_{tr}, \theta_{tr} \in L^2_\rho(Q)\right\},$$

satisfying conditions (1.6)–(1.7).

4. Uniqueness of the solution

THEOREM 4.1. *For any function $U = (u, \theta) \in D(A)$, there exists a positive constant C independent of U, such that*

$$\|U\|_B \le C\,\|AU\|_H. \tag{4.1}$$

Proof. Consider the scalar products:

$$(u_t, \mathcal{L}_1 u)_{L^2_\rho(Q^\tau)} \quad \text{and} \quad (\theta, \mathcal{L}_2\theta)_{L^2_\rho(Q^\tau)}, \tag{4.2}$$

where $Q^\tau = (0,\tau) \times \Omega$. We have from (4.2)

$$\begin{aligned}(u_t, u_{tt})_{L^2_\rho(Q^\tau)} &- a((ru_r)_r, u_t))_{L^2(Q^\tau)} + (\theta, \theta_t)_{L^2_\rho(Q^\tau)} \\ &- \varkappa((r\theta_r)_r, \theta))_{L^2(Q^\tau)} + b(ru_{rt}, \theta)_{L^2_\rho(Q^\tau)} + b(ru_t, \theta_r)_{L^2_\rho(Q^\tau)} \\ &= (u_t, \mathcal{L}_1 u))_{L^2_\rho(Q^\tau)} + (\theta, \mathcal{L}_2\theta)_{L^2_\rho(Q^\tau)}.\end{aligned} \tag{4.3}$$

Using conditions (1.3)0-(1.7), we can evaluate the first five terms on the left-hand side of (4.3) as follows

$$(u_t, u_{tt})_{L^2_\rho(Q^\tau)} = \frac{1}{2}\|u_t(r,\tau)\|^2_{L^2_\rho(\Omega)} - \frac{1}{2}\|u_1\|^2_{L^2_\rho(\Omega)}, \tag{4.4}$$

$$-a((ru_r)_r, u_t))_{L^2(Q^\tau)} = \frac{a}{2}\left\|u_r(r,\tau)\right\|^2_{L^2_\rho(\Omega)} - \frac{a}{2}\left\|\frac{\partial u_0}{\partial r}\right\|^2_{L^2_\rho(\Omega)}, \tag{4.5}$$

$$(\theta,\theta_t)_{L^2_\rho(Q^\tau)} = \frac{1}{2}\left\|\theta(r,\tau)\right\|^2_{L^2_\rho(\Omega)} - \frac{1}{2}\left\|\theta_0\right\|^2_{L^2_\rho(\Omega)}, \tag{4.6}$$

$$-\varkappa((r\theta_r)_r,\theta))_{L^2(Q^\tau)} = \varkappa\left\|\theta_r\right\|^2_{L^2_\rho(Q^\tau)}, \tag{4.7}$$

$$b(ru_{rt},\theta)_{L^2_\rho(Q^\tau)} = -b(ru_t,\theta_r)_{L^2_\rho(Q^\tau)} - 2b(u_t,\theta)_{L^2_\rho(Q^\tau)}. \tag{4.8}$$

Substitution of equalities (4.4)–(4.8) into (4.3), yields

$$\begin{aligned}
&\frac{a}{2}\left\|u_r(r,\tau)\right\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\left\|u_t(r,\tau)\right\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\left\|\theta(r,\tau)\right\|^2_{L^2_\rho(\Omega)} + \varkappa\left\|\theta_r\right\|^2_{L^2_\rho(Q^\tau)} \\
&\quad = \frac{1}{2}\left\|u_1\right\|^2_{L^2_\rho(\Omega)} + \frac{a}{2}\left\|\frac{\partial u_0}{\partial r}\right\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\left\|\theta_0\right\|^2_{L^2_\rho(\Omega)} \\
&\qquad + 2b(u_t,\theta)_{L^2_\rho(Q^\tau)} + (u_t,\mathcal{L}_1 u))_{L^2_\rho(Q^\tau)} + (\theta,\mathcal{L}_2\theta)_{L^2_\rho(Q^\tau)}.
\end{aligned} \tag{4.9}$$

