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A UNIFIED THEORY OF CONTINUITY

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Abstract. This paper further develops the theory of operations on a topological space with the aim of producing a uniform framework for the study of generalized forms of continuity. To illustrate the utility of this approach the results obtained are used to obtain many new and known characterizations of strong Θ-semi-continuity, weak continuity and almost continuity.

1. Introduction

Various unification theories in topological spaces have been studied by several authors [1, 5–11, 14, 15]. This paper develops the theory of operations on a topological space with the aim of producing a uniform framework for the study of generalized forms of continuity.

In a topological space (X,τ), $\mathrm{int}, \mathrm{cl}, \mathrm{scl}$, etc. will stand for interior, closure, semi-closure operations etc. and A^o, $\overline{A}$ will also stand for the interior of A, the closure of A for a subset A of X respectively.

Definition 1.1. Let (X,τ) be a topological space. A mapping $\varphi\colon P(X) \to P(X)$ is called an *operation* on (X,τ) if $A^o \subset \varphi(A)$ for all $A \in P(X)$ and $\varphi(\emptyset) = \emptyset$.

The class of all operations on a topological space (X,τ) will be denoted by $O(X,\tau)$.

The operations φ, $\widetilde{\varphi}$ are said to be *dual* if $\varphi(A) = X \setminus (\widetilde{\varphi}(X \setminus A))$ (equivalently $\widetilde{\varphi}(A) = X \setminus (\varphi(X \setminus A)))$ for each $A \in P(X)$.

A partial order "$\leq$" on $O(X,\tau)$ is defined by $\varphi_1 \leq \varphi_2 \Leftrightarrow \varphi_1(A) \subset \varphi_2(A)$ for each $A \in P(X)$.

An operation $\varphi \in O(X,\tau)$ is called *monotonous* if $\varphi(A) \subset \varphi(B)$ whenever $A \subset B \quad (A, B \in P(X))$.

Definition 1.2. Let $\varphi \in O(X,\tau)$, $\mathcal{U} \subset P(X)$ and $\mathcal{U}$(x)$= \{ U : U \in \mathcal{U}, x \in U \}$ for $x \in X$. Then φ is called:

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a) *regular with respect to* (shortly, w.r.t.) $\mathcal{U}$ if for each $x \in X$ and $U, V \in \mathcal{U}(x)$, there exists a $W \in \mathcal{U}(x)$ such that $\varphi(W) \subset \varphi(U) \cap \varphi(V)$.

b) *weakly finite intersection preserving (shortly, W.F.I.P.) w.r.t* $\mathcal{U}$ if whenever $U \in \mathcal{U}$, $A \in P(X)$, we have $U \cap \varphi(A) \subset \varphi(U \cap A)$.

REMARK 1.3. Monotonicity of $\varphi \in O(X, \tau)$ is not sufficient for regularity of φ w.r.t. any family $\mathcal{U} \subset P(X)$.

DEFINITION 1.4. Let $\varphi \in O(X, \tau)$ and $A, B \subset X$. Then A is called φ*-open* if $A \subset \varphi(A)$ and B is called φ*-closed* if $X \setminus B$ is φ-open.

X and $\emptyset$ are both φ-open and φ-closed sets for any $\varphi \in O(X, \tau)$.

Let (X, τ) be a topological space, $\varphi \in O(X, \tau), \mathcal{U} \subset P(X), x \in X$. We will use the following notations.

$$\varphi O(X) = \{\, U : U \subset X, U \text{ is } \varphi\text{-open} \,\}, \;\; \varphi O(X, x) = \{\, U : U \in \varphi O(X),\ x \in U \,\}$$
$$\mathcal{N}(\mathcal{U}, x) = \{\, N : N \subset X \text{ and there exists a } U \in \mathcal{U}(x) \text{ such that } U \subset N \,\}$$

DEFINITION 1.5. Let $\varphi_1, \varphi_2 \in O(X, \tau), X \in \mathcal{U} \subset P(X), A \subset X$.

a) $x \in (\mathcal{U}\text{-}\varphi_2)$ closure $A \Leftrightarrow \varphi_2(U) \cap A \neq \emptyset$ for each $U \in \mathcal{U}(x)$.
   $x \in \mathcal{U}$- closure $A \Leftrightarrow x \in (\mathcal{U}\text{-}\imath) \operatorname{cl} A$ (here $\imath$ is the identity operation).
   $x \in \varphi_{1,2} \operatorname{cl} A \Leftrightarrow x \in (\varphi_1 O(X)\text{-}\varphi_2) \operatorname{cl} A$.

b) $x \in \varphi_{1,2} \operatorname{int} A \Leftrightarrow$ there exists a $U \in \varphi_1 O(X, x)$ such that $\varphi_2(U) \subset A$.

c) A is $\varphi_{1,2}$- open $\Leftrightarrow A \subset \varphi_{1,2} \operatorname{int} A$.

d) A is $\varphi_{1,2}$-closed $\Leftrightarrow \varphi_{1,2} \operatorname{cl} A \subset A$.

Clearly, for any set $A, X \setminus \varphi_{1,2} \operatorname{int} A = \varphi_{1,2} \operatorname{cl}(X \setminus A)$ and A is $\varphi_{1,2}$-open iff $X \setminus A$ is $\varphi_{1,2}$-closed.

For $\varphi_1, \varphi_2 \in O(X, \tau)$, let $\varphi_{1,2} O(X) = \{U : U \subset X,\ U \text{ is } \varphi_{1,2}\text{-open}\}$.

Many preliminary results and the following theorem can be obtained for topological spaces from [8].

THEOREM 1.6. *Let* $\varphi_1, \varphi_2 \in O(X, \tau)$.

a) $\varphi_{1,2} O(X)$ *is a supratopology on* X *(*$\mathcal{U} \subset P(X)$ *is a supratopology iff* $\emptyset \in \mathcal{U}$*,* $X \in \mathcal{U}$ *and* $\mathcal{U}$ *is closed under arbitrary unions, [2]).*

b) *If* φ_2 *is regular w.r.t.* $\varphi_1 O(X)$ *then* $\varphi_{1,2} O(X)$ *is a topology on* X.

c) *If* $\varphi_2 \geq \imath$ *or* $\varphi_2 \geq \varphi_1$*, and if* φ_2 *is regular w.r.t.* $\varphi_1 O(X)$*, then* A *is closed (open) in the topology* $\varphi_{1,2} O(X)$ *iff* $A = \varphi_{1,2} \operatorname{cl} A$ *(*$A = \varphi_{1,2} \operatorname{int} A$*).*

EXAMPLE 1.7. Let $\varphi_1,\ \varphi_2 \in O(X, \tau)$.

