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SOME REMARKS ON THE CATEGORY SET(L), Part II

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Abstract. This paper considers some intrinsic properties of the category SET(L) of L-subsets of sets with a fixed basis L and is a continuation of our previous work [4]. Here we study properties of some abstract functors when applied to the category SET(L) as well as some special objects related to them. In the last section we consider two standard constructions, namely, inverse and direct systems in this category.

1. Introduction

The notion of a fuzzy set introduced in [5] induced many researchers to study different mathematical structures involving fuzzy sets and their generalization L-fuzzy sets [2] or just L-sets for short. In particular, some authors considered the category SET(L) of all L-subsets of all sets with a *fixed* lattice L. The purpose of our work is to study systematically some intrinsic properties of this category. The article is a continuation of our previous investigation of the category SET(L) in [4]. In particular, in [4] we considered some special morphisms and objects (as, e.g., section, retraction, isomorphism) as well as some standard constructions (as, e.g., product and coproduct of objects and morphism, pullback and pushout) in this category. The aim of this paper is to develop further the study of the category SET(L).

The paper starts with an introductory section, i.e., Preliminaries, where we recall the definition of the category SET(L) and discuss some results from [4]. The next section is devoted to some applications of abstract functors to the category SET(L). We consider, for example, the properties of the so-called set-valued hom-functor and evaluation functor. Some properties of the former are also considered in the next section devoted to some special objects in the category SET(L). Among other problems, the section deals with the properties of subobjects and quotient objects. Two last sections of the article are devoted to, accordingly, special morphisms and standard constructions in the category SET(L), where we generalize

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some results of our previous work, i.e., the properties of equalizers and consider some more concepts, i.e., inverse and direct systems.

We use standard terminology accepted in Category theory (see, e.g., [3]).

2. Preliminaries

In this section we will discuss some basic properties of the category $\mathrm{SET}(L)$. Let us start by recalling its definition (see [2]).

Suppose L is a complete lattice $(L, \leqslant)$, i.e., a partially ordered set such that for every subset $A \subset L$ the join $\bigvee A$ and the meet $\bigwedge A$ are defined. In particular, $\bigvee L =: 1_L$ and $\bigwedge L =: 0_L$. We assume that $0_L \neq 1_L$, i.e., L has at least two elements. Then the category $\mathrm{SET}(L)$ can be defined as follows.

The objects of $\mathrm{SET}(L)$ are all L-subsets of sets, i.e., mappings $X\colon \tilde{X} \to L$ where $\tilde{X}$ is an arbitrary set (maybe empty). Henceforth the objects of $\mathrm{SET}(L)$ will be denoted by X, Y or Z and arbitrary sets by $\tilde{X}$, $\tilde{Y}$ or $\tilde{Z}$. By saying that an object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ is given we will always mean that X is a mapping $X\colon \tilde{X} \to L$.

Given two objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$, the set of morphisms from X to Y $\mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$ consists of all mappings $f\colon \tilde{X} \to \tilde{Y}$ such that $X(x) \leqslant Y \circ f(x)$ for all $x \in \tilde{X}$. Given an object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$, we denote its identity morphism by e_X.

Now we will list some properties of the category $\mathrm{SET}(L)$ which we will need throughout the article and whose proofs can be found in [4]. All of them are related to special morphisms and objects in the category $\mathrm{SET}(L)$; also notice that we use “iff” for “if and only if”.

A morphism $f\colon X \to Y$ is

(1) a monomorphism iff f is injective;

(2) an epimorphism iff f is surjective;

(3) a bimorphism iff f is bijective;

(4) an isomorphism iff f is bijective and $X(x) = Y \circ f(x)$ for all $x \in \tilde{X}$;

(5) a constant morphism iff $f(\tilde{X}) = \{y_0\} \subset \tilde{Y}$;

(6) an equalizer iff f is injective and $X(x) = Y \circ f(x)$ for all $x \in \tilde{X}$.

An object X is an initial object iff $\tilde{X} = \varnothing$.

3. Special functors

In this section we will consider the properties of some abstract functors in case of one distinct category, i.e., the category $\mathrm{SET}(L)$.

Suppose we have a bifunctor $\mathtt{hom}_{\mathrm{SET}(L)}\colon \mathrm{SET}(L)^{op} \times \mathrm{SET}(L) \to \mathrm{SET}$ (notice that $\mathrm{SET}(L)^{op}$ denotes the dual category of the category $\mathrm{SET}(L)$, “$\times$” denotes the product of two categories and SET stands for the category of sets) where $\mathtt{hom}_{\mathrm{SET}(L)}(X, Y) = \mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$ and $\mathtt{hom}_{\mathrm{SET}(L)}(f, g)(h) = g \circ h \circ f$.

$$\begin{tikzcd} \mathtt{hom}_{\mathrm{SET}(L)}(X,Y) \arrow[d, no head, "\mathtt{hom}_{\mathrm{SET}(L)}(f,g)"'] \\ \mathtt{hom}_{\mathrm{SET}(L)}(X',Y') \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "h"] & & Y \arrow[d, no head, "g"] \\ X' \arrow[u, "f"] \arrow[rr, dashed, "g\circ h\circ f=", "\mathtt{hom}_{\mathrm{SET}(L)}(f,g)(h)"'] & & Y' \end{tikzcd}$$

The functor is called the set-valued hom-functor or morphism functor. For shortness sake, later on, we will denote the category $\mathrm{SET}(L)^{op} \times \mathrm{SET}(L)$ by $\mathcal{C}$.

THEOREM 3.1. *The morphism functor* $\mathtt{hom}_{\mathrm{SET}(L)}$ *is dense, not full, not faithful and not an embedding.*

Proof. Let us prove that $\mathtt{hom}_{\mathrm{SET}(L)}$ is dense, that is, for each object $\tilde{Z} \in \mathtt{Obj}\,\mathrm{SET}$ there exist such two objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ that $\mathtt{hom}_{\mathrm{SET}(L)}(X,Y)$ is isomorphic to $\tilde{Z}$. Let $\tilde{X} = \{x_0\}$, $X(x_0) = 0_L$ and $\tilde{Y} = \tilde{Z}$, $Y(y) = 1_L$ for all $y \in \tilde{Y}$. Then obviously $\mathtt{hom}_{\mathrm{SET}(L)}(X,Y) = \mathtt{Mor}_{\mathrm{SET}(L)}(X,\, Y)$ is isomorphic to $\tilde{Z}$.

Now we will verify that $\mathtt{hom}_{\mathrm{SET}(L)}$ is not full. For the proof of this property it will be enough to find such two objects (X,Y), $(X',Y') \in \mathtt{Obj}\,\mathcal{C}$ that $\mathtt{Mor}_{\mathcal{C}}((X,Y),(X',Y')) = \varnothing$ and $\mathtt{Mor}_{\mathcal{C}}(\mathtt{hom}_{\mathrm{SET}(L)}(X,Y), \mathtt{hom}_{\mathrm{SET}(L)}(X',Y')) \neq \varnothing$. It is easy to see that such two objects can be each four mappings $X' \equiv X \equiv Y' \equiv 0_L$ and $Y \equiv 1_L$ with arbitrary non-empty sets $\tilde{X}$, $\tilde{Y}$, $\tilde{X}'$, $\tilde{Y}'$. Indeed, there are no morphisms from (X,Y) to (X',Y') in the category $\mathcal{C}$, but there exists at least one morphism from $\mathtt{hom}_{\mathrm{SET}(L)}(X,Y)$ to $\mathtt{hom}_{\mathrm{SET}(L)}(X',Y')$ in the category SET.

Now let us prove that $\mathtt{hom}_{\mathrm{SET}(L)}$ is not faithful. We have to find such two objects (X,Y), $(X',Y') \in \mathtt{Obj}\,\mathcal{C}$ and such two morphisms (f,g), $(f',g')\colon (X,Y) \to (X',Y')$ that $(f,g) \neq (f',g')$ but $\mathtt{hom}_{\mathrm{SET}(L)}(f,g) = \mathtt{hom}_{\mathrm{SET}(L)}(f',g')$. Let $\tilde{X}' = \tilde{X} = \varnothing$ and $|\tilde{Y}| = |\tilde{Y}'| > 1$, $Y' \equiv 1_L$. Then $\mathtt{hom}_{\mathrm{SET}(L)}(X,Y) = \{x_0\}$, $\mathtt{hom}_{\mathrm{SET}(L)}(X',Y') = \{y_0\}$ and there exists a unique morphism $f_0 \in \mathtt{Mor}_{\mathrm{SET}}(\{x_0\}, \{y_0\})$. Obviously, one can find at least two distinct morphisms $(f,g_1),(f,g_2) \in \mathtt{Mor}_{\mathcal{C}}((X,Y),(X',Y'))$ such that both $\mathtt{hom}_{\mathrm{SET}(L)}(f,g_1)$ and $\mathtt{hom}_{\mathrm{SET}(L)}(f,g_2)$ are equal to f_0.

The last property follows immediately from the fact that $\mathtt{hom}_{\mathrm{SET}(L)}$ is not faithful. ■

Now let us try to make an L-set of $\mathtt{hom}_{\mathrm{SET}(L)}(X,Y)$ and thus to get a bifunctor $\mathtt{Hom}_{\mathrm{SET}(L)}\colon \mathcal{C} \to \mathrm{SET}(L)$.

