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FUZZY α-CONNECTEDNESS AND FUZZY α-DISCONNECTEDNESS IN FUZZY TOPOLOGICAL SPACES

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Abstract. Various types of connectedness and disconnectedness are introduced using fuzzy α-open sets. Several properties and characterizations of such spaces are discussed.

1. Introduction

The concept of fuzzy set was introduced by Zadeh in [18]. This concept has been applied by many authors to several branches of Mathematics. One of the applications was the study of fuzzy topological spaces [4, 5, 9, 10, 11, 20]. The concept of α-open set was introduced by Njastad [12] and this concept in fuzzy setting was introduced and studied by Bin Shahna [1]. In this paper we shall define various notions of fuzzy α-connectedness and fuzzy α-disconnectedness and employing the same means, we shall study questions similar to those of [6] and [3].

2. Preliminaries

In this paper we use the notion of a fuzzy topology in the original sense of Chang [4] and not in the modified sense of Lowen [10]. We shall denote a fuzzy topological space by (X,T), where X is the underlying set and T is the fuzzy topology in the sense of Chang. The symbols λ, δ, η, σ etc. are used to denote fuzzy sets and the symbol $1-\lambda$ stands for the complement of the fuzzy set λ. A fuzzy set λ in a fuzzy topological space (X,T) is called proper fuzzy set if $\lambda \neq 0$ and $\lambda \neq 1$. A subfamily B of T is called a base [2] for T if each member of T is a union of some members of B.

Let λ be a fuzzy set in a fuzzy topological space (X,T). Then we define $\operatorname{Int}\lambda = \bigvee\{\,\delta \mid \delta \leqslant \lambda, \delta \text{ is fuzzy open}\,\}$ and $\operatorname{Cl}\lambda = \bigwedge\{\,\delta \mid \delta \geqslant \lambda, \delta \text{ is fuzzy closed}\,\}$; λ is called fuzzy α-open (resp. fuzzy α-closed) [1] if $\lambda \leqslant \operatorname{Int}\operatorname{Cl}\operatorname{Int}\lambda$ (resp. $\lambda \geqslant \operatorname{Cl}\operatorname{Int}\operatorname{Cl}\lambda$). From this definition it is clear that λ is fuzzy α-open $\iff$ $1-\lambda$ is

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fuzzy α-closed. Also we define $\lambda_0 = \bigvee\{\, \delta \mid \delta \leqslant \lambda, \delta \text{ is fuzzy } \alpha\text{-open}\,\}$, $\underline{\lambda} = \bigwedge\{\, \delta \mid \delta \geqslant \lambda, \delta \text{ is fuzzy } \alpha\text{-closed}\,\}$; λ_0 and $\underline{\lambda}$ are called fuzzy α-interior and fuzzy α-closure of λ, respectively. A fuzzy set λ which is both fuzzy α-open and fuzzy α-closed is called fuzzy α-clopen set.

It is easy to verify the following relations between fuzzy α-interior and fuzzy α-closure: (a) $1-\lambda_0 = \underline{1-\lambda}$, (b) $1-\underline{\lambda} = (1-\lambda)_0$, where λ is any fuzzy set in X.

Let X and Y be any two non-empty sets and $f\colon X \to Y$ be any function. Let λ be any fuzzy set in Y. Then we define inverse image of λ, denoted by $f^{-1}(\lambda)$, as $f^{-1}(\lambda)\colon X \to I$, $f^{-1}(\lambda)(x) = \lambda[f(x)]$.

We shall denote the class of all fuzzy α-open sets of the fuzzy topological space (X,T) by $F\alpha o(X,T)$. If λ is a fuzzy set in X and δ is a fuzzy set in Y then the Cartesian product of λ and δ, denoted by $\lambda\times\delta$ [2] is defined as a fuzzy set in $X\times Y$ with

$$(\lambda\times\delta)(x,y) = \min\{\lambda(x),\delta(y)\} \quad \text{for each } (x,y) \text{ in } X\times Y.$$

Let $\{\,(X_k,T_k) \mid k\in\Gamma\,\}$ be any family of fuzzy topological spaces. Let $X = \prod_{k\in\Gamma} X_k$ and $p_k\colon X \to X_k$ be the projection map. Put $A = \{\, p_k^{-1}(\lambda) \mid \lambda \in T_k, k\in\Gamma\,\}$. Let B be the family of all finite intersections of members of A and T be the family of all unions of members of B. Then T is a fuzzy topology called the product topology [13] for X which we shall denote by $T = \prod_{k\in\Gamma} T_k$ and (X,T) is called the product fuzzy topological space. In particular, if (X,T) and (Y,S) are fuzzy topological spaces, we shall denote their product by $(X\times Y, T\times S)$.