1. If $\varphi_1 = \operatorname{int}$, $\varphi_2 = \operatorname{cl}$, then $\varphi_{1,2} O(X)$ is the topology of all Θ-open sets.
2. If $\varphi_1 = \operatorname{int}$, $\varphi_2 = \operatorname{int} \circ \operatorname{cl}$, then $\varphi_{1,2} O(X)$ is the topology τ_s having a base which is the family of all regular-open sets.
3. If $\varphi_1 = \operatorname{int} \circ \operatorname{cl} \circ \operatorname{int}$, $\varphi_2 = \imath$, then $\varphi_{1,2} O(X) = \tau^{\alpha}$, the topology of all α-open sets (A is α-open means that $A \subset (\overline{A^o})^o$).

2. The W.F.I.P. Property

For $\varphi_1, \varphi_2 \in O(X,\tau)$, if $\varphi_2 \geq \imath$ or $\varphi_2 \geq \varphi_1$, then $\varphi_1 O(X) \subset \varphi_2 O(X)$.

THEOREM 2.1. *Let (X,τ) be a topological space, $X \in \mathcal{U} \subset P(X)$, $\mathcal{A} \subset P(X)$, $\varphi_1, \varphi_2 \in O(X,\tau)$. The following are valid:*

1. *$A \subset \mathcal{U}\text{-}\mathrm{cl}\, A$ for each $A \in P(X)$.*
2. *$A \subset \varphi_1 O(X)\text{-}\mathrm{cl}\, A$ for each $A \in P(X)$.*
3. *If $A \subset B$ then $\mathcal{U}\text{-}\mathrm{cl}\, A \subset \mathcal{U}\text{-}\mathrm{cl}\, B$.*
4. *If φ_2 is W.F.I.P. w.r.t. $\mathcal{A}$, then φ_2 is W.F.I.P. w.r.t. any subfamily of $\mathcal{A}$.*
5. *If φ_2 is W.F.I.P. w.r.t. $\mathcal{A}$, then $(\mathcal{U}\text{-}\varphi_2)\,\mathrm{cl}\, A \subset \mathcal{U}\text{-}\mathrm{cl}\, A$ for each $A \in \mathcal{A}$.*
6. *If φ_2 is W.F.I.P. w.r.t. $\varphi_1 O(X)$, then $\varphi_{1,2}\,\mathrm{cl}\, A \subset \varphi_1 O(X)\text{-}\mathrm{cl}\, A$ for each $A \in \varphi_1 O(X)$.*
7. *If $\varphi_2 \geq \imath$, then $\mathcal{U}\text{-}\mathrm{cl}\, A \subset (\mathcal{U}\text{-}\varphi_2)\,\mathrm{cl}\, A$ for each $A \in P(X)$.*
8. *If $\varphi_2 \geq \imath$ or $\varphi_2 \geq \varphi_1$, then $\varphi_1 O(X)\text{-}\mathrm{cl}\, A \subset \varphi_{1,2}\,\mathrm{cl}\, A$ for each $A \in P(X)$.*
9. *If $\varphi_2 \geq \imath$ or $\varphi_2 \geq \varphi_1$, and if φ_2 is W.F.I.P. w.r.t. $\mathcal{A}$ then $\varphi_1\ O(X)\text{-}\mathrm{cl}\, A = \varphi_{1,2}\,\mathrm{cl}\, A$ for each $A \in \mathcal{A}$.*
10. *If φ_2 is W.F.I.P. w.r.t. $\mathcal{A}$ and $\varphi_2 \geq \imath$ then $\mathcal{U}\text{-}\mathrm{cl}\, A = (\mathcal{U}\text{-}\varphi_2)\,\mathrm{cl}\, A$ for each $A \in \mathcal{A}$.*

Proof. 5) Let φ_2 be W.F.I.P. w.r.t. $\mathcal{A}, A \in \mathcal{A}$ and $x \in (\mathcal{U}\text{-}\varphi_2)\,\mathrm{cl}\, A$.

$$\begin{aligned} x \in U \in \mathcal{U} &\Rightarrow \varphi_2(U) \cap A \neq \emptyset \\ &\Rightarrow \varphi_2(U) \cap A \subset \varphi_2(U \cap A) \neq \emptyset \\ &\Rightarrow U \cap A \neq \emptyset. \end{aligned}$$

So $x \in \mathcal{U}\text{-}\mathrm{cl}\, A$.

8) Let $A \in P(X)$ and $x \in \varphi_1 O(X)\text{-}\mathrm{cl}\, A$.

$$\begin{aligned} U \in \varphi_1 O(X,x) &\Rightarrow U \cap A \neq \emptyset \\ &\Rightarrow \varphi_2(U) \cap A \neq \emptyset \ \ (\text{since } U \subset \varphi_2(U) \text{ or } U \subset \varphi_1(U) \subset \varphi_2(U)). \end{aligned}$$

So $x \in \varphi_{1,2}\,\mathrm{cl}\, A$. ∎

For a topological space (X,τ), $RO(X)$, $SO(X)$, $PO(X)$ and τ^α will stand for the family of regular open sets, semi-open sets, pre-open sets, α-open sets, respectively.

EXAMPLE 2.2. Let (X,τ) be a topological space.

1. $\varphi_2 = \mathrm{cl}$ is W.F.I.P. w.r.t. τ and $\varphi_2 \geq \imath$.

   So for any $\varphi_1 \in O(X,\tau)$ and for any U $\in \tau$, we have $\varphi_{1,2}\,\mathrm{cl}\, U = \varphi_1 O(X)\text{-}\mathrm{cl}\, U$.

(a) If we take $\varphi_1 = \mathrm{int}$, $\varphi_{1,2}\,\mathrm{cl}\, A = \Theta\text{-}\mathrm{cl}\, A$, $\varphi_1 O(X)\text{-}\mathrm{cl}\, A = \tau\text{-}\mathrm{cl}\, A$ for each $A \in P(X)$. So $\Theta\text{-}\mathrm{cl}\, U = \overline{U}$ for each $U \in \tau$.

(b) If we take $\varphi_1 = \mathrm{cl} \circ \mathrm{int}$, then for each $U \in \tau$, $\varphi_{1,2}\,\mathrm{cl}\, U = \{x : x \in V \in SO(X) \Rightarrow \overline{V} \cap U \neq \emptyset\} = \Theta\text{-semi-cl}\, U = \mathrm{scl}\, U$.

2. Let $\varphi_1 = \mathrm{int}$, $\varphi_2 = \mathrm{int} \circ \mathrm{cl}$ (or $\varphi_2 = \mathrm{int} \circ \mathrm{cl} \circ \mathrm{int}$).

φ_2 is W.F.I.P. w.r.t. $\varphi_1 O(X) = \tau$, $\varphi_2 \geq \varphi_1$. For any $A \subset X$, $\varphi_{1,2}\,\mathrm{cl}\,A = \tau_s\,\mathrm{cl}\,A$, $\varphi_1 O(X)$-$\mathrm{cl}\,A = \tau\,\mathrm{cl}\,A$.