The first way is to assume that $\mathtt{Hom}_{\mathrm{SET}(L)}(X,Y) = X^*$ where X^* is a mapping from $\mathtt{Mor}_{\mathrm{SET}(L)(X,Y)}$ to L defined in the following way:

$$X^*(h) = \begin{cases} \bigvee Y(\tilde{Y}), & \tilde{Y} \neq \varnothing \\ 0_L, & \tilde{Y} = \varnothing. \end{cases}$$

Suppose we have an object $(X^{'}, Y^{'}) \in \mathtt{Obj}\,\mathcal{C}$ and a morphism $(f,g)\colon (X,Y) \to (X^{'}, Y^{'})$. Then $\mathtt{Hom}_{\mathrm{SET}(L)}(X^{'}, Y^{'}) = Y^*$ and $\mathtt{Hom}_{\mathrm{SET}(L)}(f,g)(h) = g \circ h \circ f = \overline{h}$. We need to prove that $\mathtt{Hom}_{\mathrm{SET}(L)}(f,g) \in \mathtt{Mor}_{\mathrm{SET}(L)(X^*,Y^*)}$, that is, $Y^*(\overline{h}) \geqslant X^*(h)$. Indeed, from the existence of the morphism $g \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, Y^{'})$ we have that $\bigvee Y^{'}(\tilde{Y^{'}}) \geqslant Y^{'} \circ g(y_0) \geqslant Y(y_0)$ for all $y_0 \in \tilde{Y}$. Then $\bigvee Y^{'}(\tilde{Y^{'}}) \geqslant \bigvee Y(\tilde{Y})$ and therefore $Y^*(\overline{h}) \geqslant X^*(h)$.

The second way assumes that the lattice L is endowed with an operation ${}^c\colon L \to L$ (called involution) with the following properties:

(1) for all $a, b \in L$ if $a \leqslant b$ then $b^c \leqslant a^c$;

(2) $(a^c)^c = a$ for all $a \in L$.

From these properties it can easily be derived that $(1_L)^c = 0_L$ and $(0_L)^c = 1_L$. Then $\mathtt{Hom}_{\mathrm{SET}(L)}(X,Y)$ can be defined as $X^*\colon \mathtt{Mor}_{\mathrm{SET}(L)}(X, Y) \to L$ where $X^*(h) = (\bigvee X(\tilde{X}))^c \vee (\bigvee Y(\tilde{Y}))$ with 0_L for $\bigvee \varnothing$. Suppose again we have an object $(X^{'}, Y^{'}) \in \mathtt{Obj}\,\mathcal{C}$ and a morphism $(f,g)\colon (X,Y) \to (X^{'}, Y^{'})$. Then $\mathtt{Hom}_{\mathrm{SET}(L)}(X^{'}, Y^{'}) = Y^*$, $Y^*(\tilde{h}) = (\bigvee X^{'}(\tilde{X^{'}}))^c \vee (\bigvee Y^{'}(\tilde{Y^{'}}))$ and then $\mathtt{Hom}_{\mathrm{SET}(L)}(f,g)(h) = g \circ h \circ f = \overline{h}$. We have to prove that $\mathtt{Hom}_{\mathrm{SET}(L)}(f,g) \in \mathtt{Mor}_{\mathrm{SET}(L)(X^*,Y^*)}$, that is, $Y^*(\overline{h}) \geqslant X^*(h)$.

(1) If $\tilde{X^{'}} = \varnothing$ then $Y^*(\overline{h}) = 1_L \geqslant X^*(h)$.

(2) If $\tilde{X^{'}} \neq \varnothing$ then $\tilde{X} \neq \varnothing$ and from the existence of the morphism $f\colon X^{'} \to X$ we get that $\bigvee X(\tilde{X}) \geqslant \bigvee X^{'}(\tilde{X^{'}})$ and therefore $(\bigvee X^{'}(\tilde{X^{'}}))^c \geqslant (\bigvee X(\tilde{X}))^c$.
   (a) If $\tilde{Y^{'}} = \varnothing$ then $\tilde{Y} = \varnothing$ and then $Y^*(\overline{h}) = (\bigvee X^{'}(\tilde{X^{'}}))^c \geqslant (\bigvee X(\tilde{X}))^c = X^*(h)$.
   (b) If $\tilde{Y^{'}} \neq \varnothing$ then from the existence of the morphism $g\colon Y \to Y^{'}$ we get that $\bigvee Y^{'}(\tilde{Y^{'}}) \geqslant \bigvee Y(\tilde{Y})$ and therefore $Y^*(\overline{h}) \geqslant X^*(h)$.

It is easy to see that for both ways of defining the functor $\mathtt{Hom}_{\mathrm{SET}(L)}$ the following proposition holds. (Notice that for a given object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ there exists the functor $\mathtt{Hom}_{\mathrm{SET}(L)}(\_, X)\colon \mathrm{SET}(L)^{op} \to \mathrm{SET}(L)$ defined by $\mathtt{Hom}_{\mathrm{SET}(L)}(\_, X)(Y) = \mathtt{Hom}_{\mathrm{SET}(L)}(Y, X)$, $\mathtt{Hom}_{\mathrm{SET}(L)}(\_, X)(f) = \mathtt{Hom}_{\mathrm{SET}(L)}(f, e_X)$. By analogy one can define the functor $\mathtt{hom}_{\mathrm{SET}(L)}(\_, X)$.)

Proposition 3.1. *For each $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ the triangle*

$$\begin{tikzcd} \mathrm{SET}(L)^{op} \arrow[r, "\mathtt{Hom}_{\mathrm{SET}(L)}(\_, X)"] \arrow[dr, "\mathtt{hom}_{\mathrm{SET}(L)}(\_, X)"'] & \mathrm{SET}(L) \arrow[d, "U"] \\ & \mathrm{SET} \end{tikzcd}$$

where U is a forgetful functor commutes.

We will consider some more properties of the functor $\mathtt{hom}_{\mathrm{SET}(L)}$ in the next section.

Now we will consider two more special functor types in the category $\mathrm{SET}(L)$. Let us begin with definitions (see [1]). Suppose we have a category $\mathcal{C}'$ and two functors $F_1, F_2 \colon \mathcal{C}' \to \mathcal{C}'$. Then F_1 is said to be a subfunctor of F_2 provided that there exists a natural transformation $\eta \colon F_1 \to F_2$ whose elements $\eta_A \colon F_1(A) \to F_2(A)$ are monomorphisms for all $A \in \mathtt{Obj}\,\mathcal{C}'$. Dually, F_1 is said to be a superfunctor of F_2 provided that there exists a natural transformation $\eta \colon F_1 \to F_2$ whose elements are epimorphisms. By $F_* \colon \mathcal{C}' \to \mathcal{C}'$ we will denote a functor which is a subfunctor of each functor $F \colon \mathcal{C}' \to \mathcal{C}'$ and by $F^* \colon \mathcal{C}' \to \mathcal{C}'$ such a functor that each functor $F \colon \mathcal{C}' \to \mathcal{C}'$ is a superfunctor of F^*.

THEOREM 3.2. *The category* $\mathrm{SET}(L)$ *has a unique functor* F_* *and has no functor* F^*.

Proof. Suppose $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ is an initial object in the category $\mathrm{SET}(L)$. Then the set $\tilde{X}$ is empty. Let $F \colon \mathrm{SET}(L) \to \mathrm{SET}(L)$ be such that $F(Y) = X$ for all $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and $F(f) = e_X$ for all $f \in \mathtt{Mor}\,\mathrm{SET}(L)$. Obviously, F is a functor. Suppose another functor $G \colon \mathrm{SET}(L) \to \mathrm{SET}(L)$ is given. Then for every two objects $Y, Z \in \mathtt{Obj}\,\mathrm{SET}(L)$ and each morphism $f \colon Y \to Z$ the following diagram

$$\begin{tikzcd} F(Y) \arrow[r, "\eta_Y"] \arrow[d, dash, "F(f)"'] & G(Y) \arrow[d, dash, "G(f)"] \\ F(Z) \arrow[r, "\eta_Z"'] & G(Z) \end{tikzcd}$$

where η_Y and η_Z are the unique morphisms from X to, accordingly, $G(Y)$ and $G(Z)$ commutes. Thus, $\eta = \{\eta_Y\}_{Y \in \mathtt{Obj}\,\mathrm{SET}(L)}$ is a natural transformation from F to G. Since $\tilde{X} = \varnothing$, all η_Y are monomorphisms. Therefore, F is a subfunctor of G and $F = F_*$.

Suppose there exists another functor F'_*. From the fact that F'_* is a subfunctor of F_* we derive that there exists a natural transformation $\eta \colon F'_* \to F_*$ and then $\widetilde{F'_*(Y)} = \varnothing$ for all $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$. Thus, $F'_* = F_*$.