Using Cauchy-Schwarz inequality, we estimate the last three terms in the right-hand side of (4.9) as follows

$$2b(u_t,\theta)_{L^2_\rho(Q^\tau)} \le b\left\|\theta\right\|^2_{L^2_\rho(Q^\tau)} + b\left\|u_t\right\|^2_{L^2_\rho(Q^\tau)}, \tag{4.10}$$

$$(u_t,\mathcal{L}_1 u))_{L^2_\rho(Q^\tau)} \le \frac{1}{2}\left\|u_t\right\|^2_{L^2_\rho(Q^\tau)} + \frac{1}{2}\left\|\mathcal{L}_1 u\right\|^2_{L^2_\rho(Q^\tau)}, \tag{4.11}$$

$$(\theta,\mathcal{L}_2\theta)_{L^2_\rho(Q^\tau)} \le \frac{1}{2}\left\|\theta\right\|^2_{L^2_\rho(Q^\tau)} + \frac{1}{2}\left\|\mathcal{L}_2\theta\right\|^2_{L^2_\rho(Q^\tau)}. \tag{4.12}$$

Combining (in)equalities (4.9)–(4.12), we obtain

$$\begin{aligned}
&\frac{a}{2}\left\|u_r(r,\tau)\right\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\left\|u_t(r,\tau)\right\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\left\|\theta(r,\tau)\right\|^2_{L^2_\rho(\Omega)} + \varkappa\left\|\theta_r\right\|^2_{L^2_\rho(Q^\tau)} \\
&\quad \le \frac{1}{2}\left\|u_1\right\|^2_{L^2_\rho(\Omega)} + \frac{a}{2}\left\|\frac{\partial u_0}{\partial r}\right\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\left\|\theta_0\right\|^2_{L^2_\rho(\Omega)} (b+\frac{1}{2})\left\|u_t\right\|^2_{L^2_\rho(Q^\tau)} \\
&\qquad + (b+\frac{1}{2})\left\|\theta\right\|^2_{L^2_\rho(Q^\tau)} + \frac{1}{2}\left\|\mathcal{L}_2\theta\right\|^2_{L^2_\rho(Q^\tau)} + \frac{1}{2}\left\|\mathcal{L}_1 u\right\|^2_{L^2_\rho(Q^\tau)}.
\end{aligned} \tag{4.13}$$

It is easy to check that

$$\frac{1}{2}\left\|u(r,\tau)\right\|^2_{L^2_\rho(\Omega)} \le \frac{1}{2}\left\|u_0\right\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\left\|u\right\|^2_{L^2_\rho(Q^\tau)} + \frac{1}{2}\left\|u_t\right\|^2_{L^2_\rho(Q^\tau)}, \tag{4.14}$$

If we sum side to side (4.13) and (4.14), we get

$$\begin{aligned}
&\left\|u_r(r,\tau)\right\|^2_{L^2_\rho(\Omega)} + \left\|u_t(r,\tau)\right\|^2_{L^2_\rho(\Omega)} + \left\|u(r,\tau)\right\|^2_{L^2_\rho(\Omega)} + \left\|\theta(r,\tau)\right\|^2_{L^2_\rho(\Omega)} + \left\|\theta_r\right\|^2_{L^2_\rho(Q^\tau)} \\
&\quad \le c\Big(\left\|u_1\right\|^2_{L^2_\rho(\Omega)} + \left\|\theta_0\right\|^2_{L^2_\rho(\Omega)} + \left\|u_0\right\|^2_{W^1_{2,\rho}(\Omega)} + \left\|\mathcal{L}_2\theta\right\|^2_{L^2_\rho(Q^\tau)} \\
&\qquad + \left\|\mathcal{L}_1 u\right\|^2_{L^2_\rho(Q^\tau)} + \left\|u\right\|^2_{L^2_\rho(Q^\tau)} + \left\|u_t\right\|^2_{L^2_\rho(Q^\tau)} + \left\|\theta\right\|^2_{L^2_\rho(Q^\tau)}\Big),
\end{aligned} \tag{4.15}$$