So if $U \in \tau$, $\tau_s\,\mathrm{cl}\,U = \tau\,\mathrm{cl}\,U$. We know that if a family $\mathcal{A} \subset P(X)$ is a supratopology, then

$$\tau_{\mathcal{A}} = \{U \subset X : A \in \mathcal{A} \Rightarrow U \cap A \in \mathcal{A}\}$$

is a topology on X [17].

If $\varphi \in O(X, \tau)$ is monotonous then $\varphi O(X)$ is a supratopology. But the converse is not true.

Example 2.3. Let τ be the usual topology on $\mathbb{R}$ and $\varphi\colon P(\mathbb{R}) \to P(\mathbb{R})$ be defined as follows:

$\varphi(A) = \overline{A}$ if $A^o = \emptyset$ and $\varphi(A) = A$ if $A^o \neq \emptyset$.

Then $\varphi \in O(\mathbb{R}, \tau)$, $\varphi O(\mathbb{R}) = P(\mathbb{R})$. But φ is not monotonous. For example, for the sets $A = (0,1) \cap \mathbb{Q}$, $B = (0,1)$ we have $A \subset B$ but $\varphi(A) \not\subset \varphi(B)$.

If $\varphi O(X)$ is a supratopology for $\varphi \in O(X, \tau)$ and if φ is W.F.I.P. w.r.t. τ then we have $\tau \subset \tau_{\varphi O(X)}$.

Let (X, τ) be a topological space and let $\mathcal{A}$ be a supratopology on X containing τ. If we define the mapping $\varphi\colon P(X) \to P(X)$ as $\varphi(A) = \mathcal{A}$-$\mathrm{int}\,A$ (i.e, φ is the interior operation with respect to $\mathcal{A}$) then $\varphi \in O(X, \tau)$, φ is monotonous and $\varphi O(X) = \mathcal{A}$. In this case if $\varphi = \mathcal{A}$-int is W.F.I.P. w.r.t. some subfamily $\mathcal{W}$ of $P(X)$ then $\mathcal{W} \subset \tau_{\mathcal{A}} = \tau_{\varphi O(X)}$.

For any $\mathcal{A} \subset P(X)$ containing X, the mapping $\varphi\colon P(X) \to P(X)$ defined by $\varphi(A) = \mathcal{A}$-$\mathrm{cl}\,A$ is an operation on (X, τ) for any topology τ on X.

Theorem 2.4. *Let $\mathcal{A}$ be a supratopology on X and $\tau_{\mathcal{A}}$, the topology mentioned above. If we take $\varphi_2 = \mathcal{A}$-cl, then φ_2 is W.F.I.P. w.r.t. $\tau_{\mathcal{A}}$ and $\varphi_2 \geq \imath$.*

Proof. Straightforward. ■

Corollary 2.5. *Let $\varphi \in O(X, \tau)$. If $\varphi O(X)$ is a supratopology and if we take $\varphi_2 = \varphi O(X)$-cl, then for any $\varphi_1 \in O(X, \tau)$ and for any $U \in \tau_{\varphi O(X)}$ we have $\varphi_{1,2} clU = \varphi_1 O(X)$-$\mathrm{cl}\,U$.*

Example 2.6. Take $\varphi_1,\ \varphi_2 \in O(X, \tau)$.

1. If we take $\varphi_1 = \mathrm{cl} \circ \mathrm{int}$, then $\varphi_1 O(X) = SO(X)$ and $\tau_{SO(X)} = \tau^\alpha$. For $\varphi_2 = \varphi_1 O(X)$-cl $=$ scl, we get $\varphi_{1,2}\,\mathrm{cl}\,U = \text{semi-}\Theta\text{-}\,\mathrm{cl}\,U = \mathrm{scl}\,U$ for each α-open set U (so for $U \in \tau,\ U \in \tau_s,\ U \in \tau_\theta$).
2. If we take $\varphi_1 = \mathrm{int} \circ \mathrm{cl}$, $\varphi_2 = \varphi_1 O(X)$-cl $=$ pcl, we get

$$\begin{aligned}\varphi_{1,2}\,\mathrm{cl}\,U &= \{\, x : x \in V \in PO(X) \Rightarrow (\mathrm{pcl}\,V) \cap U \neq \emptyset \,\}\\ &= \varphi_1 O(X)\text{-}\,\mathrm{cl}\,U = \mathrm{pcl}\,U \text{ for each } U \in \tau_{PO(X)}.\end{aligned}$$

3. If we take $\mathcal{A} = SO(X)$, $\varphi_2 = SO(X)$-cl $=$ scl, then $\varphi_2 \geq \imath$, φ_2 is W.F.I.P. w.r.t. τ^α. If we take $\varphi_1 = \operatorname{int}$, we get
$$\begin{aligned}\varphi_{1,2}\operatorname{cl} A &= \{\, x \in X : x \in U \in \tau \Rightarrow \operatorname{scl} U \cap A \neq \emptyset \,\}\\ &= \{\, x \in X : x \in U \in \tau \Rightarrow \overline{U}^o \cap A \neq \emptyset \,\}\\ &= \tau_s \operatorname{cl} A = \varphi_1 O(X)\text{- cl}\, A = \tau\text{- cl}\, A\end{aligned}$$
for each $A \in \tau^\alpha$ by using Theorem 2.1(9) and the property $\operatorname{scl} U = U \cup \overline{U}^o = \overline{U}^o$ for $U \in \tau$ [3].
4. Let $\varphi_1 = \operatorname{int} \circ \operatorname{cl} \circ \operatorname{int}$, $\varphi_2 = \operatorname{scl}$. Then φ_2 is W.F.I.P. w.r.t. $\varphi_1 O(X) = \tau^\alpha$. For each $A \in \tau^\alpha = \varphi_1 O(X)$ we have
$\varphi_{1,2}\operatorname{cl} A = \{x \in X : x \in U \in \tau^\alpha \Rightarrow (\operatorname{scl} U) \cap A \neq \emptyset\} = \{x \in X : x \in U \in \tau^\alpha \Rightarrow \overline{U}^o \cap A \neq \emptyset\} = \{x \in X : x \in U \in \tau \Rightarrow \overline{U}^o \cap A \neq \emptyset\} = \tau_s\text{- cl}\, A = \varphi_1 O(X)\text{- cl}\, A = \tau^\alpha \operatorname{cl} A$.
5. If we take $\varphi_2 = \operatorname{pcl}$, then $\varphi_2 \geq \imath$, φ_2 is W.F.I.P. w.r.t. $\tau_{PO(X)}$ (and w.r.t. $\tau, \tau_s, \tau_\theta$).
If we take $\varphi_1 = \operatorname{int}$, we get $\varphi_{1,2}\operatorname{cl} A = \{x \in X : x \in T \in \tau \Rightarrow (\operatorname{pcl} T) \cap A \neq \emptyset\} = \{x \in X : x \in T \in \tau \Rightarrow \overline{T^o} \cap A \neq \emptyset\} = \{x \in X : x \in T \in \tau \Rightarrow \overline{T} \cap A \neq \emptyset\} = \Theta\text{- cl}\, A = \varphi_1 O(X)\text{- cl}\, A = \tau \operatorname{cl} A$ for each A$\in \tau_{PO(X)}$ (so for A$\in \tau, A \in \tau_s, A \in \tau_\theta$). We used here the property $\operatorname{pcl} A = A \cup \overline{A^o}$ for any $A \subset X$ [3].