Suppose there exists a functor F^*. Then F_* is a superfunctor of F^* and there exists a natural transformation $\eta \colon F_* \to F^*$ whose elements are epimorphisms, therefore, $\widetilde{F^*(Y)} = \varnothing$ for all $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$. Suppose $F(Y) = Z_0 \in \mathtt{Obj}\,\mathrm{SET}(L)$, $\tilde{Z}_0 \neq \varnothing$ for all $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and $F(f) = e_{Z_0}$ for all $f \in \mathtt{Mor}\,\mathrm{SET}(L)$. Then F is a superfunctor of F^* and there exists a natural transformation $\eta \colon F \to F^*$. Choose any two objects $Y_1, Y_2 \in \mathtt{Obj}\,\mathrm{SET}(L)$ and a morphism $f \colon Y_1 \to Y_2$. Then there exist such two morphisms η_{Y_1}, η_{Y_2} that the following diagram

$$\begin{tikzcd} F(Y_1) \arrow[r, "\eta_{Y_1}"] \arrow[d, dash, "F(f)"'] & F^*(Y_1) \arrow[d, dash, "F^*(f)"] \\ F(Y_2) \arrow[r, "\eta_{Y_2}"'] & F^*(Y_2) \end{tikzcd}$$

commutes. Since the sets $\mathtt{Mor}_{\mathrm{SET}(L)}(F(Y_1), F^*(Y_1))$, $\mathtt{Mor}_{\mathrm{SET}(L)}(F(Y_2), F^*(Y_2))$ are empty, we immediately get a contradiction. ∎

The last functor we will consider here is a bifunctor $E\colon \mathrm{SET}(L)^{\mathrm{SET}(L)} \times \mathrm{SET}(L) \to \mathrm{SET}(L)$ (notice that $\mathrm{SET}(L)^{\mathrm{SET}(L)}$ denotes the (quasi)category of all functors $F\colon \mathrm{SET}(L) \to \mathrm{SET}(L)$) defined in the following way: $E(F, X) = F(X)$ for all $F \in \mathtt{Obj}\,\mathrm{SET}(L)^{\mathrm{SET}(L)}$, $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ and $E(\eta, f) = G(f) \circ \eta_X = \eta_Y \circ F(f)$ for all $(\eta, f)\colon (F, X) \to (G, Y)$. The functor is called the evaluation functor for $\mathrm{SET}(L)^{\mathrm{SET}(L)}$. For shortness sake, later on, we will denote the category $\mathrm{SET}(L)^{\mathrm{SET}(L)} \times \mathrm{SET}(L)$ by $\mathcal{C}$.

THEOREM 3.3. *The evaluation functor E is dense, not full, not faithful and not an embedding.*

Proof. For each object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ $E(e_{\mathrm{SET}(L)}, X) = e_{\mathrm{SET}(L)}(X) = X$ where $e_{\mathrm{SET}(L)}$ is the identity functor on $\mathrm{SET}(L)$. Thus, E is dense.

Now let us prove that E is not full. Let $F(Y) = X \in \mathtt{Obj}\,\mathrm{SET}(L)$, $G(Y) = Z \in \mathtt{Obj}\,\mathrm{SET}(L)$ for all $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and $F(f) = e_X$, $G(f) = e_Z$ for all $f \in \mathtt{Mor}\,\mathrm{SET}(L)$ where $\tilde{X} = \{x_0\}$, $X(x_0) = 0_L$ and $\tilde{Z} = \{z_0\}$, $Z(z_0) = 1_L$. Suppose $\tilde{X_1} = \{x_1\}$, $X_1(x_1) = 1_L$ and $\tilde{X_2} = \{x_2\}$, $X_2(x_2) = 0_L$. Then $E(F, X_1) = X$, $E(G, X_2) = Z$ and the set $\mathtt{Mor}_{\mathrm{SET}(L)}(X, Z)$ is not empty. The fact that $\mathtt{Mor}_{\mathcal{C}((F,X_1),(G,X_2))} = \varnothing$ implies that E is not full.

Now we will verify that E is not faithful. Let $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ be such that $\tilde{X} = \varnothing$. Then let $F(Y) = G(Y) = X$ for all $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and $F(f) = G(f) = e_X$ for all $f \in \mathtt{Mor}\,\mathrm{SET}(L)$. Suppose X_1, $X_2 \in \mathtt{Obj}\,\mathrm{SET}(L)$ are such two objects that $|\tilde{X_1}| = |\tilde{X_2}| > 1$ and $X_2(x) = 1_L$ for all $x \in \tilde{X_2}$. Then $\mathtt{Mor}_{\mathrm{SET}(L)}(F(X_1), G(X_2)) = \{f\}$ and there exist at least two distinct morphisms $g_1, g_2\colon X_1 \to X_2$ such that $E(\eta, g_1) = E(\eta, g_2) = f$ where $\eta\colon F \to G$ is the identity natural transformation.

The last property follows immediately from the fact that E is not faithful. ∎

4. Special objects

This section is devoted to some special objects in the category $\mathrm{SET}(L)$. To begin with, we will consider some objects related to the functor $\mathtt{hom}_{\mathrm{SET}(L)}$ discussed in the previous section. In fact, all these objects can be defined through this functor, but here we will use another "internal" characterizations of them. Let us begin by considering separator and coseparator in the category $\mathrm{SET}(L)$.

THEOREM 4.1. *An object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ is a separator iff $\tilde{X} \neq \varnothing$ and $X(x) = 0_L$ for all $x \in \tilde{X}$.*

Proof. Let us prove the necessity first and therefore assume that X is a separator. Then for every two objects Y, $Z \in \mathtt{Obj}\,\mathrm{SET}(L)$, whenever $f, g\colon Y \to Z$ are two distinct morphisms, there exists such a morphism $h\colon X \to Y$ that $f \circ h \neq g \circ h$.

Obviously, $\tilde{X} \neq \varnothing$. Suppose there exists a point $x_0 \in \tilde{X}$ such that $X(x_0) \neq 0_L$. Then let $\tilde{Y} = \{y_1\}$, $Y(y_1) = 0_L$ and $\tilde{Z} = \{z_1, z_2\}$, $z_1 \neq z_2$, $Z(z_1) = Z(z_2) = 1_L$.

In this case every two mappings $f, g\colon \tilde{Y} \to \tilde{Z}$ will be morphisms and thus, we can take $f(y_1) = z_1$, $g(y_1) = z_2$. Obviously, $f \neq g$. Since $\mathtt{Mor}_{\mathrm{SET}(L)}(X, Y) = \varnothing$, we conclude that X is not a separator that contradicts our former assumption.

The sufficiency is obvious. ∎

Given a separator $X \in \mathtt{Obj}\,\mathrm{SET}(L)$, we can get a faithful functor from $\mathrm{SET}(L)$ to SET, i.e., $\mathtt{hom}_{\mathrm{SET}(L)}(X, \_)$.

Theorem 4.2. *An object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ is a coseparator iff there exist such two points $x_1, x_2 \in \tilde{X}$, $x_1 \neq x_2$ that $X(x_1) = X(x_2) = 1_L$.*

Proof. Let us prove the necessity first and therefore assume that X is a coseparator. Then for every two objects $Y, Z \in \mathtt{Obj}\,\mathrm{SET}(L)$, whenever $f, g\colon Y \to Z$ are two distinct morphisms, there exists such a morphism $h\colon Z \to X$ that $h \circ f \neq h \circ g$.

Suppose $\tilde{Y} = \{y_1\}$, $Y(y_1) = 1_L$ and $\tilde{Z} = \{z_1, z_2\}$, $z_1 \neq z_2$, $Z(z_1) = Z(z_2) = 1_L$. In this case every two mappings $f, g\colon \tilde{Y} \to \tilde{Z}$ will be morphisms and thus we can take $f(y_1) = z_1$, $g(y_1) = z_2$. Obviously, $f \neq g$. Then there exists such a morphism $h\colon Z \to X$ that $h \circ f \neq h \circ g$, therefore, $h \circ f(y_1) = x_1 \neq x_2 = h \circ g(y_1)$ and $X(x_1) = X(x_2) = 1_L$.

The sufficiency is obvious. ∎

Now we will consider projective and injective objects in the category $\mathrm{SET}(L)$.

Theorem 4.3. *An object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ is a projective object iff $X(x) = 0_L$ for all $x \in \tilde{X}$.*

Proof. Let us prove the necessity first and therefore assume that X is a projective object. Then for every two objects $Y, Z \in \mathtt{Obj}\,\mathrm{SET}(L)$, each epimorphism $f\colon Y \to Z$ and each morphism $g\colon X \to Z$ there exists a morphism $h\colon X \to Y$ such that the triangle

$$\begin{tikzcd} & X \arrow[dl, "h"'] \arrow[d, dash, "g"] \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

commutes.

Suppose there exists such a point $x_0 \in \tilde{X}$ that $X(x_0) \neq 0_L$. Then let $Z = X$, $g = e_X$ and $\tilde{Y} = \tilde{X}$, $Y(x) = 0_L$ for all $x \in \tilde{X}$, $f = id_{\tilde{X}}$. Obviously, $f \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, Z)$ and is surjective, therefore, f is an epimorphism in $\mathrm{SET}(L)$. Since $\mathtt{Mor}_{\mathrm{SET}(L)}(X, Y) = \varnothing$, we conclude that X is not a projective object that contradicts our former assumption.

Now let us prove the sufficiency. Since f is an epimorphism, f is surjective and then we can define h as follows: $h(x) = y \in f^{-1} \circ g(x)$ for all $x \in \tilde{X}$. Obviously, $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$ and $g = f \circ h$. ∎

The theorem implies the following result.