where

$$c = \frac{\max(a, 2b+2)}{\min(a, 1, 2\varkappa)}.$$

Applying Gronwall's inequality [4] to (4.15), we obtain

$$\|u(r,\tau)\|^2_{W^{1,1}_{2,\rho}(\Omega)} + \|\theta(r,\tau)\|^2_{L^2_\rho(\Omega)} + \|\theta_r\|^2_{L^2_\rho(Q^\tau)}$$
$$\leq ce^{cT}\left(\|u_1\|^2_{L^2_\rho(\Omega)} + \|\theta_0\|^2_{L^2_\rho(\Omega)} + \|u_0\|^2_{W^1_{2,\rho}(\Omega)} + \|f\|^2_{L^2_\rho(Q^\tau)} + \|g\|^2_{L^2_\rho(Q^\tau)}\right). \quad (4.16)$$

Replacing the left-hand side of (4.16) by its upper bound with respect to τ over $(0,T)$, gives the desired estimate (4.1), with $C = \sqrt{c}\exp(\frac{cT}{2})$. ∎

It can be proved in a standard way that the operator $A\colon B = B_1 \times B_2 \to H = H_1 \times H_2$ is closable. Let $\overline{A}$ be its closure.

Proposition 4.2. *The operator $A\colon B \to H$ has a closure.*

Proof. The proof can be established in a similar way as in [12]. ∎

These are some consequences of Theorem 4.1.

Corollary 4.3. *There exists a positive constant C such that*

$$\|U\|_B \leq C\left\|\overline{A}U\right\|_H, \qquad \forall U \in D(\overline{A}). \quad (4.17)$$

Inequality (4.17) leads to the following results:

Corollary 4.4. *A strong solution of (1.1)–(1.7) is unique and depends continuously on $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) \in H$, where $\mathcal{H}_1 = \{f, u_0, u_1\}$ and $\mathcal{H}_2 = \{g, \theta_0\}$.*

Corollary 4.5. *The range $R(\overline{A})$ of $\overline{A}$ is closed in H and $R(\overline{A}) = \overline{R(A)}$.*

This last corollary shows that in order to prove that problem (1.1)–(1.7) has a strong solution for arbitrary $(\mathcal{H}_1, \mathcal{H}_2) \in H$, it is sufficient to prove that the range of A is dense in H; that is $\overline{R(A)} = H$.

5. Solvability of the problem

Proposition 5.1. *If, for some function $W = (w_1, w_2) \in (L^2_\rho(Q))^2$ and for all elements $U \in D_0(A) = \{U/U \in D(A) : \ell_1 u = \ell_2 u = \ell_3\theta = 0\}$, we have*

$$(\mathcal{L}_1 u, w_1)_{L^2_\rho(Q)} + (\mathcal{L}_2\theta, w_2)_{L^2_\rho(Q)} = 0, \quad (5.1)$$

then W vanishes almost everywhere in Q.

Proof. Since relation (5.1) holds for any element of $D_0(A)$, we then take an element $U = (u,\theta)$ with a special form given by

$$U = \begin{cases} (0,0), & 0 \leq t \leq s \\ (\int_s^t (\tau - t)u_{\tau\tau}\,d\tau, \int_s^t \theta_\tau\,d\tau), & s \leq t \leq T \end{cases} \quad (5.2)$$

such that (u_{tt}, θ_t) is a solution of the system

$$\begin{cases} ru_{tt} = E_1(r,t), \\ r\theta_t = E_2(r,t), \end{cases} \quad (5.3)$$

where $E_1(r,t)=\int_t^T w_1(r,\tau)\,d\tau$, and $E_2(r,t)=\int_t^T w_2(r,\tau)\,d\tau$. It is clear that

$$\begin{cases} w_1=-ru_{ttt} \\ w_2=-r\theta_{tt}. \end{cases} \tag{5.4}$$

By virtue of relations (5.2) and (5.3), the function $U=(u,\theta)\in(L^2_\rho(Q))^2$. In fact U possesses a higher order of smoothness.