It can be seen from the above examples that we may obtain many equalities for many different types of closure.

Lemma 2.7. *Let $\varphi_1,\ \varphi_2 \in O(X,\tau)$. If for each U$\in \varphi_1 O(X)$ we have $\varphi_2(U) \in \varphi_1 O(X)$ and $\varphi_2(\varphi_2(U)) \subset \varphi_2(U)$, then for each $U \in \varphi_1 O(X)$, $\varphi_2(U)$ is $\varphi_{1,2}$-open.*

The definition of a base of a supratopology, given in [2], is similar to that of a base of a topology.

Theorem 2.8. *Take $\varphi_1,\ \varphi_2 \in O(X,\tau)$. If $\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$, and if $\varphi_2(U)$ is $\varphi_{1,2}$-open for each $U \in \varphi_1 O(X)$, then the family $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$ is a base for the supratopology $\varphi_{1,2} O(X)$.*

Proof. Firstly, $\mathcal{B} \subset \varphi_{1,2} O(X)$ by hypothesis. Let A be a $\varphi_{1,2}$-open set. Then
$$\begin{aligned} x \in A &\Rightarrow x \in \varphi_{1,2} \operatorname{int} A\\ &\Rightarrow \text{there exists a } U_x \in \varphi_1 O(X,x) \text{ such that } \varphi_2(U_x) \subset A\\ &\Rightarrow x \in \varphi_2(U_x) \subset A.\end{aligned}$$
Hence, $A = \bigcup\{\varphi_2(U_x) : x \in A\}$. ∎

Example 2.9. Take $\varphi_1,\ \varphi_2 \in O(X,\tau)$.
1. If $\varphi_1 = \operatorname{int}$, $\varphi_2 = \operatorname{int} \circ \operatorname{cl}$, then A is $\varphi_{1,2}$-open $\Leftrightarrow A \in \tau_s$.
For $U \in \varphi_1 O(X) = \tau$, $\varphi_2(U) = \overline{U}^o \in \tau_s$.
$\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\} = \{\overline{U}^o : U \in \tau\} = RO(X)$.
This gives us the known result that $RO(X)$ is a base for τ_s.

2. If $\varphi_1 = \mathrm{cl} \circ \mathrm{int}$, $\varphi_2 = \mathrm{scl}$, then $\varphi_1 O(X) = SO(X)$. For $U \in SO(X)$, $\varphi_2(U) = \mathrm{scl}\, U \in SO(X)$, $\varphi_2 \geq \imath$.

   Thus A is $\varphi_{1,2}$-open iff A is semi-Θ-open.

   Hence, $\mathcal{B} = \{\mathrm{scl}\, U : U \in SO(X)\} = SR(X)$ is a base for the supratopology of semi-Θ-open sets.

3. $\varphi_{1,2}\psi_{1,2}$-continuities

We accept throughout that the operations φ_i $(i = 1, 2, 3, 4)$, (ψ_i $(i = 1, 2, 3, 4)$) are defined on a topological space $(X, \tau)((Y, \vartheta)$, resp.), and that f is a function from (X, τ) to (Y, ϑ).

The following definition, Theorem 3.2 and Theorem 3.3 are given for fuzzy topological spaces in [6].

Definition 3.1. A function $f\colon (X, \tau) \to (Y, \vartheta)$ is called *$\varphi_{1,2}\psi_{1,2}$-continuous* if for each $x \in X$ and for each V$\in \psi_1 O(Y, f(x))$, there exists a $U \in \varphi_1 O(X, x)$ such that $f(\varphi_2(U)) \subset \psi_2(V)$.

Theorem 3.2.*For a function $f\colon (X, \tau) \to (Y, \vartheta)$ the following are equivalent:*

(a) *f is $\varphi_{1,2}\psi_{1,2}$ continuous.*

(b) *For each $V \in \psi_1 O(Y)$ we have $f^{-1}(V) \subset \varphi_{1,2}\, \mathrm{int}\, f^{-1}(\psi_2(V))$.*

(c) *For each $x \in X$ and for each $V \in \psi_1 O(Y, f(x))$, $x \in \varphi_{1,2}\, \mathrm{int}\, f^{-1}(\psi_2(V))$.*

(d) *For each $A \in P(X)$ we have $f(\varphi_{1,2}\, \mathrm{cl}\, A) \subset \psi_{1,2}\, \mathrm{cl}\, f(A)$.*

(e) *For each $B \in P(Y)$ we have $\varphi_{1,2}\, \mathrm{cl}\, f^{-1}(B) \subset f^{-1}(\psi_{1,2}\, \mathrm{cl}(B))$.*

(f) *For each $B \in P(Y)$ we have $f^{-1}(\psi_{1,2}\, \mathrm{int}\, B) \subset \varphi_{1,2}\, \mathrm{int}\, f^{-1}(B)$.*

Theorem 3.3.*Let $f\colon (X, \tau) \to (Y, \vartheta)$. The following are valid:*

(a) *If f is $\varphi_{1,2}\psi_{1,2}$-continuous then the inverse image of each $\psi_{1,2}$-open set is $\varphi_{1,2}$-open.*

(b) *The inverse image of each $\psi_{1,2}$-open set is $\varphi_{1,2}$-open iff the inverse image of each $\psi_{1,2}$-closed set is $\varphi_{1,2}$-closed.*

(c) *If $\varphi_1 \leq \varphi_3$, $\varphi_2 \geq \varphi_4$, $\psi_1 \geq \psi_3$, $\psi_2 \leq \psi_4$ and f is $\varphi_{1,2}\psi_{1,2}$-continuous then f is $\varphi_{3,4}\psi_{3,4}$-continuous.*

(d) *If for each $x \in X$ and for each $V \in \psi_1 O(Y, f(x))$, there exists a $\varphi_{1,2}$-open set U containing x such that $f(U) \subset \psi_2(V)$, then f is $\varphi_{1,2}\psi_{1,2}$-continuous.*

Theorem 3.4. *If $\widetilde{\psi_2}$ is the dual operator of ψ_2, then $f\colon (X, \tau) \to (Y, \vartheta)$ is $\varphi_{1,2}\psi_{1,2}$-continuous iff $\varphi_{1,2}\, \mathrm{cl}(f^{-1}(\widetilde{\psi_2}(K))) \subset f^{-1}(K)$ for each ψ_1-closed subset K of Y.*

