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A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL DERIVATIVES

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Abstract. A subspace $\mathcal{D}'_*(P)$ of the space of distributions has been analized and the Laplace transform of their elements applied to solve, in a prescribed domain, the system of linear partial differential equations with fractional derivatives, as well. To this system belongs a mathematical model of a visco elastic rod submited to the axial force of the form $F(t) = B + A\theta(t - t_0)$, where A and B are constants and θ is the Heaviside functions.

1. Introduction

Laplace transform (in short LT) of numerical functions has been elaborated as a powerful mathematical theory very useful in solving mathematical models (cf. [3]). However it has been belived that LT has two important shortcomings. First, application of the Laplace transform call for some growth conditions of the elements to which it is applied. Secondly, there is no simple characterisation of the functions which are LT-s of the numerical functions, property important for the applications. The situation with the LT of generalizedfunctions was similar (cf. [14], [15]).

Recently H. Komatsu [8], [9] overcame successfuly the all shortcomings of the mentioned LT-s defining the LT of hyperfunctions in one dimensional case. Notions of the Laplace hyperfunctions and hiperfunctions belong to an abstract mathematical theory, therefore it can not easily be accepted by people working in applications. In [13] one can find elaborated a theory the LT of a subspace $\mathcal{D}'_*(P)$ of the space of distributions. In this paper we use this theory to solve a system of partial differential equations with fractional derivatives, which is a mathematical model of a viscoelastic rod, and a coefficient of which is the discontinuous function of the form $B + A\theta(t - t_0)$.

2. A method of solutions of mathematical models

We recall some definitions and facts from [13] adapted to the need of forthcoming equations.

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2.1. LT of tempered distributions

By $\mathcal{S}'(\mathbb{R})$ is denoted the space of tempered distributions and by $\mathcal{S}'_+ = \{T \in \mathcal{S}'(\mathbb{R});\ T|_{(-\infty,0)} = 0\}$, ($T|_{(-\infty,0)}$ is the restriction of T on $(-\infty,0)$). The LT of an element $f \in \mathcal{S}'_+$ is defined by

$$\widehat{f}(z) = \mathcal{L}(f)(z) = \langle f(t), e^{-zt}\rangle, \quad z \in \mathbb{R}_+ + i\mathbb{R}. \tag{2.1}$$

For the properties of so defined LT one can consult [14] and [15].

Let $\mathcal{H}^{(\alpha,\beta)}(\mathbb{R}_+)$, $\alpha, \beta \in \overline{\mathbb{R}}_+$, denote the set of holomorphic functions f on $\mathbb{R}_+ + i\mathbb{R}$ which satisfy the following growth condition

$$|f(z)| \le M(1+|z|^2)^{\alpha/2}(1+x^{-\beta}),\ z = x+iy \in \mathbb{R}_+ + i\mathbb{R}.$$

We set $\mathcal{H}(\mathbb{R}_+) = \bigcup_{\alpha\ge 0,\beta\ge 0} \mathcal{H}^{(\alpha,\beta)}(\mathbb{R}_+)$. Then we have the following:

Proposition A. *Algebras $\mathcal{H}(\mathbb{R}_+)$ and $\mathcal{S}_+$ are isomorphic. This isomorphism is realized by the LT.*

If $\sigma \ge 0$, $f \in \mathcal{S}_+$ and $g = e^{\sigma t} f(t)$, then $\mathcal{L}(g)(s) = \langle f(t), e^{(s-\sigma)t}\rangle$, $\operatorname{Re} s > \sigma$.

Let $F(s)$ be a holomorphic function for $\operatorname{Re} s > \omega$. The function $F(\zeta+\omega)$ is holomorphic for $\operatorname{Re}\zeta > 0$. If $F(\zeta+\omega) \in \mathcal{H}(\mathbb{R}_+)$, then there exists $f \in \mathcal{S}'_+)$ such that

$$\mathcal{L}(e^{\omega t} f)(s) = F(s). \tag{2.2}$$

2.2. Space $\mathcal{D}'_\omega(I)$

Let $\mathcal{A}$ be the vector space, subspace of $e^{\omega t}\mathcal{S}'_+$:

$$\mathcal{A} = \{T \in e^{\omega t}\mathcal{S}'(\mathbb{R});\ T|_{(-\infty,b)} = 0\}. \tag{2.3}$$

In $e^{\omega t}\mathcal{S}'_+$ we define an equivalence relation: $f \sim g \iff f - g \in \mathcal{A}$. Let $\mathcal{B}$ denote

$$\mathcal{B} = e^{\omega t}\mathcal{S}'_+/\mathcal{A}, \tag{2.4}$$

then $b \in \mathcal{B} \iff b = class(T) \equiv cl(T)$, where $T \in e^{\omega t}\mathcal{S}'_+$.