Let (X,T) be any fuzzy topological space and A be any non-empty subset of X. Define $T/A = \{\,\lambda/A \mid \lambda\in T\,\}$. Then it is well known that T/A is a fuzzy topology in A and the fuzzy topological space $(A, T/A)$ is called fuzzy subspace [7] of (X,T).

A fuzzy point p [16] in (X,T) is a fuzzy set in (X,T) such that

$$p(x) = \begin{cases} t, & \text{for } x = x_0 \text{ where } 0<t<1,\\ 0, & \text{otherwise.}\end{cases}$$

p is said to have the support x_0 and value t. A fuzzy set in (X,T) is called a fuzzy singleton [8] if it takes the value zero (0) for all points in X except one. A fuzzy singleton with value 1 is called a crisp singleton.

A fuzzy topological space (X,T) is said to be fuzzy α-open T_1 (in short fαoT_1) if for every pair of non-zero fuzzy sets γ, β with $\gamma \nleqslant \beta$ there exists a fuzzy α-open set λ such that $\beta\leqslant\lambda$ and $\gamma\nleqslant\lambda$.

A fuzzy topological space (X,T) is said to be fuzzy α-open compact (in short fαo compact) if for each family $B \subset F\alpha o(X,T)$ such that $\bigvee\{\,\delta \mid \delta\in B\,\} = 1$, there exists a finite subfamily B_0 of B such that $\bigvee\{\,\lambda \mid \lambda\in B_0\,\} = 1$.

A fuzzy topological space (X,T) is said to be product related [2] to a fuzzy topological space (Y,S) if for any fuzzy set γ in X and ς in Y whenever $(1-\lambda)\ngeqslant\gamma$ and $(1-\delta)\ngeqslant\varsigma$ implies $[(1-\lambda)\times 1]\vee[1\times(1-\delta)] \geqslant \gamma\times\varsigma$, where λ is a fuzzy open set in X and δ is a fuzzy open set in Y, there exist a fuzzy open set λ_1 in X and a fuzzy open set δ_1 in Y such that

$(1-\lambda_1) \geqslant \gamma$ or $(1-\delta_1) \geqslant \varsigma$ and $[(1-\lambda_1)\times 1]\vee[1\times(1-\delta_1)] = [(1-\lambda)\times 1]\vee[1\times(1-\delta)]$.

Note. In point set topology it is well known that the closure of the product is the product of the closures. However, it is shown in [2] that this is not true in fuzzy setting. This difficulty is removed if we assume the fuzzy topological spaces (X, T) and (Y, S) are product related. This is the reason for introducing the concept of product relation defined above.

Throughout the paper for any subset $A \subset X$ we shall denote the characteristic function of A by μ_A.

3. Fuzzy α-connectedness

Definition. A fuzzy topological space (X, T) is said to be fuzzy α-connected if (X, T) has no proper fuzzy set λ which is both fuzzy α-open and fuzzy α-closed.

Proposition 1. *A fuzzy topological space (X, T) is fuzzy α-connected iff it has no non-zero fuzzy α-open sets λ_1 and λ_2 such that $\lambda_1 + \lambda_2 = 1$.*

Proof. If such λ_1 and λ_2 exist, then λ_1 is a proper fuzzy set which is both fuzzy α-open and fuzzy α-closed. To prove the converse suppose that (X, T) is not fuzzy α-connected. Then it has a proper fuzzy set λ_1 (say) which is both fuzzy α-open and fuzzy α-closed. Now put $\lambda_2 = 1 - \lambda_1$. Then λ_2 is a fuzzy α-open set such that $\lambda_2 \neq 0$ and $\lambda_1 + \lambda_2 = 1$. ■

Corollary. *A fuzzy topological space (X, T) is fuzzy α-connected iff it has no non-zero fuzzy sets λ_1 and λ_2 such that $\lambda_1 + \lambda_2 = \underline{\lambda_1} + \lambda_2 = \lambda_1 + \underline{\lambda_2} = 1$.*

Proposition 2. *Let A be a fuzzy α-connected subset of X and λ_1 and λ_2 be non-zero fuzzy α-open sets in (X, T) such that $\lambda_1 + \lambda_2 = 1$. Then either $\lambda_1/A = 1$ or $\lambda_2/A = 1$.*