Theorem 4.4. *If $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ is a separator then X is a projective object.*

Given a separator $X \in \mathtt{Obj}\,\mathrm{SET}(L)$, we can get a functor from $\mathrm{SET}(L)$ to SET which is faithful and preserves epimorphisms, i.e., $\mathtt{hom}_{\mathrm{SET}(L)}(X, \_)$.

THEOREM 4.5. *An object* $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ *is an injective object iff* $\tilde{X} \neq \varnothing$ *and* $X(x) = 1_L$ *for all* $x \in \tilde{X}$.

Proof. Let us prove the necessity first and therefore assume that X is an injective object. Then for every two objects $Y, Z \in \mathtt{Obj}\,\mathrm{SET}(L)$, each monomorphism $f\colon Y \to Z$ and each morphism $g\colon Y \to X$ there exists a morphism $h\colon Z \to X$ such that the triangle

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"'] & Z \arrow[dl, "h"] \\ X & \end{tikzcd}$$

commutes.

Suppose $\tilde{X} = \varnothing$. Then let $\tilde{Y} = \varnothing$ and $\tilde{Z} \neq \varnothing$. Obviously, there exists a monomorphism $f\colon Y \to Z$ and a morphism $g\colon Y \to X$. Since the set $\mathtt{Mor}_{\mathrm{SET}(L)}(Z, X)$ is empty, we conclude that X is not an injective object that contradicts our former assumption.

Now suppose there exists such a point $x_0 \in \tilde{X}$ that $X(x_0) \neq 1_L$. Then let $\tilde{Y} = \{y_0\}$, $Y(y_0) = 0_L$ and $g(y_0) = x_0$. Obviously, $g \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, X)$. Let $\tilde{Z} = \{z_0\}$, $Z(z_0) = 1_L$ and $f(y_0) = z_0$. Obviously, $f \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, Z)$. Since f is injective, f is a monomorphism in the category $\mathrm{SET}(L)$. If there exists such a morphism $h\colon Z \to X$ that $h \circ f = g$ then $x_0 = g(y_0) = h \circ f(y_0) = h(z_0)$, therefore, $X \circ h(z_0) \not\geqslant Z(z_0)$ and h is not a morphism. Thus, X is not an injective object.

Now let us prove the sufficiency. Since f is a monomorphism, f is injective and then we can define h as follows:

$$h(z) = \begin{cases} g \circ f^{-1}(z), & z \in f(\tilde{Y}) \\ x \in \tilde{X}, & z \notin f(\tilde{Y}). \end{cases}$$

Obviously, $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, X)$ and $g = h \circ f$. ∎

Lastly, we will consider some properties of subobjects and quotient objects in the category $\mathrm{SET}(L)$. Let us start with the former.

Suppose we have an object $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and two arbitrary subobjects of Y $(X, f), (Z, g)$ where $f\colon X \to Y$ and $g\colon Z \to Y$ are some monomorphisms. Then (X, f) is said to be smaller than (Z, g) – denoted by $(X, f) \leqslant (Z, g)$ – iff there exists some morphism $h\colon X \to Z$ such that the triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \\ Z \arrow[ur, "g"'] & \end{tikzcd}$$

commutes.

If $(X, f) \leqslant (Z, g)$ and $(Z, g) \leqslant (X, f)$ then (X, f) and (Z, g) are said to be isomorphic subobjects of Y; denoted by $(X, f) \approx (Z, g)$.

THEOREM 4.6. $(X, f) \leqslant (Z, g)$ *iff the following conditions are fulfilled:*

(1) $f(\tilde{X}) \subset g(\tilde{Z})$;

(2) $X(x) \leqslant Z(z)$ *for* $x \in \tilde{X}$, $z \in \tilde{Z}$, $f(x) = g(z)$.

Proof. Let us prove the necessity first and therefore assume that $(X, f) \leqslant (Z, g)$.

Suppose there exists such $y_0 \in f(\tilde{X})$ that $y_0 \notin g(\tilde{Z})$. Then there exists some $x_0 \in \tilde{X}$ such that $f(x_0) = y_0$ and for all mappings $h \colon \tilde{X} \to \tilde{Z}$ $g \circ h(x_0) \neq y_0 = f(x_0)$. Thus, $(X, f) \nleqslant (Z, g)$.

Suppose there exist such $x_0 \in \tilde{X}$ and $z_0 \in \tilde{Z}$ that $f(x_0) = g(z_0)$ but $X(x_0) \nleqslant Z(z_0)$. Since $(X, f) \leqslant (Z, g)$, there exists some morphism $h \colon X \to Z$ such that $g \circ h = f$. Thus, $g(z_0) = f(x_0) = g \circ h(x_0)$ and then $z_0 = h(x_0)$, since g is a monomorphism and therefore injective. Since h is a morphism, $Z \circ h(x_0) = Z(z_0) \geqslant X(x_0)$ that contradicts our former assumption.

Now let us prove the sufficiency and therefore assume that all conditions of the theorem are fulfilled. Let $h(x) = g^{-1} \circ f(x)$ for all $x \in \tilde{X}$. Then $g \circ h(x) = g \circ g^{-1} \circ f(x) = f(x)$, since g is injective. Further, the second condition of the theorem implies that $Z \circ h(x) \geqslant X(x)$, since $g \circ h(x) = f(x)$. Therefore, $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, Z)$. ∎

The theorem implies the following result.

THEOREM 4.7. $(X, f) \approx (Z, g)$ *iff the following conditions are fulfilled:*

(1) $f(\tilde{X}) = g(\tilde{Z})$;

(2) $X(x) = Z(z)$ *for* $x \in \tilde{X}$, $z \in \tilde{Z}$, $f(x) = g(z)$.

Since $\approx$ is an equivalence relation on the class of all subobjects of Y, the class can be partitioned into equivalence classes of isomorphic subobjects. By choosing one representative from each class we get a system of representatives called a representative class of subobjects of Y. It can be easily seen that the pairs (X, h) where $\tilde{X} \subset \tilde{Y}$, $X(x) \leqslant Y(x)$ for all $x \in \tilde{X}$ and h is the inclusion map form a representative class of subobjects of Y, since $(X_1, h_1) \approx (X_2, h_2)$ iff $X_1 = X_2$ and for each subobject (Z, g) of Y there exists a pair $(X, h) \approx (Z, g)$ where $\tilde{X} = g(\tilde{Z})$ and $X(x) = Z(g^{-1}(x))$ for all $x \in \tilde{X}$. Since the class of all pairs (X, h) is a set, the following theorem holds.

THEOREM 4.8. *The category* SET(L) *is well-powered, i.e., each object* $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ *has a representative class of subobjects that is a set.*

Now suppose we have an object $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and two arbitrary quotient objects of Y $(f, X), (g, Z)$ where $f \colon Y \to X$ and $g \colon Y \to Z$ are some epimorphisms. Then (f, X) is said to be larger than (g, Z) – denoted by $(f, X) \geqslant (g, Z)$ – iff there exists some morphism $h\colon X \to Z$ such that the triangle

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "g"'] & X \arrow[d, "h"] \\ & Z \end{tikzcd}$$

commutes.

If $(f, X) \geqslant (g, Z)$ and $(g, Z) \geqslant (f, X)$ then (f, X) and (g, Z) are said to be isomorphic quotient objects of Y; denoted by $(f, X) \approx (g, Z)$.

THEOREM 4.9. $(f, X) \geqslant (g, Z)$ *iff the following conditions are fulfilled:*

(1) for all $y_1, y_2 \in \tilde{Y}$ $f(y_1) = f(y_2)$ *implies* $g(y_1) = g(y_2)$*;*

(2) $X(f(y)) \leqslant Z(g(y))$ *for all* $y \in \tilde{Y}$*.*

Proof. Let us prove the necessity first and therefore assume that $(f, X) \geqslant (g, Z)$.

Suppose there exist such $y_1, y_2 \in \tilde{Y}$ that $f(y_1) = f(y_2)$ and $g(y_1) \neq g(y_2)$. Since $(f, X) \geqslant (g, Z)$, there exists some morphism $h\colon X \to Z$ such that $h \circ f = g$. Thus, $g(y_1) = h \circ f(y_1) = h \circ f(y_2) = g(y_2)$ that contradicts our former assumption.

Suppose there exists such $y_0 \in \tilde{Y}$ that $X(f(y_0)) \nleqslant Z(g(y_0))$. Since $(f, X) \geqslant (g, Z)$, there exists some morphism $h\colon X \to Z$ such that $h \circ f = g$. Thus, $Z(g(y_0)) = Z(h \circ f(y_0)) \geqslant X(f(y_0))$ that contradicts out former assumption.

Now let us prove the sufficiency and therefore assume that all conditions of the theorem are fulfilled. Let $h(x) = g(y)$, $y \in f^{-1}(x)$. If x_1, $x_2 \in \tilde{X}$ and $x_1 = x_2$ then $h(x_1) = g(y_1)$, $f(y_1) = x_1$ and $h(x_2) = g(y_2)$, $f(y_2) = x_2$. Since $f(y_1) = x_1 = x_2 = f(y_2)$ implies $g(y_1) = g(y_2)$ then $h(x_1) = h(x_2)$ and the definition of h is correct. For each $y \in \tilde{Y}$ it follows that $h \circ f(y) = g(y_0)$, $y_0 \in f^{-1} \circ f(y)$. Since $f(y_0) = f(y)$ implies $g(y_0) = g(y)$ then $h \circ f(y) = g(y)$. Further, for an arbitrary point $x_0 \in \tilde{X}$ $Z \circ h(x_0) = Z \circ g(y_0)$, $y_0 \in f^{-1}(x_0)$ and $Z(g(y_0)) \geqslant X(f(y_0)) = X(x_0)$. Therefore, $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, Z)$. ■

The theorem implies the following result.