Theorem 3.5. *If ψ_1 is the dual operator of ψ_2, then $f\colon (X, \tau) \to (Y, \vartheta)$ is $\varphi_{1,2}\psi_{1,2}$-continuous iff $\varphi_{1,2}\, \mathrm{cl}\, f^{-1}(\psi_1(K)) \subset f^{-1}(K)$ for each ψ_1-closed set K of Y iff for each ψ_1-open set U, $f^{-1}(U) \subset \varphi_{1,2}\, \mathrm{int}\, f^{-1}(\psi_2(U))$.*

THEOREM 3.6. *If $\psi_2 \geq \psi_1$ or $\psi_2 \geq \imath$, and if $\psi_2(V)$ is $\psi_{1,2}$-open for each $V \in \psi_1 O(Y)$, then the following are equivalent for a function $f\colon (X,\tau) \to (Y,\vartheta)$.*

(a) *f is $\varphi_{1,2}\psi_{1,2}$-continuous.*

(b) *The inverse image of each $\psi_{1,2}$-open set is $\varphi_{1,2}$-open.*

(c) *$f^{-1}(\psi_2(V))$ is $\varphi_{1,2}$-open for each $V \in \psi_1 O(Y)$.*

Proof. $(a) \Rightarrow (b)$. Known from Theorem 3.3.

$(b) \Rightarrow (c)$. Obvious from the hypothesis.

$(c) \Rightarrow (a)$. Let $x \in X$ and $V \in \psi_1 O(Y, f(x))$. Since $\psi_1 \leq \psi_2$ or $\imath \leq \psi_2$ we have $f(x) \in V \subset \psi_2(V)$ so $x \in f^{-1}(\psi_2(V))$ and $f^{-1}(\psi_2(V))$ is $\varphi_{1,2}$-open. There exists an $U \in \varphi_1 O(X,x)$ such that $\varphi_2(U) \subset f^{-1}(\psi_2(V))$. Hence $f(\varphi_2(V)) \subset \psi_2(V)$, so f is $\varphi_{1,2}\psi_{1,2}$-continuous. ■

THEOREM 3.7.*If $\psi_2 = \imath$, then a function $f\colon (X,\tau) \to (Y,\vartheta)$ is $\varphi_{1,2}\psi_{1,2}$-continuous iff $f^{-1}(V)$ is $\varphi_{1,2}$-open for each $V \in \psi_1 O(Y)$.*

THEOREM 3.8.*If ψ_1 is the dual of ψ_2 and $\psi_1(B) \in \psi_1 O(Y)$, $\psi_{1,2}\operatorname{cl}(\psi_1(B)) \subset \psi_2(B)$ for each $B \subset Y$, then the following are equivalent.*

(a) *f is $\varphi_{1,2}\psi_{1,2}$-continuous.*

(b) *$\varphi_{1,2}\operatorname{cl}(f^{-1}(V)) \subset f^{-1}(\psi_{1,2}\operatorname{cl}(V))$ for each $V \in \psi_1 O(Y)$.*

(c) *$f^{-1}(\psi_{1,2}\operatorname{int}(K)) \subset \varphi_{1,2}\operatorname{int} f^{-1}(K)$ for each ψ_1-closed set K.*

THEOREM 3.9. *If φ_2 and ψ_2 are monotonous then the following are equivalent for a function $f\colon (X,\tau) \to (Y,\vartheta)$.*

(a) *f is $\varphi_{1,2}\psi_{1,2}$-continuous.*

(b) *For each $x \in X$ and for each $N \in \mathcal{N}(\psi_1 O(Y), f(x))$, there exists an $M \in \mathcal{N}(\varphi_1 O(X), x)$ such that $f(\varphi_2(M)) \subset \psi_2(N)$.*

(c) *For each $x \in X$ and for each $N \in \mathcal{N}(\psi_1 O(Y), f(x))$, $x \in \varphi_{1,2}\operatorname{int} f^{-1}(\psi_2(N))$.*

THEOREM 3.10 *If ψ_2 is monotonous and $\varphi_2 = \imath$, then any function $f\colon (X,\tau) \to (Y,\vartheta)$ is $\varphi_{1,2}\psi_{1,2}$-continuous iff for each $x \in X$ and for each $N \in \mathcal{N}(\psi_1 O(Y), f(x))$ we have $f^{-1}(\psi_2(N)) \in \mathcal{N}(\varphi_1 O(X), x)$.*

REMARK 3.11. Clearly if a family $\mathcal{B}$ is a base for the supratopology $\psi_{1,2}O(Y)$, then $f^{-1}(V) \in \varphi_{1,2}O(X)$ for each $V \in \psi_{1,2}O(Y)$ iff $f^{-1}(V) \in \varphi_{1,2}O(X)$ for each $V \in \mathcal{B}$ iff $f^{-1}(Y \setminus V)$ is $\varphi_{1,2}$-closed for each $V \in \mathcal{B}$.

THEOREM 3.12. *If $\varphi_2 \geq \varphi_1$ or $\varphi_2 \geq \imath$, and if $\varphi_2(U)$ is $\varphi_{1,2}$-open for each $U \in \varphi_1 O(X)$ then (noting that $\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\}$ is a base of the supratopology $\varphi_{1,2}O(X)$ by Theorem 2.8.), the following are equivalent.*

(a) *f is $\varphi_{1,2}\psi_{1,2}$-continuous.*

(b) *For each $x \in X$ and for each $V \in \psi_1 O(Y, f(x))$, there exists a $B \in \mathcal{B}$ containing x such that $f(B) \subset \psi_2(V)$.*

(c) *For each $x \in X$ and for each $V \in \psi_1 O(Y, f(x))$, there exists a $\varphi_{1,2}$-open set W containing x such that $f(W) \subset \psi_2(V)$.*

Proof. $(b) \Rightarrow (c) \Rightarrow (a)$. Clear from the hypotheses and Theorem 3.3(d).

$(a) \Rightarrow (b)$. Let $x \in X$ and $V \in \psi_1 O(Y, f(x))$. Then there exists a $U \in \varphi_1 O(X, x)$ such that $f(\varphi_2(U)) \subset \psi_2(V)$ and $x \in \varphi_2(U)$. Since $\varphi_2(U) \in \mathcal{B}$, we obtain the desired result. ■

EXAMPLE 3.13. Let $\varphi_1 = \mathrm{cl} \circ \mathrm{int}$, $\varphi_2 = \mathrm{scl}$ on (X, τ) and $\psi_1 = \mathrm{int}$, $\psi_2 = \imath$ on (Y, ϑ). Then:

$f\colon (X, \tau) \to (Y, \vartheta)$ is $\varphi_{1,2}\psi_{1,2}$-continuous iff f is strongly Θ-semi-continuous (strong Θ-semi-continuity was defined in [4], and called s-Θ-continuity in [13]).