Proof. Follows from Proposition 1. ■

Proposition 3. *Let $(A, T/A)$ be a fuzzy subspace of the fuzzy topological space (X, T) and let λ be a fuzzy set in A. Further, let δ be the fuzzy set in X defined as*

$$\delta(x) = \begin{cases} \lambda(x), & \text{if } x \in A, \\ 0, & \text{if } x \in X \setminus A. \end{cases}$$

Then $(\underline{\lambda})_{T/A} = (\underline{\delta})_T/A$, where $(\underline{\lambda})_{T/A}$ is the fuzzy α-closure of λ with respect to T/A and $(\underline{\delta})_T$ is the fuzzy α-closure of δ with respect to T.

Definition. Fuzzy sets λ_1 and λ_2 in a fuzzy topological space (X, T) are said to be fuzzy α-separated if $\underline{\lambda_1} + \lambda_2 \leqslant 1$ and $\lambda_1 + \underline{\lambda_2} \leqslant 1$.

Proposition 4. *Let $\{A_k\}_{k\in\Gamma}$ be a family of fuzzy α-connected subsets of X such that for each $k, l \in \Gamma$ with $k \neq l$, μ_{A_k} and μ_{A_l} are not fuzzy α-separated from each other. Then $\bigcup_{k\in\Gamma} A_k$ is a fuzzy α-connected subset of X.*

Proof. Suppose $Y = \bigcup_{k\in\Gamma} A_k$ is not fuzzy α-connected. Then there exist fuzzy α-open sets δ and σ in Y such that $\delta + \sigma = 1$, $\delta \neq 0, 1$, $\sigma \neq 0, 1$. Fix $k_0 \in \Gamma$. Then A_{k_0} is a fuzzy α-connected subset of Y as it is so in X. Then by Proposition 2, either $\delta/A_{k_0} = 1$ or $\sigma/A_{k_0} = 1$. Without loss of generality we shall assume that $\delta/A_{k_0} = 1$ (i).

Define two fuzzy sets λ_1 and λ_2 as follows:

$$\lambda_1(x) = \begin{cases} \delta(x), & \text{if } x \in Y, \\ 0, & \text{if } x \in X \setminus Y; \end{cases} \qquad \lambda_2(x) = \begin{cases} \sigma(x), & \text{if } x \in Y, \\ 0, & \text{if } x \in X \setminus Y. \end{cases}$$

Then by Proposition 3,

$$(\underline{\delta})_{T/Y} = (\underline{\lambda_1})_T/Y \qquad \text{and} \qquad (\underline{\sigma})_{T/Y} = (\underline{\lambda_2})_T/Y. \tag{ii}$$

Now (i) implies that $\mu_{A_{k_0}} \leqslant \lambda_1$ and so $\underline{\mu_{A_{k_0}}} \leqslant \underline{\lambda_1}$. (iii)

Let $k \in \Gamma \setminus \{k_0\}$. Since A_k is a fuzzy α-connected subset of Y, either $\delta/A_k = 1$ or $\sigma/A_k = 1$. We shall show that $\mu_{A_k}/A_k \neq \sigma/A_k$. Suppose that $\mu_{A_k}/A_k = \sigma/A_k$. Then $\mu_{A_k} \leqslant \lambda_2$ and hence $\underline{\mu_{A_k}} \leqslant \underline{\lambda_2}$. (iv).

Since $\delta + \sigma = \underline{\delta} + \sigma = \delta + \underline{\sigma} = 1$, $\lambda_1 + \underline{\lambda_2} \leqslant 1$ and $\underline{\lambda_1} + \lambda_2 \leqslant 1$ (by (ii) and definitions of λ_1 and λ_2), (iii) and (iv) imply that

$$\underline{\mu_{A_{k_0}}} + \mu_{A_k} \leqslant \underline{\lambda_1} + \lambda_2 \leqslant 1 \quad \text{and} \quad \mu_{A_{k_0}} + \underline{\mu_{A_k}} \leqslant \lambda_1 + \underline{\lambda_2} \leqslant 1.$$

This gives a contradiction as $\mu_{A_{k_0}}$ and μ_{A_k} are not fuzzy α-separated from each other.