THEOREM 4.10. $(f, X) \approx (g, Z)$ *iff the following conditions are fulfilled:*

(1) for all $y_1, y_2 \in \tilde{Y}$ $f(y_1) = f(y_2)$ *iff* $g(y_1) = g(y_2)$*;*

(2) $X(f(y)) = Z(g(y))$ *for all* $y \in \tilde{Y}$*.*

Since $\approx$ is an equivalence relation on the class of all quotient objects of Y, the class can be partitioned into equivalence classes of isomorphic quotient objects. By choosing one representative from each class we get a system of representatives called a representative class of quotient objects of Y. It can be easily seen that the pairs (h, X) where $\tilde{X} = \tilde{Y}/Q$ for some equivalence relation Q on $\tilde{Y}$, $X([y]) \geqslant \bigvee Y([y])$ where $[y]$ denotes the equivalence class generated by y and h is the induced quotient map form a representative class of quotient objects of Y, since $(h_1, X_1) \approx (h_2, X_2)$ iff $X_1 = X_2$ and for each quotient object (g, Z) of Y there exists a pair $(h, X) \approx$

(g, Z) where $\tilde{X} = \tilde{Y}/Q$ with $(y_1, y_2) \in Q$ iff $g(y_1) = g(y_2)$ for all $y_1, y_2 \in \tilde{Y}$ and $X([y]) = Z(g(y))$ for all $y \in \tilde{Y}$. Since the class of all pairs (h, X) is a set, the following theorem holds.

Theorem 4.11. *The category* $\mathrm{SET}(L)$ *is co-(well-powered), i.e., each object* $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ *has a representative class of quotient objects that is a set.*

5. Special morphisms

In this section we will consider the properties of some special morphisms in the category $\mathrm{SET}(L)$. To begin with, we will note that the category $\mathrm{SET}(L)$ is not balanced, that is, there exists a bimorphism $f \in \mathtt{Mor}\,\mathrm{SET}(L)$ which is not an isomorphism. Consider, for example, a morphism $f\colon X \to Y$ where $\tilde{X} = \{x_0\}$, $X(x_0) = 0_L$, $\tilde{Y} = \{y_0\}$, $Y(y_0) = 1_L$ and $f(x_0) = y_0$. Since f is bijective, f is a bimorphism but $Y \circ f(x_0) \neq X(x_0)$, therefore, f is not an isomorphism.

We will continue by considering a coconstant morphism in the category $\mathrm{SET}(L)$.

Theorem 5.1. *A morphism* $f\colon X \to Y$ *is a coconstant morphism iff* $\tilde{X} = \varnothing$.

Proof. Let us prove the necessity first and therefore assume that f is a coconstant morphism. Then for each object $Z \in \mathtt{Obj}\,\mathrm{SET}(L)$ and every two morphisms $g, h\colon Y \to Z$ it follows that $g \circ f = h \circ f$.

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[dr, "X"'] & \tilde{Y} \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[d, "Y"] & \tilde{Z} \arrow[dl, "Z"] \\ & L & \end{tikzcd}$$

Suppose there exists a point $x_0 \in \tilde{X}$. Then $f(x_0) = y_0 \in \tilde{Y}$. Let $\tilde{Z} = \{z_1, z_2\}$, $z_1 \neq z_2$ and $Z(z_1) = Z(z_2) = 1_L$. In this case every mappings $g, h\colon \tilde{Y} \to \tilde{Z}$ will be morphisms and thus we can take $g(y_0) = z_1$ and $h(y_0) = z_2$. It is easy to see that $g \circ f \neq h \circ f$.

The sufficiency is obvious. ∎

From this theorem, the theorem about constant morphism in the category $\mathrm{SET}(L)$ and the definition of zero morphism (notice that a morphism is said to be a zero morphism provided that it is both constant and coconstant) the following theorem can be derived.

Theorem 5.2. *A morphism* $f\colon X \to Y$ *is a zero morphism iff* $\tilde{X} = \varnothing$.

Obviously, there exist such two objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ that the set $\mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$ contains no zero morphism, therefore, the category $\mathrm{SET}(L)$ is not pointed.

Now we will consider a coequalizer in the category $\mathrm{SET}(L)$. Suppose we have two morphisms $f, g\colon X \to Y$. We will find a coequalizer of f and g. Following the definition we need such a pair (h, Z) where $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, Z)$ that

(1) $h \circ f = h \circ g$;

(2) for each object $W \in \mathtt{Obj}\,\mathrm{SET}(L)$ and each morphism $h^{'}: Y \to W$ such that $h^{'} \circ f = h^{'} \circ g$ there exists a unique morphism $m \colon Z \to W$ such that the triangle

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "h'"'] & Z \arrow[dl, dashed, "m"] \\ W & \end{tikzcd}$$

commutes.

(Notice that the definition of coequalizer implies that (h, Z) is a quotient object of Y.)

Let Q be the smallest equivalence relation on $\tilde{Y}$ that contains all pairs $(f(x), g(x))$ for $x \in \tilde{X}$. Then let $\tilde{Z} = \tilde{Y}/Q = \{[y] | y \in \tilde{Y}\}$ where $[y]$ denotes the equivalence class generated by y and $Z([y_0]) = \vee\{Y(y) | y \in [y_0]\}$ for all $[y_0] \in \tilde{Z}$. The mapping $h \colon \tilde{Y} \to \tilde{Z}$ will be the induced quotient map, therefore, $h(y) = [y]$ for all $y \in \tilde{Y}$. Obviously, $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, Z)$.

Proposition 5.1. *(h, Z) is a coequalizer of f and g.*

Proof. Obviously, $h \circ f = h \circ g$. Suppose there exists another morphism $h^{'}: Y \to W$ such that $h^{'} \circ f = h^{'} \circ g$. From the definition of Q we derive that $h^{'}([y]) \subset \{w_y\} \subset \tilde{W}, w_y \in \tilde{W}$ for all $y \in \tilde{Y}$, therefore, there exists a unique mapping $m \colon \tilde{Z} \to \tilde{W}$ such that $m \circ h = h^{'}$, i.e., $m([y]) = h^{'}(y)$ for all $[y] \in \tilde{Z}$. Let us prove that $m \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, W)$, that is, $W \circ m([y]) \geqslant Z([y])$ for all $[y] \in \tilde{Z}$. Fix an arbitrary point $[y_0] \in \tilde{Z}$. Since $h^{'} \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, W)$, it follows that $W \circ h^{'}(y_0) = W \circ h^{'}(y) \geqslant Y(y)$ for all $y \in [y_0]$ and then $W \circ m([y_0]) = W \circ h^{'}(y_0) \geqslant \vee\{Y(y) | y \in [y_0]\} = Z([y_0])$. Thus, m is indeed a morphism. ∎

Since f and g were two arbitrary morphisms, the following result holds.

Theorem 5.3. *The category $\mathrm{SET}(L)$ has coequalizers, i.e., every two morphisms $f, g \colon X \to Y$ have a coequalizer.*

The notion of coequalizer gives rise to defining a special kind of epimorphisms called regular epimorphisms. Given an arbitrary morphism $h \colon Y \to Z$, the pair (h, Z) is called a regular quotient object of Y and h is called a regular epimorphism iff there exist such two morphisms $f, g \colon X \to Y$ that (h, Z) is a coequalizer of f and g. The following theorem shows the necessary and sufficient conditions for a morphism to be a regular epimorphism.

Theorem 5.4. *A morphism $h \colon Y \to Z$ is a regular epimorphism iff the following conditions are fulfilled:*

(1) h is surjective;

(2) $Z(z) = \vee\{Y(y) | h(y) = z\}$ for all $z \in \tilde{Z}$.

Proof. Let us prove the necessity first and therefore assume that h is a regular epimorphism. Then (h, Z) is a coequalizer of some morphisms $f, g \colon X \to Y$ and thus, a quotient object of Y. Therefore h is an epimorphism that implies that h is surjective.

Now let us prove that $Z(z) = \vee\{Y(y)|h(y) = z\}$ for all $z \in \tilde{Z}$. Let Q be such an equivalence relation on $\tilde{Y}$ that for every two points $y_1, y_2 \in \tilde{Y}$ $(y_1, y_2) \in Q$ iff $h(y_1) = h(y_2)$. Let $\tilde{W} = \tilde{Y}/Q = \{[y]|y \in \tilde{Y}\}$ and $W([y_0]) = \vee\{Y(y)|y \in [y_0]\}$ for all $[y_0] \in \tilde{W}$. The mapping h' will be the induced quotient map, therefore, $h'(y) = [y]$ for all $y \in \tilde{Y}$. Obviously, $h' \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, W)$ and $h' \circ f = h' \circ g$. Then there exists such a morphism $m\colon Z \to W$ that $h' = m \circ h$. Now suppose we have an arbitray point $z_0 \in \tilde{Z}$. Since $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, Z)$, it follows that $Z(z_0) \geqslant \vee\{Y(y)|h(y) = z_0\}$. The fact that $m \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, W)$ implies that $W \circ m(z_0) \geqslant Z(z_0)$. The mapping h is surjective, therefore, there exists such a point $y_0 \in \tilde{Y}$ that $h(y_0) = z_0$. Then $W \circ m(z_0) = W \circ m \circ h(y_0) = W \circ h'(y_0) = W([y_0]) = \vee\{Y(y)|y \in [y_0]\} = \vee\{Y(y)|h(y) = h(y_0) = z_0\} \geqslant Z(z_0)$. Thus, $Z(z_0) = \vee\{Y(y)|h(y) = z_0\}$.