$\varphi_2 \geq \imath$, $\varphi_2(U) = \mathrm{scl}\, U \in \varphi_1 O(X) = SO(X)$ and $\varphi_2(\varphi_2(U)) = \mathrm{scl}(\mathrm{scl}\, U) = \mathrm{scl}\, U = \varphi_2(U)$ for each $U \in \varphi_1 O(X)$.

$$\mathcal{B} = \{\varphi_2(U) : U \in \varphi_1 O(X)\} = \{sclU : U \in SO(X)\}$$
$$= SR(X) = \text{the family of sets which are semi-open and semi-closed.}$$

$\psi_2 \geq \imath$, $\psi_{1,2} O(Y) = \vartheta$, $\psi_2(V)$ is $\psi_{1,2}$-open for each $V \in \psi_1 O(Y)$, $\widetilde{\psi_2} = \imath$ is the dual of ψ_2, $\psi_{1,2}\,\mathrm{cl}\, B = \overline{B}$ for each $B \in P(Y)$ and $\varphi_{1,2}\,\mathrm{cl}\, A = \text{semi-}\Theta\text{-}\mathrm{cl}\, A$ [13].

Now, by using Lemma 2.7, Theorems 2.8, 3.2, 3.3(b), 3.4, 3.6, 3.7, 3.9 and 3.12, we get the following result which generalizes Theorem 2.1 in [4].

THEOREM 3.14. *For any function $f\colon (X, \tau) \to (Y, \vartheta)$, the following are equivalent:*

(a) *f is strongly Θ-semi-continuous.*
(b) *For each $x \in X$ and for each $V \in \vartheta(f(x))$ we have $x \in \text{semi-}\Theta\text{-}\mathrm{int}\, f^{-1}(V)$.*
(c) *For each $B \in P(Y)$ we have $f^{-1}(B^o) \subset \text{semi-}\Theta\text{-}\mathrm{int}\, f^{-1}(B)$.*
(d) *Semi-Θ-$\mathrm{cl}(f^{-1}(K)) \subset f^{-1}(K)$ for each closed subset K of Y.*
(e) *$f^{-1}(V)$ is semi-Θ-open for each open subset V of Y.*
(f) *$f^{-1}(K)$ is semi-Θ-closed for each closed subset K of Y.*
(g) *$f(\text{semi-}\Theta\text{-}\mathrm{cl}\, A) \subset \overline{f(A)}$ for each $A \subset X$.*
(h) *Semi-Θ-$\mathrm{cl}(f^{-1}(B)) \subset f^{-1}(\overline{B})$ for each $B \subset Y$.*
(i) *For each $x \in X$ and for each $V \in \vartheta(f(x))$, there exists a semi-regular set U containing x such that $f(U) \subset V$.*
(j) *For each $x \in X$ and for each $V \in \vartheta(f(x))$, there exists a semi-Θ-open set W such that $f(W) \subset V$.*
(k) *For each $x \in X$ and for each $N \in \mathcal{N}(\vartheta, f(x))$, there exists an $M \in \mathcal{N}(SO(X), x)$ such that $f(\mathrm{scl}\, M) \subset N$.*
(l) *For each $x \in X$ and for each $N \in \mathcal{N}(\vartheta, f(x))$ we have $x \in \text{semi-}\theta\text{-}\mathrm{int}(f^{-1}(N))$.*

Example 3.15. Let $\varphi_1 = \mathrm{int}$, $\varphi_2 = \imath$ on (X, τ) and $\psi_1 = \mathrm{int}$, $\psi_2 = \mathrm{cl}$ on (Y, ϑ). Then:

f is $\varphi_{1,2}\psi_{1,2}$-continuous iff f is weakly continuous (weak-continuity was defined by Levine [12]).

φ_2 and ψ_2 are monotonous, ψ_1 is the dual of ψ_2, $\psi_2 \geq \imath$ and ψ_2 is W.F.I.P. w.r.t. $\psi_1 O(Y) = \vartheta$.

$\psi_{1,2} \operatorname{cl} V = \Theta\text{-}\operatorname{cl} V = \psi_1 O(Y)\text{-}\operatorname{cl} V = \vartheta \operatorname{cl} V = \overline{V} = \psi_2(V)$ for each $V \in \psi_1 O(Y) = \vartheta$.

For each $B \in P(Y)$ we have $\psi_1(B) = B^o \in \psi_1 O(Y) = \vartheta$ and $\psi_{1,2} \operatorname{cl}(\psi_1(B)) = \Theta\text{-}\operatorname{cl} B^o = \overline{B^o} \subset \overline{B} = \psi_2(B)$.

Now, by using Theorems 2.1(9), 3.2, 3.5, 3.8 and 3.9 we obtain the following result.

THEOREM 3.16. *For any function $f\colon (X,\tau) \to (Y,\vartheta)$, the following are equivalent:*

(a) *f is weakly continuous.*
(b) *For each $x \in X$ and for each $V \in \vartheta(f(x))$, there exists an $U \in \tau(x)$ such that $f(U) \subset \Theta\text{-}\operatorname{cl} V$.*
(c) *$f^{-1}(V) \subset (f^{-1}(\overline{V}))^o$ for each $V \in \vartheta$.*
(d) *$f^{-1}(V) \subset (f^{-1}(\Theta\text{-}\operatorname{cl} V))^o$ for each $V \in \vartheta$.*
(e) *For each $x \in X$ and for each $V \in \vartheta(f(x))$, we have $x \in (f^{-1}(\overline{V}))^o$.*
(f) *For each $x \in X$ and for each $V \in \vartheta(f(x))$, we have $x \in (f^{-1}(\Theta\text{-}\operatorname{cl} V))^o$.*
(g) *$f(\overline{A}) \subset \Theta\text{-}\operatorname{cl}(f(A))$ for each $A \subset X$.*
(h) *$\overline{f^{-1}(B)} \subset f^{-1}(\Theta\text{-}\operatorname{cl} B)$ for each $B \subset Y$.*
(i) *$f^{-1}(\Theta\text{-}\operatorname{int} B) \subset (f^{-1}(B))^o$ for each $B \subset Y$.*
(j) *$\overline{f^{-1}(K^o)} \subset f^{-1}(K)$ for each closed subset K of Y.*
(k) *$\overline{f^{-1}(V)} \subset f^{-1}(\overline{V})$ for each $V \in \vartheta$.*
(l) *$\overline{f^{-1}(V)} \subset f^{-1}(\Theta\text{-}\operatorname{cl} V)$ for each $V \in \vartheta$.*
(m) *For each $x \in X$ and for each $N \in \mathcal{N}(\vartheta, f(x))$, there exists an $M \in \mathcal{N}(\tau, x)$ s.t. $f(M) \subset \overline{N}$.*
(n) *For each $x \in X$ and for each $N \in \mathcal{N}(\vartheta, f(x))$ we have $x \in (f^{-1}(\overline{N}))^o$.*

If we use the properties:

$\Theta\text{-}\operatorname{cl} V = \overline{V} = \vartheta_s \operatorname{cl} V = \vartheta^\alpha \operatorname{cl} V$ for each $V \in \vartheta$ (Examples 2.2 and 2.6),

$\Theta\text{-}\operatorname{int} K = K^o = \vartheta_s \operatorname{int} K = \vartheta^\alpha \operatorname{int} K$ for each closed subset K of Y,

then we obtain some other characterizations of weak continuity.