This contradiction shows that $\mu_{A_k}/A_k \neq \sigma/A_k$ and hence $\mu_{A_k}/A_k = \delta/A_k$ for $k \in \Gamma$ implies $\delta = \mu_{A_k}/Y$. But $\delta + \sigma = 1$. So, $\sigma(x) = 0$ for all $x \in Y$. That is, $\sigma = 0$, which is a contradiction since $\sigma \neq 0$. So our assumption is wrong. Hence the proposition is proved. ∎

Corollary 1. *Let $\{A_k\}_{k\in\Gamma}$ be a family of fuzzy α-connected subsets of a fuzzy topological space (X, T) and $\bigcap_{k\in\Gamma} A_k \neq \emptyset$. Then $\bigcup_{k\in\Gamma} A_k$ is a fuzzy α-connected subset of X.*

Corollary 2. *If $\{A_n\}_{n=1}^{\infty}$ is a sequence of fuzzy α-connected subsets of a fuzzy topological space (X, T) such that μ_{A_n} and $\mu_{A_{n+1}}$ are not fuzzy α-separated from each other for $n = 1, 2, \dots$, then $\bigcup_{n=1}^{\infty} A_n$ is a fuzzy α-connected subset of X.*

The following proposition is easy to establish.

Proposition 5. *Let A and B be subsets of a fuzzy topological space (X, T) such that $\mu_A \leqslant \mu_B \leqslant \underline{\mu_A}$. If A is a fuzzy α-connected subset of X, then B is of the same kind, too.*

4. Fuzzy super α-connectedness

Definition. A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy regular α-open set if $\lambda = (\underline{\lambda})_0$.

DEFINITION. A fuzzy topological space (X,T) is called fuzzy super α-connected if there is no proper fuzzy regular α-open subset in it.

In the following proposition we give several characterizations of fuzzy super α-connected spaces.

PROPOSITION 6. *The following are equivalent for a fuzzy topological space* (X,T).

a. (X,T) *is fuzzy super* α*-connected.*

b. $\underline{\lambda}=1$ *whenever* λ *is a non-zero fuzzy* α*-open set in* (X,T).

c. $\lambda_0=0$ *whenever* λ *is a fuzzy* α*-closed set in* (X,T) *such that* $\lambda\neq 1$.

d. (X,T) *does not have non-zero fuzzy* α*-open sets* λ_1 *and* λ_2 *such that* $\lambda_1+\lambda_2\leqslant 1$.

e. (X,T) *does not have non-zero fuzzy sets* λ_1 *and* λ_2 *such that* $\underline{\lambda_1}+\lambda_2=\lambda_1+\underline{\lambda_2}=1$.

f. (X,T) *does not have fuzzy closed sets* δ_1 *and* δ_2 *such that* $\delta_1+\delta_2\geqslant 1$.

DEFINITION. A fuzzy subspace $(A,T/A)$ of a fuzzy topological space (X,T) is called fuzzy super α-connected subset of (X,T) if it is a fuzzy super α-connected topological space as a fuzzy subspace of (X,T).

PROPOSITION 7. *Let* (X,T) *be a fuzzy topological space. If* $A\subset Y\subset X$*, then* A *is a fuzzy super* α*-connected subset of* (X,T) *iff it is a fuzzy super* α*-connected subset of the fuzzy subspace* $(Y,T/Y)$ *of* (X,T).

PROPOSITION 8. *Let* A *be a fuzzy super* α*-connected subset of a fuzzy topological space* (X,T)*. If there exist fuzzy closed sets* δ_1 *and* δ_2 *in* X *such that* $(\delta_1)_0+\delta_2=\delta_1+(\delta_2)_0=1$*, then* $\delta_1/A=1$ *or* $\delta_2/A=1$.

PROPOSITION 9. *Let* (X,T) *be a fuzzy topological space and* $A\subset X$ *be a fuzzy super* α*-connected subset of* X *such that* μ_A *is fuzzy* α*-open set in* X*. If* λ *is a fuzzy regular* α*-open set in* (X,T)*, then either* $\mu_A\leqslant\lambda$ *or* $\mu_A\leqslant 1-\lambda$.

Proof. Follows from Proposition 8. ∎

PROPOSITION 10. *Let* $\{A_k\}_{k\in\Gamma}$ *be a family of subsets of a fuzzy topological space* (X,T) *such that each* μ_{A_k} *is fuzzy* α*-open. If* $\bigcap_{k\in\Gamma}A_k\neq\emptyset$ *and each* A_k *is a fuzzy super* α*-connected subset of* X*, then* $\bigcup_{k\in\Gamma}A_k$ *is also a fuzzy super* α*-connected subset of* (X,T).