Now let us prove the sufficiency and therefore assume that all the conditions of the theorem are fulfilled. Obviously, one can find a set $\tilde{X}$ and such two mappings $f, g\colon \tilde{X} \to \tilde{Y}$ that Q defined in the first part of the theorem is the smallest equivalence relation that contains all pairs $(f(x), g(x))$ for $x \in \tilde{X}$ (for every $y_1, y_2 \in \tilde{Y}$ such that $h(y_1) = h(y_2)$ take a point $x_{y_1 y_2}$ and define $f(x_{y_1 y_2}) = y_1$ and $g(x_{y_1 y_2}) = y_2$). Then let $X(x) = 0_L$ for all $x \in \tilde{X}$. Thus, $f, g \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$. The proof that (h, Z) is a coequalizer of f and g is the same as in the proposition 5.1. ∎

The theorem implies that there are epimorphisms in the category SET(L) that are not regular epimorphisms.

Now let us consider a generalization of coequalizer, i.e., multiple coequalizer in the category SET(L). Suppose we have two objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and a non-empty indexed family of morphisms $(f_i)_{i\in I}$ contained in the set $\mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$. A pair (h, Z) where $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, Z)$ called a multiple coequalizer of $(f_i)_{i\in I}$ provided that

(1) $h \circ f_i = h \circ f_j$ for all $i, j \in I$;

(2) for each object $W \in \mathtt{Obj}\,\mathrm{SET}(L)$ and each morphism $h'\colon Y \to W$ such that $h' \circ f_i = h' \circ f_j$ for all $i, j \in I$ there exists a unique morphism $m\colon Z \to W$ such that $m \circ h = h'$.

Let Q be the smallest equivalence relation on $\tilde{Y}$ that contains all pairs $(f_i(x), f_j(x))$ for $i, j \in I$ and $x \in \tilde{X}$. Then let $\tilde{Z} = \tilde{Y}/Q = \{[y]|y \in \tilde{Y}\}$ and $Z([y_0]) = \vee\{Y(y)|y \in [y_0]\}$ for all $[y_0] \in \tilde{Z}$. The mapping $h\colon \tilde{Y} \to \tilde{Z}$ will be the induced quotient map. One can easily prove that (h, Z) is a multiple coequalizer of $(f_i)_{i\in I}$. Thus, the following theorem holds.

THEOREM 5.5. *The category* SET(L) *has multiple coequalizers, i.e., each non-empty indexed family of morphisms* $(f_i)_{i\in I}$ *contained in* $\mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$ *has a multiple coequalizer.*

Lastly, we will consider one special case of coequalizers, i.e., cokernels in the category SET(L). Suppose we have a morphism $f\colon X \to Y$. If $g_0\colon X \to Y$ is the unique zero morphism from X to Y then (if it exists) a coequalizer of f and g_0 is called the cokernel of f. From our previous investigations it follows that if g_0 is a zero morphism then $\tilde{X} = \varnothing$ and then $f = g_0$. Thus, the cokernel of f is a pair (h, Z) where h is bijective and $Z \circ h(y) = Y(y)$ for all $y \in \tilde{Y}$. Therefore, h is an isomorphism. Since the category SET(L) is not pointed, there exist such morphisms in this category which have no cokernel. Further, suppose we have a morphism $h\colon Y \to Z$. Then (h, Z) is called a normal quotient object of Y and h is called a normal epimorphism provided that there exists such a morphism $f\colon X \to Y$ that (h, Z) is the cokernel of f. The following theorem shows that in the category SET(L) normal epimorphisms and isomorphisms are equivalent.

Theorem 5.6. *A morphism $h\colon Y \to Z$ is a normal epimorphism iff h is an isomorphism.*

Proof. Let us prove the necessity first and therefore assume that h is a normal epimorpsim. Then it follows that (h, Z) is a coequalizer of some $f, g_0 \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$ where $\tilde{X} = \varnothing$. Since $e_Y \circ f = e_Y \circ g_0$, there exists such a morphism $m\colon Z \to Y$ that $m \circ h = e_y$. Hence, h is a section. The fact that (h, Z) is a subobject of Y implies that h is an epimorphism and then h is an isomorphism.

Now let us prove the sufficiency and therefore assume that h is an isomorphism. Then h is bijective and $Z \circ h(y) = Y(y)$ for all $y \in \tilde{Y}$. Obviously, $h \circ f = h \circ g_0$ where $f, g_0 \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$, $\tilde{X} = \varnothing$. For each morphism $h^{'}: Y \to W$ such that $h^{'} \circ f = h^{'} \circ g_0$ there exists a unique mapping $m\colon \tilde{Z} \to \tilde{W}$ such that $m \circ h = h^{'}$, i.e., $m(z) = h^{'} \circ h^{-1}(z)$ for all $z \in \tilde{Z}$. Then $W \circ m(z) = W \circ h^{'} \circ h^{-1}(z) \geqslant Y \circ h^{-1}(z) = Z \circ h \circ h^{-1}(z) = Z(z)$ and therefore $m \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, W)$. ■

Now let us consider the dual of coequalizer, i.e., equalizer in the category SET(L). Suppose we have two morphisms $f, g\colon X \to Y$. We will find an equalizer of f and g. Following the definition, we need such a pair (Z, h) where $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, X)$ that

(1) $f \circ h = g \circ h$;

(2) for each object $W \in \mathtt{Obj}\,\mathrm{SET}(L)$ and each morphism $h^{'}: W \to X$ such that $f \circ h^{'} = g \circ h^{'}$ there exists a unique morphism $m\colon W \to Z$ such that the triangle

$$\begin{tikzcd} Z \arrow[r, "h"] & X \\ & W \arrow[u, "h'"'] \arrow[ul, dashed, "m"] \end{tikzcd}$$

commutes.

(Notice that the definition of equalizer implies that (Z, h) is a subobject of X.)

Let $\tilde{Z} = \{x | x \in \tilde{X}, f(x) = g(x)\}$ and $Z(x) = X(x)$ for all $x \in \tilde{Z}$. The mapping $h\colon \tilde{Z} \to \tilde{X}$ will be the inclusion map. Obviously, $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, X)$.

Proposition 5.2. *(Z, h) is an equalizer of f and g.*

Proof. Obviously, $f \circ h = g \circ h$. Suppose there exists another morphism $h^{'}: W \to X$ such that $f \circ h^{'} = g \circ h^{'}$. Then there exists a unique mapping $m: \tilde{W} \to \tilde{Z}$ such that $h \circ m = h^{'}$, i.e., $m(w) = h^{-1} \circ h^{'}(w)$ for all $w \in \tilde{W}$. Fix an arbitrary point $w_0 \in \tilde{W}$. Then $Z \circ m(w_0) = X \circ h \circ m(w_0) = X \circ h^{'}(w_0) \geqslant W(w_0)$. Thus, m is indeed a morphism. ∎

Since f and g were two arbitrary morphisms, the following result holds.

THEOREM 5.7. *The category* SET(L) *has equalizers, i.e., every two morphisms* $f, g: X \to Y$ *have an equalizer.*

By analogy with coequalizers a special kind of monomorphisms, i.e., regular monomorphisms can be defined. From the theorem about the properties of equalizers in the category SET(L) we can derive the necessary and sufficient conditions for an arbitrary morphism $h: Z \to X$ to be a regular monomorphism.

THEOREM 5.8. *A morphism* $h: Z \to X$ *is a regular monomorphism iff the following conditions are fulfilled:*

(1) h *is injective;*

(2) $X \circ h(x) = Z(x)$ *for all* $x \in \tilde{X}$.

The theorem implies that there are monomorphisms in the category SET(L) that are not regular monomorphisms.

Now let us consider a generalization of equalizer, i.e., multiple equalizer in the category SET(L) whose definition can be obtained by dualizing the notion of multiple coequalizer.

Suppose we have two objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and a non-empty indexed family of morphisms $(f_i)_{i \in I}$ contained in the set $\mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$. Let $\tilde{Z} = \{x | x \in \tilde{X}, f_i(x) = f_j(x) \text{ for all } i, j \in I\}$ and $Z(x) = X(x)$ for all $x \in \tilde{Z}$. The mapping $h: \tilde{Z} \to \tilde{X}$ will be the inclusion map. Obviously, $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, X)$. One can easily prove that (Z, h) is a multiple equalizer of $(f_i)_{i \in I}$. Thus, the following theorem holds.