Using lemma 3.2 of [11], the function $W=(w_1,w_2)$ represented by (5.4) is in $(L^2_\rho(Q))^2$.

Replacing the functions w_1 and w_2 given by (5.4) in the relation (5.1), we obtain

$$\begin{aligned} &-(u_{tt},u_{ttt})_{L^2_\rho(Q)}+a(u_{ttt},(ru_r)_r)_{L^2(Q)}-b(r\theta_r,u_{ttt})_{L^2_\rho(Q)} \\ &\qquad -(\theta_t,\theta_{tt})_{L^2_\rho(Q)}+\varkappa(\theta_{tt},(r\theta_r)_r)_{L^2(Q)}-b(r\theta_{tt},u_{tr})_{L^2_\rho(Q)}=0. \end{aligned} \tag{5.5}$$

Taking into account the special form of U given by (5.2) and (5.3), using conditions (1.6)–(1.7), and integrating by parts each term of (5.5), gives

$$-(u_{tt},u_{ttt})_{L^2_\rho(Q)}=\frac{1}{2}\left\|u_{tt}(r,s)\right\|^2_{L^2_\rho(\Omega)}, \tag{5.6}$$

$$a(u_{ttt},(ru_r)_r)_{L^2(Q)}=\frac{a}{2}\left\|u_{rt}(r,T)\right\|^2_{L^2_\rho(\Omega)}, \tag{5.7}$$

$$-b(r\theta_r,u_{ttt})_{L^2_\rho(Q)}=b(r\theta_{rt},u_{tt})_{L^2_\rho(Q)}, \tag{5.8}$$

$$-(\theta_t,\theta_{tt})_{L^2_\rho(Q)}=\frac{1}{2}\left\|\theta_t(r,s)\right\|^2_{L^2_\rho(\Omega)}, \tag{5.9}$$

$$\varkappa(\theta_{tt},(r\theta_r)_r)_{L^2(Q)}=\varkappa\left\|\theta_{rt}\right\|^2_{L^2_\rho(Q_s)}, \tag{5.10}$$

$$-b(r\theta_{tt},u_{tr})_{L^2_\rho(Q)}=-b(r\theta_{rt},u_{tt})_{L^2_\rho(Q_s)}-2b(\theta_t,u_{tt})_{L^2_\rho(Q_s)}. \tag{5.11}$$

Combining equalities (5.5)–(5.11), we get

$$\begin{aligned} \frac{1}{2}\left\|\theta_t(r,s)\right\|^2_{L^2_\rho(\Omega)}+\frac{1}{2}\left\|u_{tt}(r,s)\right\|^2_{L^2_\rho(\Omega)}+\frac{a}{2}\left\|u_{rt}(r,T)\right\|^2_{L^2_\rho(\Omega)}+\varkappa\left\|\theta_{rt}\right\|^2_{L^2_\rho(Q_s)} \\ =2b(\theta_t,u_{tt})_{L^2_\rho(Q_s)}, \end{aligned} \tag{5.12}$$

where $Q_s=\Omega\times[s,T]$. By using Cauchy-Schwarz inequality and discarding the two last terms of the left-hand side of (5.12), we obtain

$$\begin{aligned} &\left\|\theta_t(r,s)\right\|^2_{L^2_\rho(\Omega)}+\left\|u_{tt}(r,s)\right\|^2_{L^2_\rho(\Omega)} \\ &\qquad\le 2b\int_s^T\left(\int_0^1 r\theta_t^2(r,t)\,dr+\int_0^1 ru_{tt}^2(r,t)\,dr\right)dt. \end{aligned} \tag{5.13}$$

From (5.13), it follows that

$$-\frac{d}{ds}\left\{e^{2bs}\int_s^T\left(\int_0^1 r\theta_t^2(r,t)\,dr+\int_0^1 ru_{tt}^2(r,t)\,dr\right)dt\right\}\le 0. \tag{5.14}$$