PROPOSITION 11. *If* A *and* B *are subsets of a fuzzy topological space* (X,T) *and* $\mu_A\leqslant\mu_B\leqslant\underline{\mu_A}$ *and if* A *is a fuzzy super* α*-connected subset of* (X,T)*, then so is* B.

PROPOSITION 12. *Let* (X,T) *and* (Y,S) *be any two fuzzy super* α*-connected spaces which are product related. Then* $(X\times Y,T\times S)$ *is a fuzzy super* α*-connected space.*

Proof. Suppose that $(X \times Y, T \times S)$ is not fuzzy super α-connected. Then there exists $\lambda_1, \lambda_2 \in F\alpha o(X,T)$ and $\eta_1, \eta_2 \in F\alpha o(Y,S)$ such that $\lambda_1 \times \eta_1 \neq 0$, $\lambda_2 \times \eta_2 \neq 0$ and

$$(\lambda_1 \times \eta_1)(x,y) + (\lambda_2 \times \eta_2)(x,y) \leqslant 1 \quad \text{for all} \quad (x,y) \in X \times Y. \tag{i}$$

As (X,T) and (Y,S) are product related, $\lambda_1 \times \eta_1$ and $\lambda_2 \times \eta_2$ are fuzzy α-open sets in $(X \times Y, T \times S)$. Now $\lambda_1 \times \eta_1 = p_X^{-1}(\lambda_1) \wedge p_Y^{-1}(\eta_1)$; p_X is the projection map of $X \times Y$ onto X etc. So, $\min\{\lambda_1(x), \eta_1(y)\} + \min\{\lambda_2(x), \eta_2(y)\} \leqslant 1$ for every $(x,y) \in X \times Y$. Now from (i) we have that for any $(x,y) \in X \times Y$ either (ii) $\lambda_1(x) + \lambda_2(x) \leqslant 1$ or (iii) $\lambda_1(x) + \eta_1(y) \leqslant 1$ or (iv) $\eta_1(y) + \lambda_2(x) \leqslant 1$ or (v) $\eta_1(y) + \eta_2(y) \leqslant 1$. Now $\lambda_1 \wedge \lambda_2 \in F\alpha o(X,T)$ and $\eta_1 \wedge \eta_2 \in F\alpha o(Y,S)$.

As (X,T) and (Y,S) are fuzzy super α-connected topological spaces, if $\lambda_1 \wedge \lambda_2 \neq 0$, $\eta_1 \wedge \eta_2 \neq 0$, then there exist $x_1 \in X$ and $y_1 \in Y$ such that $(\lambda_1 \wedge \lambda_2)(x_1) > \frac{1}{2}$ and $(\eta_1 \wedge \eta_2)(y_1) > \frac{1}{2}$. So $\lambda_1(x_1) > \frac{1}{2}$, $\lambda_2(x_1) > \frac{1}{2}$, $\eta_1(y_1) > \frac{1}{2}$, $\eta_2(y_1) > \frac{1}{2}$. Therefore if $x = x_1$, $y = y_1$ then none of the above four possibilities will be true. If $\lambda_1 \wedge \lambda_2 = 0$, then for each $x \in X$ either $\lambda_1(x) = 0$ or $\lambda_2(x) = 0$. So for every $x \in X$, $\lambda_1(x) + \lambda_2(x) \leqslant 1$. Note that $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ as both $\lambda_1 \times \eta_1$ and $\lambda_2 \times \eta_2$ are $\neq 0$, which implies that (X,T) is not fuzzy super α-connected. Similarly $\eta_1 \wedge \eta_2 = 0$ will imply that (Y,S) is not fuzzy super α-connected. This gives a contradiction as both (X,T) and (Y,S) are fuzzy super α-connected. So our assumption that $(X \times Y, T \times S)$ is not fuzzy super α-connected is false. ■

5. Fuzzy strongly α-connectedness

Definition. A fuzzy topological space (X,T) is said to be fuzzy strongly α-connected if it has no non-zero fuzzy α-closed sets f and k such that $f + k \leqslant 1$. If (X,T) is not fuzzy strongly α-connected, then it is called fuzzy weakly α-connected.