THEOREM 3.17. *The following are equivalent for any function $f\colon (X,\tau) \to (Y,\vartheta)$:*

(a) *f is weakly continuous.*
(b) *For each $x \in X$ and for each $V \in \vartheta(f(x))$, there exists a $U \in \tau(x)$ such that $f(U) \subset \vartheta_s \operatorname{cl} V$.*
(c) *$f^{-1}(V) \subset (f^{-1}(\vartheta_s clV))^o$ for each $V \in \vartheta$.*

(d) *For each* $x \in X$ *and for each* $V \in \vartheta(f(x))$*, we have* $x \in (f^{-1}(\vartheta_s \operatorname{cl} V))^o$.

(e) $\overline{f^{-1}(\Theta\text{-}\operatorname{int} K)} \subset f^{-1}(K)$ *for each closed subset* K *of* Y.

(f) $\overline{f^{-1}(\vartheta_s \operatorname{int} K)} \subset f^{-1}(K)$ *for each closed subset* K *of* Y.

(g) $\overline{f^{-1}(V)} \subset f^{-1}(\vartheta_s \operatorname{cl} V)$ *for each* $v \in \vartheta$.

(h) *For each* $x \in X$ *and for each* $V \in \vartheta(f(x))$*, there exists a* $U \in \tau(x)$ *such that* $f(U) \subset \vartheta^\alpha \operatorname{cl} V$.

(i) $f^{-1}(V) \subset (f^{-1}(\vartheta^\alpha clV))^o$ *for each* $V \in \vartheta$.

(j) *For each* $x \in X$ *and for each* $V \in \vartheta(f(x))$*, we have* $x \in (f^{-1}(\vartheta^\alpha \operatorname{cl} V))^o$.

(k) $\overline{f^{-1}(\vartheta^\alpha \operatorname{int} K))} \subset f^{-1}(K)$ *for each closed subset* K *of* Y.

Example 3.18. Let $\varphi_1 = \operatorname{int}$, $\varphi_2 = \imath$ on (X,τ) and $\psi_1 = \operatorname{int}$, $\psi_2 = \operatorname{int} \circ \operatorname{cl}$ on (Y,ϑ). Then:

$f\colon (X,\tau) \to (Y,\vartheta)$ is $\varphi_{1,2}\psi_{1,2}$-continuous iff f is almost-continuous (almost-continuity was defined in [16]);

$\varphi_{1,2} \operatorname{int} A = \tau\text{-}\operatorname{int} A = A^o$, $\varphi_{1,2} \operatorname{cl} A = \overline{A}$ for any $A \subset X$;

$\widetilde{\psi_2} = \operatorname{cl} \circ \operatorname{int}$ is the dual operation of ψ_2, φ_2, ψ_2 are monotonous and $\psi_2 \geq \psi_1$;

ψ_2 is W.F.I.P. w.r.t. $\psi_1 O(Y) = \vartheta$;

$\psi_2(V) = \overline{V}^o$ is $\psi_{1,2}$-open for each $V \in \psi_1 O(Y)$, $\mathcal{B} = \{\psi_2(V) : V \in \vartheta\} = RO(Y)$ and $RO(Y)$ is a base for the supratopology $\psi_{1,2}O(Y) = \vartheta_s$;

$\psi_{1,2} \operatorname{int} B = \delta\text{-}\operatorname{int} B = \vartheta_s\text{-}\operatorname{int} B$ and $\psi_{1,2} \operatorname{cl} V = \delta\text{-}\operatorname{cl} V = \vartheta_s \operatorname{cl} V$ for any $B \subset Y$.

For $V \in \vartheta$, $\psi_{1,2} \operatorname{cl} V = \delta\text{-}\operatorname{cl} V = \vartheta_s \operatorname{cl} V = \psi_1 O(Y)\text{-}\operatorname{cl} V = \vartheta \operatorname{cl} V = \overline{V}$.

Now, by using Theorems 2.1, 2.8, 3.2, 3.3(b), 3.4, 3.6, 3.9, 3.10, and Remark 3.11, we get the following result.

Theorem 3.19. *The following are equivalent for any function* $f\colon (X,\tau) \to (Y,\vartheta)$*:*

(a) f *is almost continuous.*

(b) $f^{-1}(V) \subset (f^{-1}(\overline{V}^o))^o$ *for each* $V \in \vartheta$.

(c) *For each* $x \in X$ *and each* $V \in \vartheta$ *containing* $f(x)$ *we have* $x \in (f^{-1}(\overline{V}^o))^o$.

(d) $f(\overline{A}) \subset \vartheta_s\text{-}\operatorname{cl} f(A)$ *for each* $A \in P(X)$.

(e) $\overline{f^{-1}(B)} \subset f^{-1}(\vartheta_s\text{-}\operatorname{cl} B)$ *for each* $B \in P(Y)$.

(f) $f^{-1}(\vartheta_s\text{-}\operatorname{int} B) \subset (f^{-1}(B))^o$ *for each* $B \in P(Y)$.

(g) $\overline{f^{-1}(\overline{K^o})} \subset f^{-1}(K)$ *for each closed subset* K *of* Y.

(h) $f^{-1}(V)$ *is open for each* $V \in \vartheta_s$.

(i) $f^{-1}(\overline{V}^o)$ *is open for each* $V \in \vartheta$.

(j) $f^{-1}(V)$ *is open for each* $V \in RO(Y)$.

(k) $f^{-1}(K)$ *is closed for each regular-closed subset* K *of* Y.

(l) *$f^{-1}(K)$ is closed for each ϑ_s-closed subset K of Y.*

(m) *For each $x \in X$ and for each $N \in \mathcal{N}(\vartheta, f(x))$, there exists an $M \in \mathcal{N}(\tau, x)$ s.t. $f(M) \subset \overline{N}^o$.*

(n) *For each $x \in X$ and for each $N \in \mathcal{N}(\vartheta, f(x))$ we have $x \in (f^{-1}(\overline{N}^o))^o$.*

For $V \in \vartheta$, $\overline{V}^o = V \cup \overline{V}^o = \operatorname{scl} V$ 3. So, for $V \in \vartheta$, $\overline{V}^o = \operatorname{scl} V = \Theta\text{-semi-}\operatorname{cl} V = \text{semi-}\Theta\text{-}\operatorname{cl} V$ (from Examples 2.2 and 2.6).