THEOREM 5.9. *The category* SET(L) *has multiple equalizers, i.e., each non-empty indexed family of morphisms* $(f_i)_{i \in I}$ *contained in* $\mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$ *has a multiple equalizer.*

Lastly, let us consider one special case of equalizers, i.e., kernels in the category SET(L). Suppose we have a morphism $f: X \to Y$. If $g_0: X \to Y$ is the unique zero morphism from X to Y then (if it exists) an equalizer of f and g_0 is called the kernel of f. If g_0 is a zero morphism then $\tilde{X} = \varnothing$ and then the kernel of f is a pair (Z, h) where $\tilde{Z} = \varnothing$ and $h: \tilde{Z} \to \tilde{X}$ is the empty mapping. Obviously, there exist such morphisms in the category SET(L) which have no kernel. By analogy with normal epimorphisms we can obtain normal monomorphisms. It is easy to see that the only normal monomorphism in the category SET(L) is the morphism $h: X \to Y$ where $\tilde{X} = \tilde{Y} = \varnothing$.

6. Standard constructions

In this section we will consider two standard constructions in the category SET(L), i.e., inverse and direct systems. We will start with the former (see [1]).

Suppose $\mathcal{C}'$ is a category and Ω is a partially-ordered and directed set. For each $\omega \in \Omega$ choose an object $X_\omega \in \mathtt{Obj}\,\mathcal{C}'$ such that $X_{\omega_1} \neq X_{\omega_2}$ for $\omega_1 \neq \omega_2$ and for every $\omega_1, \omega_2 \in \Omega$ such that $\omega_1 \leqslant \omega_2$ choose a morphism $f^{\omega_2}_{\omega_1} \colon X_{\omega_2} \to X_{\omega_1}$. If the triple $S = \{X_\omega, f^{\omega_2}_{\omega_1}, \Omega\}$ is a subcategory of $\mathcal{C}'$ then S is called an inverse system. (Notice that from the properties of S it follows that for every $\omega_1, \omega_2, \omega_3 \in \Omega$ such that $\omega_1 \leqslant \omega_2 \leqslant \omega_3$ $f^{\omega_2}_{\omega_1} \circ f^{\omega_3}_{\omega_2} = f^{\omega_3}_{\omega_1}$ and for each $\omega \in \Omega$ $f^{\omega}_{\omega} = e_{X_\omega}$.) A pair $\{X, (h_\omega)_{\omega\in\Omega}\}$ where $X \in \mathtt{Obj}\,\mathcal{C}'$ and $h_\omega \in \mathtt{Mor}'_{\mathcal{C}}(X, X_\omega)$ for all $\omega \in \Omega$ is called a limit of S iff the following conditions are fulfilled:

(1) $f^{\omega_2}_{\omega_1} \circ h_{\omega_2} = h_{\omega_1}$ for all $\omega_1, \omega_2 \in \Omega, \omega_1 \leqslant \omega_2$;

(2) for each pair $\{Y, (g_\omega)_{\omega\in\Omega}\}$ where $Y \in \mathtt{Obj}\,\mathcal{C}'$, $g_\omega \in \mathtt{Mor}'_{\mathcal{C}}(Y, X_\omega)$ for all $\omega \in \Omega$ and $f^{\omega_2}_{\omega_1} \circ g_{\omega_2} = g_{\omega_1}$ for all $\omega_1, \omega_2 \in \Omega, \omega_1 \leqslant \omega_2$ where exists a unique morphism $f \colon Y \to X$ such that $h_\omega \circ f = g_\omega$ for all $\omega \in \Omega$.

Theorem 6.1. *In the category* SET(L) *each inverse system has a limit.*

Proof. Suppose we have an inverse system $S = \{X_\omega, f^{\omega_2}_{\omega_1}, \Omega\}$. We have to find a limit of S, i.e., a pair $\{X, (h_\omega)_{\omega\in\Omega}\}$. We will proceed as in [FF]. Let $\tilde{X} = \{\{x\} = (x_\omega)_{\omega\in\Omega} \in \prod\limits_{\omega\in\Omega} \widetilde{X_\omega} \mid f^{\omega_2}_{\omega_1}(x_{\omega_2}) = x_{\omega_1}$ for all $\omega_1, \omega_2 \in \Omega, \omega_1 \leqslant \omega_2\}$ and $X(\{x\}) = \bigwedge\limits_{x_\omega \in \{x\}} X_\omega(x_\omega)$. For each $\omega \in \Omega$ let h_ω be the restriction to $\tilde{X}$ of the projective mapping $p_\omega \colon \prod\limits_{\omega\in\Omega} \widetilde{X_\omega} \to \widetilde{X_\omega} \colon \{x\} \mapsto x_\omega$. Since $X_\omega \circ h_\omega(\{x\}) = X_\omega(x_\omega) \geqslant \bigwedge\limits_{x_\omega \in \{x\}} X_\omega(x_\omega) = X(\{x\})$, the mapping h_ω is a morphism. For every $\omega_1, \omega_2 \in \Omega, \omega_1 \leqslant \omega_2$ $f^{\omega_2}_{\omega_1} \circ h_{\omega_2}(\{x\}) = f^{\omega_2}_{\omega_1}(x_{\omega_2}) = x_{\omega_1} = h_{\omega_1}(\{x\})$.

Suppose we have another pair $\{Y, (g_\omega)_{\omega\in\Omega}\}$ and $f^{\omega_2}_{\omega_1} \circ g_{\omega_2} = g_{\omega_1}$ for all $\omega_1, \omega_2 \in \Omega, \omega_1 \leqslant \omega_2$. Let f be the diagonal mapping $\mathop{\triangle}\limits_{\omega\in\Omega} g_\omega \colon \tilde{Y} \to \prod\limits_{\omega\in\Omega} \widetilde{X_\omega} \colon y \mapsto (g_\omega(y))_{\omega\in\Omega}$. Obviously, $p_\omega \circ f = g_\omega$ for all $\omega \in \Omega$. Fix an arbitrary point $y_0 \in \tilde{Y}$. Then $f^{\omega_2}_{\omega_1} \circ p_{\omega_2} \circ f(y_0) = f^{\omega_2}_{\omega_1} \circ g_{\omega_2}(y_0) = g_{\omega_1}(y_0) = p_{\omega_1} \circ f(y_0)$. Thus, $f(y_0) \in \tilde{X}$ and $f \colon \tilde{Y} \to \tilde{X}$. It is easy to see that $h_\omega \circ f = g_\omega$ for all $\omega \in \Omega$. Let us prove that $f \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, X)$, that is, $X \circ f(y) \geqslant Y(y)$ for all $y \in \tilde{Y}$. Suppose $y_0 \in \tilde{Y}$. Then $X_\omega \circ g_\omega(y_0) \geqslant Y(y_0)$ and then $\bigwedge\limits_{\omega\in\Omega} X_\omega \circ g_\omega(y_0) \geqslant Y(y_0)$. Therefore, $X \circ f(y_0) = \bigwedge\limits_{\omega\in\Omega} X_\omega \circ g_\omega(y_0) \geqslant Y(y_0)$. For each morphism $f' \colon Y \to X$ such that $h_\omega \circ f' = g_\omega$ for all $\omega \in \Omega$ it follows that $f' = f$, since $g_\omega = h_\omega \circ f' = p_\omega \circ f'$. Thus, f is the only morphism with the required property. ■

Now let us consider the dual of inverse system, i.e., direct system in the category SET(L).

Suppose $\mathcal{C}'$ is a category and Ω is a partially-ordered and directed set. For each $\omega \in \Omega$ choose an object $X_\omega \in \mathtt{Obj}\,\mathcal{C}'$ such that $X_{\omega_1} \neq X_{\omega_2}$ for $\omega_1 \neq \omega_2$ and for every $\omega_1, \omega_2 \in \Omega$ such that $\omega_1 \leqslant \omega_2$ choose a morphism $f^{\omega_1}_{\omega_2} \colon X_{\omega_1} \to X_{\omega_2}$. If the triple $S = \{X_\omega, f^{\omega_1}_{\omega_2}, \Omega\}$ is a subcategory of $\mathcal{C}'$ then S is called a direct system. (Notice that from the properties of S it follows that for every $\omega_1, \omega_2, \omega_3 \in \Omega$ such that $\omega_1 \leqslant \omega_2 \leqslant \omega_3$ $f^{\omega_2}_{\omega_3} \circ f^{\omega_1}_{\omega_2} = f^{\omega_1}_{\omega_3}$ and for each $\omega \in \Omega$ $f^\omega_\omega = e_{X_\omega}$.) A pair $\{(h_\omega)_{\omega \in \Omega}, X\}$ where $X \in \mathtt{Obj}\,\mathcal{C}'$ and $h_\omega \in \mathtt{Mor}'_{\mathcal{C}}(X_\omega, X)$ for all $\omega \in \Omega$ is called a limit of S iff the following conditions are fulfilled:

(1) $h_{\omega_2} \circ f^{\omega_1}_{\omega_2} = h_{\omega_1}$ for all $\omega_1, \omega_2 \in \Omega, \omega_1 \leqslant \omega_2$;

(2) for each pair $\{(g_\omega)_{\omega \in \Omega}, Y\}$ where $Y \in \mathtt{Obj}\,\mathcal{C}'$, $g_\omega \in \mathtt{Mor}'_{\mathcal{C}}(X_\omega, Y)$ for all $\omega \in \Omega$ and $g_{\omega_2} \circ f^{\omega_1}_{\omega_2} = g_{\omega_1}$ for all $\omega_1, \omega_2 \in \Omega, \omega_1 \leqslant \omega_2$ where exists a unique morphism $f \colon X \to Y$ such that $f \circ h_\omega = g_\omega$ for all $\omega \in \Omega$.