For any fuzzy topological space (X,T) it is easy to prove the following properties.

a. (X,T) is fuzzy strongly α-connected iff it has no non-zero fuzzy α-open sets λ_1 and λ_2 such that $\lambda_1 \neq 1$, $\lambda_2 \neq 1$ and $\lambda_1 + \lambda_2 \geqslant 1$.

b. Let $(A, T/A)$ be a fuzzy subspace of a fuzzy strongly α-connected space (X,T). Then A is fuzzy strongly α-connected iff for any fuzzy α-open sets λ_1 and λ_2 in (X,T), $\mu_A \leqslant \lambda_1 + \lambda_2$ implies either $\mu_A \leqslant \lambda_1$ or $\mu_A \leqslant \lambda_2$.

c. Let (X,T) be a fuzzy strongly α-connected space. Let A be a subset of X such that μ_A is fuzzy α-closed in (X,T). Then A is fuzzy strongly α-connected subset of X.

Regarding the product of strongly α-connected spaces we prove the following

Proposition 13. *Let (X,T) and (Y,S) be fuzzy strongly α-connected spaces. Assume that they are product related. Then $(X \times Y, T \times S)$ is a fuzzy strongly α-connected space.*

Proof. Suppose that the space $(X \times Y, T \times S)$ is not fuzzy strongly α-connected. Since members of $F\alpha o(X \times Y, T \times S)$ are precisely of the type (by Theorem 1.5

of [1]) "$\lambda \times \delta$", where $\lambda \in F\alpha o(X,T)$, $\delta \in F\alpha o(Y,S)$, there exist non-zero fuzzy sets $\lambda_1, \lambda_3 \in F\alpha o(X,T)$ and $\lambda_2, \lambda_4 \in F\alpha o(Y,S)$ such that $\lambda_1 \times \lambda_2 \neq 1$, $\lambda_3 \times \lambda_4 \neq 1$ and for every $x \in X$, $y \in Y$,

$$\min\{\lambda_1(x), \lambda_2(y)\} + \min\{\lambda_3(x), \lambda_4(y)\} \geqslant 1. \tag{1}$$

Clearly, $\lambda_1 \vee \lambda_3 \in F\alpha o(X,T)$ and $\lambda_2 \vee \lambda_4 \in F\alpha o(Y,S)$. Given that (X,T) and (Y,S) are fuzzy strongly α-connected, so if $\lambda_1 \vee \lambda_3 \neq 1$, and $\lambda_2 \vee \lambda_4 \neq 1$, then there is $x_1 \in X$ and $y_1 \in Y$ such that $(\lambda_1 \vee \lambda_3)(x_1) < \frac{1}{2}$ and $(\lambda_2 \vee \lambda_4)(y_1) < \frac{1}{2}$ which implies $\lambda_1(x_1) < \frac{1}{2}$, $\lambda_3(x_1) < \frac{1}{2}$, $\lambda_2(y_1) < \frac{1}{2}$ and $\lambda_4(y_1) < \frac{1}{2}$. So for $x = x_1$ and $y = y_1$, (1) does not hold. If $\lambda_1 \vee \lambda_3 = 1$, then for each $x \in X$,

$$\lambda_1(x) = 1 \quad \text{or} \quad \lambda_3(x) = 1. \tag{2}$$

Now we show that $\lambda_1 \neq 1$. Suppose $\lambda_1 = 1$. Then $\lambda_1 \times \lambda_2 \neq 1$ and (Y,S) is fuzzy strongly α-connected implies that there exists $y_0 \in Y$ such that $\lambda_2(y_0) < \frac{1}{2}$. Now $\lambda_3 \times \lambda_4 \neq 1$. So either $\lambda_3 \neq 1$ or $\lambda_4 \neq 1$.

Case 1. If $\lambda_3 \neq 1$, then as (X,T) is fuzzy strongly α-connected there is $x_0 \in X$ such that $\lambda_3(x_0) < \frac{1}{2}$. So for $x = x_0$, $y = y_0$, (1) is not true.