For any closed subset K of Y, $\overline{K^o} = \text{semi-}\operatorname{int} K = \Theta\text{-semi-}\operatorname{int} K = \text{semi-}\Theta\text{-}\operatorname{int} K$.

Now, by using the above equalities, we obtain some other characterizations of almost continuity.

THEOREM 3.20. *The following are equivalent for any function $f\colon (X,\tau) \to (Y,\vartheta)$:*

(a) *f is almost continuous.*

(b) *For each $x \in X$ and for each $V \in \vartheta(f(x))$, there exists a $U \in \tau(x)$ such that $f(U) \subset \operatorname{scl} V$.*

(c) *For each $x \in X$ and for each $V \in \vartheta(f(x))$, there exists a $U \in \tau(x)$ such that $f(U) \subset \Theta\text{-semi-}\operatorname{cl} V$.*

(d) *For each $x \in X$ and for each $V \in \vartheta(f(x))$, there exists a $U \in \tau(x)$ such that $f(U) \subset \text{semi-}\Theta\text{-}\operatorname{cl} V$.*

(e) *$f^{-1}(V) \subset (f^{-1}(\operatorname{scl} V))^o$ for each $V \in \vartheta$.*

(f) *$f^{-1}(V) \subset (f^{-1}(\Theta\text{-semi-}\operatorname{cl} V))^o$ for each $V \in \vartheta$.*

(g) *$f^{-1}(V) \subset (f^{-1}(\text{semi-}\Theta\text{-}\operatorname{cl} V))^o$ for each $V \in \vartheta$.*

(h) *For each $x \in X$ and each $V \in \vartheta$ containing $f(x)$ we have $x \in (f^{-1}(\operatorname{scl} V))^o$.*

(i) *For each $x \in X$ and each $V \in \vartheta$ containing $f(x)$ we have $x \in (f^{-1}(\Theta\text{-semi-}\operatorname{cl} V))^o$.*

(j) *For each $x \in X$ and each $V \in \vartheta$ containing $f(x)$ we have $x \in (f^{-1}(\text{semi-}\Theta\text{-}\operatorname{cl} V))^o$.*

(k) *$\overline{f^{-1}(\text{semi-int}K)} \subset f^{-1}(K)$ for each closed subset K of Y.*

(l) *$\overline{f^{-1}(\Theta\text{-semi-int}K)} \subset f^{-1}(K)$ for each closed subset K of Y.*

(m) *$\overline{f^{-1}(\text{semi-}\Theta\text{-}\operatorname{int} K)} \subset f^{-1}(K)$ for each closed subset K of Y.*

(n) *$f^{-1}(\operatorname{scl} V)$ is open for each $V \in \vartheta$.*

(p) *$f^{-1}(\Theta\text{-semi-}\operatorname{cl} V)$ is open for each $V \in \vartheta$.*

(q) *$f^{-1}(\text{semi-}\Theta\text{-}\operatorname{cl} V)$ is open for each $V \in \vartheta$.*

EXAMPLE 3.21. Let $\varphi_1 = \operatorname{int}$, $\varphi_2 = \imath$, $\varphi_3 = \operatorname{int}$ and $\varphi_4 = \imath$ be operations on (X,τ), and $\psi_1 = \operatorname{int}$, $\psi_2 = \operatorname{int} \circ \operatorname{cl}$, $\psi_3 = \operatorname{int}$ and $\psi_4 = \operatorname{cl}$. Then

$\varphi_1 \le \varphi_3$, $\varphi_2 \ge \varphi_4$, $\psi_1 \ge \psi_3$ and $\psi_2 \le \psi_4$.

By using Theorem 3.3 and Example 3.15 it is seen that almost-continuity implies weak-continuity.

Continuing in this manner, we may obtain the definitions of many types of continuity as well as many equivalent conditions.

REFERENCES

[**1**] M. E. Abd El-Monsef, F. M. Zeyada and A. S. Mashour, *Operations on topologies and their applications on some types of coverings*, Ann. Soc. Sci. Bruxelles, **97**,4 (1983), 155–172.

[**2**] M. E. Abd El-Monsef, E. F. Lashien, *Local discrete extensions of supratopologies*, Tamkang J. Math. **21**, 1 (1990), 1–6.

[**3**] D. Andrijević, *On the topology generated by pre-open sets*, Mat. Vesnik, **39** (1987), 367–376.

[**4**] S. Jafari, T. Noiri, *On strongly Θ-semi-continuous functions*, Indian J. Pure Appl. Math. **29**, 11 (1998), 1195–1201.

[**5**] D. S. Janković, *On functions with α-closed graphs*, Glasnik Matematički, **18**, 38 (1983), 141–148.

[**6**] A. Kandil, E. E. Kerre, A. A. Nouh, *Operations and mappings on fuzzy topological spaces*, Ann. Soc. Sci. Bruxelles **105**, 4 (1991), 165–168.

[**7**] S. Kasahara, *Operation-compact spaces*, Math. Japonica **24**, 1 (1979), 97–105.

[**8**] E. E. Kerre, A. A. Nouh, A. Kandil, *Operations on the class of fuzzy sets on a universe endowed with a fuzzy topology*, J. Math. Anal. Appl. **180** (1993), 325–341.

[**9**] J. K. Kohli, *A class of mappings containing all continuous and all semiconnected mappings*, Proc. Amer. Math. Soc. **72**, 1 (1978), 175–181.

[**10**] J. K. Kohli, *A framework including the theories of continuties and certain non-continuous functions*, Note di Mat. **10**, 1 (1990), 37–45.

[**11**] J. K. Kohli, *Change of topology, characterizations and product theorems for semi-locally P-spaces*, Houston J. Math. **17**, 3 (1991), 335–349.

[**12**] N. Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly **68** (1961), 44–46.

[**13**] M. N. Mukherjee, C. K. Basu, *On semi-Θ-closed sets, semi-Θ-connectedness and some associated mappings*, Bull. Cal. Mat. Soc. **83** (1991), 227–238.

[**14**] M. N. Mukherjee, G. Sengupta, *On π-closedness: A unified theory*, Anal. Sti. Univ. "A⊂. I. Cuza" Iaşi, **39**, s. 1.a, Mat. 3 (1993), 1–10.

[**15**] G. Sengupta, M. N. Mukherjee, *On π-closedness, a unified theory II*, Boll. U. M. I. **7**, 8-B (1994), 999–1013.

[**16**] M. K. Singal, A. R. Singal, *Almost-continuous mappings*, Yokohama Math. J. **16** (1968), 63–73.

[**17**] T. H. Yalvaç, *Relations between new topologies obtained from old ones*, Acta Math. Hungar. **64**, 3 (1994), 231–235.

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