Integrating (5.14) over $[s,T]$ and using the fact that

$$\int_s^T\left(\int_0^1 r\theta_t^2(r,t)\,dr+\int_0^1 ru_{tt}^2(r,t)\,dr\right)dt\ |_{s=T}=0,$$

it follows that

$$e^{2bs}\left(\int_s^T \int_0^1 r\theta_t^2(r,t)\,dr\,dt + \int_s^T \int_0^1 ru_{tt}^2(r,t)\,dr\,dt\right) \leq 0. \tag{5.15}$$

Hence, we deduce from (5.15) that $W = (w_1, w_2) = (0,0)$ almost everywhere in Q_s. Proceeding in this way step by step, we prove that $W = 0$ almost everywhere in Q. ∎

We now prove the following theorem which gives the existence of a strong solution of problem (1.1)–(1.7).

THEOREM 5.2. *For any* $(f,g) \in (L^2_\rho(Q))^2$ *and any* $u_0 \in W^1_{2,\rho}(\Omega)$, $u_1 \in L^2_\rho(Q)$, $\theta_0 \in L^2_\rho(Q)$, *there exists a unique strong solution* $U = \overline{A}^{-1}\mathcal{H} = \overline{A^{-1}}\mathcal{H}$ *of the problem (1.1)–(1.7), where* $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) \in H$, $\mathcal{H}_1 = \{f, u_0, u_1\}$, $\mathcal{H}_2 = \{g, \theta_0\}$, $U = (u, \theta)$ *and*

$$\|U\|_B \leq C\,\|AU\|_H\,,$$

for a positive constant C, *independent of* U.

Proof. It is sufficient to prove that the range $R(A)$ of A is dense in $H = H_1 \times H_2$.

Suppose that for some $\Psi = (\mathcal{G}_1, \mathcal{G}_2) = (\{w_1, w_3, w_4\}, \{w_2, w_5\}) \in H$, the orthogonal of $R(A)$, so that

$$\begin{aligned}(AU, \Psi)_H &= (\{L_1u, L_2\theta\}, \{\mathcal{G}_1, \mathcal{G}_2\})_H \\ &= (\{(\mathcal{L}_1u, \ell_1u, \ell_2u), (\mathcal{L}_2\theta, \ell_3\theta), (\{w_1, w_3, w_4\}, \{w_2, w_5\})\})_H \\ &= (\mathcal{L}_1u, w_1)_{L^2_\rho(Q)} + (\ell_1u, w_3)_{W^1_\rho(\Omega)} + (\ell_2u, w_4)_{L^2_\rho(\Omega)} \\ &\quad + (\mathcal{L}_2\theta, w_2)_{L^2_\rho(Q)} + (\ell_3\theta, w_5)_{L^2_\rho(\Omega)} = 0. \end{aligned} \tag{5.16}$$

We must show that $\Psi = \mathbf{0}$. Putting $U \in D_0(A)$ in (5.16), we get

$$(\mathcal{L}_1u, w_1)_{L^2_\rho(Q)} + (\mathcal{L}_2\theta, w_2)_{L^2_\rho(Q)} = 0, \qquad \forall U \in D(A). \tag{5.17}$$

Hence proposition 5.1 implies that $w_1 = w_2 = 0$.

The relation (5.17), implies that

$$(\ell_1u, w_3)_{W^1_\rho(\Omega)} + (\ell_2u, w_4)_{L^2_\rho(\Omega)} + (\ell_3\theta, w_5)_{L^2_\rho(\Omega)} = 0, \tag{5.18}$$

for all $U \in D(A)$.

Since the three quantities in (5.18) vanish independently and since the ranges of the trace operators ℓ_1, ℓ_2, and ℓ_3 are respectively everywhere dense in the spaces $W^1_{2,\rho}(\Omega)$, $L^2_\rho(Q)$, and $L^2_\rho(Q)$, therefore it follows, from (5.18), that $w_3 = w_4 = w_5 = 0$. Hence $\overline{R(A)} = H$. ∎

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