Case 2. If $\lambda_4 \neq 1$, then since $\lambda_2 \neq 1$ and (Y,S) is fuzzy strongly α-connected there is $y_1 \in Y$ such that $\lambda_2(y_1) + \lambda_4(y_1) < 1$. So for any $x \in X$ and $y = y_1$,

$$\min\{\lambda_1(x), \lambda_2(y)\} + \min\{\lambda_3(x), \lambda_4(y)\} \leqslant \lambda_2(y_1) + \lambda_4(y_1) < 1.$$

This is a contradiction because of (1). Thus $\lambda_1 = 1$ is not possible. Similarly, we can prove that $\lambda_3 \neq 1$. By (2), $\lambda_1 + \lambda_3 \geqslant 1$. So (X,T) is not fuzzy strongly α-connected, which is a contradiction. Therefore $\lambda_1 \vee \lambda_3 = 1$ is not possible. Similarly, we can show that $\lambda_2 \vee \lambda_4 = 1$ is not possible. This proves that our assumption is wrong. Thus the proposition is proved. ■

Example. This example taken from [6] shows that an infinite fuzzy product of fuzzy strongly α-connected spaces need not be fuzzy strongly α-connected.

Let $X_n = [0,1]$, $n = 1, 2, \ldots$, and $T_n = \{0, 1, \frac{n}{2(n+1)}\}$, $n = 1, 2, \ldots$. Clearly, (X_n, T_n) is fuzzy strongly α-connected for all $n = 1, 2, \ldots$. Let T be the product fuzzy topology on $X = \prod_{n=1}^{\infty} X_n$. Then (X,T) is not strongly fuzzy α-connected, since T contains a member $\bigvee_{n=1}^{\infty} p_n^{-1}(\lambda_n) \neq 1$ such that $\bigvee_{n=1}^{\infty} p_n^{-1}(\lambda_n)(x) = \frac{1}{2}$ for $x \in X$, where $\lambda_n = \frac{n}{2(n+1)}$ and $p_n \colon X \to X_n$ is the projection map.

6. Extremally fuzzy α-disconnectedness

Definition. A fuzzy topological space (X,T) is said to be extremally fuzzy α-disconnected if $\underline{\lambda}$ is fuzzy α-open for every $\lambda \in F\alpha o(X,T)$.

Using the techniques adopted in [3] we present the characterizations and properties of such spaces as follows.

Proposition 14. *For any fuzzy topological space (X,T) the following are equivalent:*

(a) (X,T) *is extremally fuzzy* α*-disconnected.*

(b) *For each fuzzy* α*-closed set* λ, λ_0 *is fuzzy* α*-closed.*

(c) *For each fuzzy* α*-open set* λ, *we have* $\underline{\lambda} + \underline{(1 - \underline{\lambda})} = 1$.

(d) *For every pair of fuzzy* α*-open sets* λ, δ *in* (X,T) *with* $\underline{\lambda} + \delta = 1$, *we have* $\underline{\lambda} + \underline{\delta} = 1$.

Proof. $(a) \implies (b)$. Let λ be any fuzzy α-closed set. We claim λ_0 is fuzzy α-closed. Now $1 - \lambda_0 = \underline{1 - \lambda}$. Since λ is fuzzy α-closed, $1 - \lambda$ is fuzzy α-open and $1 - \lambda_0 = \underline{(1 - \lambda)}$ and by (a) we get $\underline{(1 - \lambda)}$ is fuzzy α-open. That is λ_0 is fuzzy α-closed.

$(b) \implies (c)$. Suppose that λ is any fuzzy α-open set. Now $1 - \underline{\lambda} = (1 - \lambda)_0$. Therefore,

$$\begin{aligned} \underline{\lambda} + \underline{(1 - \underline{\lambda})} &= \underline{\lambda} + \underline{(1 - \lambda)_0} = \underline{\lambda} + (1 - \lambda)_0 \quad \text{[by (b)]} \\ &= \underline{\lambda} + (1 - \underline{\lambda}) = 1. \end{aligned}$$

$(c) \implies (d)$. Suppose λ and δ be any two fuzzy α-open sets in (X,T) such that $\underline{\lambda} + \delta = 1$ (1). Then by (c), $\underline{\lambda} + \underline{(1 - \underline{\lambda})} = 1 = \underline{\lambda} + \delta$ implies $\delta = \underline{(1 - \underline{\lambda})}$ (2). But from (1), $\delta = 1 - \underline{\lambda}$ and so from (2), $1 - \underline{\lambda} = \underline{(1 - \underline{\lambda})}$. That is $1 - \underline{\lambda}$ is fuzzy α-closed and so $\underline{\delta} = 1 - \underline{\lambda}$. That is $\underline{\delta} + \underline{\lambda} = 1$.