THEOREM 6.2. *In the category* SET(L) *each direct system has a limit.*

Proof. Suppose we have an inverse system $S = \{X_\omega, f^{\omega_1}_{\omega_2}, \Omega\}$. We have to find a limit of S, i.e., a pair $\{(h_\omega)_{\omega \in \Omega}, X\}$. Let Q be such a relation on $\bigoplus\limits_{\omega \in \Omega} \widetilde{X_\omega}$ that for every two points $x_{\omega_1}, y_{\omega_2} \in \bigoplus\limits_{\omega \in \Omega} \widetilde{X_\omega}$ $(x_{\omega_1}, y_{\omega_2}) \in Q$ iff there exists such $\omega \in \Omega$ that $\omega_1 \leqslant \omega, \omega_2 \leqslant \omega$ and $f^{\omega_1}_\omega(x_{\omega_1}) = f^{\omega_2}_\omega(y_{\omega_2})$. Let us verify that Q is an equivalence relation on $\bigoplus\limits_{\omega \in \Omega} \widetilde{X_\omega}$.

(1) $(x_\omega, x_\omega) \in Q$ for all $x_\omega \in \bigoplus\limits_{\omega \in \Omega} \widetilde{X_\omega}$, since $f^\omega_\omega(x_\omega) = x_\omega$.

(2) Obviously, $(x_{\omega_1}, y_{\omega_2}) \in Q$ implies $(y_{\omega_2}, x_{\omega_1}) \in Q$ for all $x_{\omega_1}, y_{\omega_2} \in \bigoplus\limits_{\omega \in \Omega} \widetilde{X_\omega}$.

(3) Suppose $(x_{\omega_1}, y_{\omega_2}) \in Q$ and $(y_{\omega_2}, z_{\omega_3}) \in Q$ for some $x_{\omega_1}, y_{\omega_2}, z_{\omega_3} \in \bigoplus\limits_{\omega \in \Omega} \widetilde{X_\omega}$. Then there exist such two points $\omega^1_0, w^2_0 \in \Omega$ that $\omega_1 \leqslant \omega^1_0, \omega_2 \leqslant \omega^1_0, f^{\omega_1}_{\omega^1_0}(x_{\omega_1}) = f^{\omega_2}_{\omega^1_0}(y_{\omega_2})$ and $\omega_2 \leqslant \omega^2_0, \omega_3 \leqslant \omega^2_0, f^{\omega_2}_{\omega^2_0}(y_{\omega_2}) = f^{\omega_3}_{\omega^2_0}(z_{\omega_3})$. Since the set Ω is directed, there exists such a point $\omega^3_0 \in \Omega$ that $\omega^1_0 \leqslant \omega^3_0$ and $\omega^2_0 \leqslant \omega^3_0$. Then $f^{\omega^1_0}_{\omega^3_0} \circ f^{\omega_1}_{\omega^1_0}(x_{\omega_1}) = f^{\omega^1_0}_{\omega^3_0} \circ f^{\omega_2}_{\omega^1_0}(y_{\omega_2})$ and $f^{\omega^2_0}_{\omega^3_0} \circ f^{\omega_2}_{\omega^2_0}(y_{\omega_2}) = f^{\omega^2_0}_{\omega^3_0} \circ f^{\omega_3}_{\omega^2_0}(z_{\omega_3})$. From the properties of S it follows that $f^{\omega^1_0}_{\omega^3_0} \circ f^{\omega_1}_{\omega^1_0}(x_{\omega_1}) = f^{\omega_1}_{\omega^3_0}(x_{\omega_1}), f^{\omega^1_0}_{\omega^3_0} \circ f^{\omega_2}_{\omega^1_0}(y_{\omega_2}) = f^{\omega_2}_{\omega^3_0}(y_{\omega_2}), f^{\omega^2_0}_{\omega^3_0} \circ f^{\omega_2}_{\omega^2_0}(y_{\omega_2}) = f^{\omega_2}_{\omega^3_0}(y_{\omega_2})$ and $f^{\omega^2_0}_{\omega^3_0} \circ f^{\omega_3}_{\omega^2_0}(z_{\omega_3}) = f^{\omega_3}_{\omega^3_0}(z_{\omega_3})$. Thus, $f^{\omega_1}_{\omega^3_0}(x_{\omega_1}) = f^{\omega_2}_{\omega^3_0}(y_{\omega_2}) = f^{\omega_3}_{\omega^3_0}(z_{\omega_3})$ and $(x_{\omega_1}, z_{\omega_3}) \in Q$.

Let $\tilde{X} = \bigoplus\limits_{\omega \in \Omega} \widetilde{X_\omega}/Q = \{[x_\omega] | x_\omega \in \bigoplus\limits_{\omega \in \Omega} \widetilde{X_\omega}\}$ and $X([x_\omega]) = \vee\{X_{\omega'}(y_{\omega'}) | y_{\omega'} \in [x_\omega]\}$. For each $\omega \in \Omega$ the mapping h_ω will be the induced quotient map, i.e., $h_\omega(x_\omega) = [x_\omega]$ for all $x_\omega \in \widetilde{X_\omega}$. Since $X \circ h_\omega(x_\omega) = X([x_\omega]) \geqslant X_\omega(x_\omega)$ then $h_\omega \in \mathtt{Mor}_{\mathrm{SET}(L)}(X_\omega, X)$. Suppose we have $\omega_1, \omega_2 \in \Omega, \omega_1 \leqslant \omega_2$. Then for each $x_{\omega_1} \in \widetilde{X_{\omega_1}}$ it follows that $(x_{\omega_1}, f^{\omega_1}_{\omega_2}(x_{\omega_1})) \in Q$, since $\omega_1 \leqslant \omega_2, \omega_2 \leqslant \omega_2$ and $f^{\omega_1}_{\omega_2}(x_{\omega_1}) =$

$f_{\omega_2}^{\omega_2} \circ f_{\omega_2}^{\omega_1}(x_{\omega_1})$. Therefore, $h_{\omega_2} \circ f_{\omega_2}^{\omega_1}(x_{\omega_1}) = [f_{\omega_2}^{\omega_1}(x_{\omega_1})] = [x_{\omega_1}] = h_{\omega_1}(x_{\omega_1})$ and the required property is fulfilled.

Suppose we have another pair $\{(g_\omega)_{\omega\in\Omega}, Y\}$ and $g_{\omega_2} \circ f_{\omega_2}^{\omega_1} = g_{\omega_1}$ for all $\omega_1, \omega_2 \in \Omega, \omega_1 \leqslant \omega_2$. Let $f\colon \tilde{X} \to \tilde{Y}$ be such that $f([x_\omega]) = g_\omega(x_\omega)$ for all $[x_w] \in \tilde{X}$. We have to verify that the definition of f is correct. Suppose $[x_{\omega_1}] = [y_{\omega_2}]$ for some $x_{\omega_1}, y_{\omega_2} \in \tilde{X}$. Since $(x_{\omega_1}, y_{\omega_2}) \in Q$, there exists such $\omega \in \Omega$ that $\omega_1 \leqslant \omega, \omega_2 \leqslant \omega$ and $f_\omega^{\omega_1}(x_{\omega_1}) = f_\omega^{\omega_2}(y_{\omega_2})$. Then $g_\omega \circ f_\omega^{\omega_2}(y_{\omega_2}) = g_{\omega_2}(y_{\omega_2})$ and $g_\omega \circ f_\omega^{\omega_1}(x_{\omega_1}) = g_{\omega_1}(x_{\omega_1})$. Since $g_\omega \circ f_\omega^{\omega_2}(y_{\omega_2}) = g_\omega \circ f_\omega^{\omega_1}(x_{\omega_1})$ then $g_{\omega_2}(y_{\omega_2}) = g_{\omega_1}(x_{\omega_1})$. Thus, $f([x_{\omega_1}]) = f([y_{\omega_2}])$.

Obviously, $f \circ h_\omega = g_\omega$ for all $\omega \in \Omega$. Let us prove that $f \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$, that is, $Y \circ f([x_\omega]) \geqslant X([x_\omega])$ for all $[x_\omega] \in \tilde{X}$. Suppose we have $[x_{\omega_0}] \in \tilde{X}$. Then for all $y_\omega \in [x_{\omega_0}]$ $g_\omega(y_\omega) = f \circ h_\omega(y_\omega) = f([y_\omega]) = f([x_{\omega_0}]) = g_\omega(x_{\omega_0})$. Thus, $Y \circ f([x_{\omega_0}]) = Y \circ g_\omega(x_{\omega_0}) = Y \circ g_\omega(y_\omega) \geqslant X_\omega(y_\omega)$ for all $y_\omega \in [x_{\omega_0}]$, since $g_\omega \in \mathtt{Mor}_{\mathrm{SET}(L)}(X_\omega, Y)$. Then $Y \circ f([x_{\omega_0}]) \geqslant \vee\{X_\omega(y_\omega) | y_\omega \in [x_{\omega_0}]\} = X([x_{\omega_0}])$. Thus, f is indeed a morphism. For each morphism $f' : X \to Y$ such that $f' \circ h_\omega = g_\omega$ for all $\omega \in \Omega$ it follows that $f' \circ h_\omega(x_\omega) = f'([x_\omega]) = g_\omega(x_\omega)$ and then $f' = f$. ■

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