$(d) \implies (a)$. Let λ be any fuzzy α-open set in (X,T) and put $\delta = 1 - \underline{\lambda}$. From the construction of δ it follows that $\underline{\lambda} + \delta = 1$. Therefore by (d) we have $\underline{\lambda} + \underline{\delta} = 1$ and hence $\underline{\lambda}$ is fuzzy α-open in (X,T). That is (X,T) is extremally fuzzy α-disconnected. ■

7. Totally fuzzy α-disconnectedness

Definition. A fuzzy topological space (X,T) is said to be totally fuzzy α-disconnected if for every pair of fuzzy points $p \neq q$ in X, there exist non-zero fuzzy α-open sets λ, δ such that $\lambda + \delta = 1$, λ contains p and δ contains q.

Suppose $A \subset X$. A is said to be totally fuzzy α-disconnected subset of X if $(A, T/A)$ as a fuzzy subspace of (X,T) is totally fuzzy α-disconnected.

Proposition 15. *The maximal fuzzy* α*-connected subsets of a totally fuzzy* α*-disconnected space* (X,T) *are singleton sets.*

Proof. Let $(Y, T/Y)$ be a subspace of (X,T). It suffices to show that $(Y, T/Y)$ is totally fuzzy α-disconnected whenever it contains more than one point. Let x_1 and x_2 be any two distinct points in Y. Define

$$p\colon Y \to I \text{ as } p(x) = \begin{cases} 1/3, & x = x_1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$q\colon Y \to I \text{ as } q(x) = \begin{cases} 2/3, & x = x_2, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, p and q are distinct fuzzy points in Y. Also define

$$p^*\colon X \to I \text{ as } p^*(x) = \begin{cases} 1/3, & x = x_1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$q^*\colon X \to I \text{ as } q^*(x) = \begin{cases} 2/3, & x = x_2, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, p^* and q^* are distinct fuzzy points in X such that $p^*/Y = p$ and $q^*/Y = q$.

Since (X,T) is totally fuzzy α-disconnected space there exists non-zero fuzzy α-open sets λ, δ in (X,T) such that λ contains p^*, δ contains q^* and $\lambda + \delta = 1$. Then $\lambda_1 = \lambda/Y$, $\delta_1 = \delta/Y$ are non-zero fuzzy α-open sets in $(Y, T/Y)$ such that $\lambda_1 + \delta_1 = 1$, λ_1 contains p and δ_1 contains q. This shows that $(Y, T/Y)$ is totally fuzzy α-disconnected. ∎

PROPOSITION 16. *Every fuzzy subspace of a totally fuzzy α-disconnected space is totally fuzzy α-disconnected.*

PROPOSITION 17. *Let (X,T) be a $f\alpha oT_1$-space. If (X,T) has a base whose members are fuzzy α-clopen, then (X,T) is totally fuzzy α-disconnected.*

PROPOSITION 18. *Let (X,T) be $f\alpha o$ compact and $f\alpha oT_1$-space. Then (X,T) is totally fuzzy α-disconnected iff (X,T) has a base whose members are fuzzy α-clopen.*

The following remarks give the link to the existing notions of (dis-)connectedness in the fuzzy sense as introduced in lot of papers such as [3, 6, 11, 17, 19, 20].

REMARK 1. In [17] it is defined that a fuzzy set $\sigma \in I^X$ is called disconnected (o-disconnected) in the fuzzy topological space (X,T) if there exist closed (open) fuzzy sets λ, δ such that $\sigma \wedge \lambda \neq 0$, $\sigma \wedge \delta \neq 0$, $\sigma \leqslant \lambda \vee \delta$, $\sigma \wedge \lambda \wedge \delta = 0$. It is connected (o-connected) if it is not disconnected (not o-disconnected).

If T is a topology on X, the fuzzy topology $\tilde{T}$ induced by T is defined by $\tilde{T} = \{\, \lambda \in I^X \mid \lambda^{-1}\{(0,1]\} \in T \,\}$. Zheng [19] stated a theorem that in a fuzzy topological space "induced" by a topology, the notions of connectedness and o-connectedness coincide. However, in [17] Wuyts has shown that this theorem is false and suggested some modifications.

From this, one can observe that our concept of fuzzy α-connectedness/fuzzy α-disconnectedness etc. are independent.

REMARK 2. In [15] Nanda has introduced the concept of super connectedness as a fuzzified version of its corresponding concept in [14] for ordinary topological spaces. A fuzzy topological space (X,T) is said to be super connected if every fuzzy open set is connected. Again this concept is also independent to our concept of connectedness as we speak about connectedness for the fuzzy topological space (X,T) and not for any specific fuzzy set in X.

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