Definition 1. Let $I = [a,b)$, $a \ge 0$, $b < \infty$, $a < b$. $\mathcal{D}'_\omega(I)$ is a subspace of $e^{\omega t}\mathcal{D}'((-\infty,b))$,

$$\mathcal{D}'_\omega(I) = \{T \in e^{\omega t}\mathcal{D}'((-\infty,b));\ \exists T^0 \in e^{\omega t}\mathcal{S}'_+,\ T^0|_{(-\infty,b)} = T\}. \tag{2.5}$$

It is eaily seen that a distribution defined by $e^{\omega t} f(t)$, $f(t) = 0$, $t < 0$ and $f(t) \subset L_{loc}([0,b))$ belong to $\mathcal{D}'_\omega(I)$.

Definition 2. The LT of $f \in \mathcal{D}'(I)_\omega$ is defined by

$$\mathcal{L}(\mathcal{D}'(I)) = \mathcal{L}(e^{\omega t}\mathcal{S}'_+)/\mathcal{L}(\mathcal{A}). \tag{2.6}$$

Hence, if $f \in \mathcal{D}'_\omega(I)$, then

$$\mathcal{L}(f) = cl(\mathcal{L}(f^0)),\ \text{where}\ f^0 \in e^{\omega t}\mathcal{S}'_+,\ f^0|_{(-\infty,b)} = f. \tag{2.7}$$

Let $f \in \mathbf{C}^{(p)}((-\infty,b))$, $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; H be a function such that $H(x) = 0$, $x \in (-\infty,a)$; $H(x) = 1$, $x \in [a,b)$. Let $[Hf]$ denote the regular distribution defined by Hf, $[Hf] \in \mathcal{D}'_*(I)$. By $[f^a_{(p)}]$, $p \in \mathbb{N}$, we denote the distribution defined by the function $f_a^{(p)}$; $f_a^{(p)}(x) = f^{(p)}(x)$, $x \in (a,b)$, $f_a^{(p)}(x) = 0$, $x \in (-\infty, a)$ and is not defined for $x = a$. Then

$$\begin{aligned} D^p[Hf] &= [f_a^{(p)}] + f^{(p-1)}(a)\delta(x-a) + \cdots + f(a)\delta^{(p-1)}(x-a) \\ &= [f_a^{(p)}] + R_p(f) = [H_a f^{(p)}] + R_p(f) = [Hf^{(p)}] + R_p(f), \end{aligned} \tag{2.8}$$

where $D^p[Hf]$ is the derivative of order p in the sense of distributions.

For $f^{(\alpha)}, 0 < \alpha < 1$,

$$\left[f_a^{(\alpha)}\right] = \frac{1}{\Gamma(1-\alpha)} D[(H_a f) * (\theta(\tau)\tau^{-\alpha})] - \frac{1}{\Gamma(1-\alpha)} \lim_{t \to a^+} \int_a^t \frac{f(t-\tau)}{\tau^\alpha}\, d\tau\, \delta(t-a), \tag{2.9}$$

where θ is the Heviside function. If $f = O(t^{-(\beta-\alpha)})$, $t \to 0$, $0 < \alpha < \beta < 1$, then

$$\lim_{t \to a^+} \int_a^t \frac{f(t-\tau)}{\tau^\alpha}\, d\tau = 0. \tag{2.10}$$

3. Solutions of a mathematical model

3.1. Mathematical model of lateral vibration of a visco-elastic rod

Consider a rod BC of length $l = 1$ simply supported at both ends as shown in Fig. 1. The axis of the rod is initially straight. At the end C the rod is loaded by a compressive forc F that is function of time t and whose action line coincides with the rod axis in the undeformed (initial) state.

Fig. 1. Coordinate system and local configuration

If we consider the planar motion with the force $F = B + A\theta(t - t_0)$, $t_0 > 0$, (θ is Heaviside's function), then the following system

$$\begin{aligned} \frac{\partial^2 m}{\partial \xi^2} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial t^2} = 0; \\ \frac{\partial^2 u}{\partial \xi^2} + \mu_1 D_t^\alpha \frac{\partial^2 u}{\partial \xi^2} + \mu_2 D_t^\beta \frac{\partial^2 u}{\partial \xi^2} = m + \mu\, D_t^\alpha m, \end{aligned} \tag{3.1}$$

$0 < t$, $0 < \xi < 1$, with the boundary conditions:

$$m(0,t) = 0; \;\; m(1,t) = 0; \;\; u(0,t) = 0, \;\; u(1,t) = 0, \; t \geq 0, \tag{3.2}$$

and $\lambda = B + A\theta(t - t_0)$ is the mathematical model of this motion (cf. [1]). Similar models one can find in [2], [6], [7], [10], [11].

In [1] system (3.1), (3.2) has been treated but for $\lambda = B + A\delta(t - t_0)$, where δ is Dirac's distribution. Here we will consider a system some more general than system (3.1)

$$\begin{aligned} \frac{\partial^2 m}{\partial \xi^2} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial t^2} &= g(t) \sin\, k\pi\xi\,,\ k \in \mathbb{N}, \\ \frac{\partial^2 u}{\partial \xi^2} + \mu_1\, D_t^\alpha \frac{\partial^2 u}{\partial \xi^2} + \mu_2 D_t^\beta\, \frac{\partial^2 u}{\partial \xi^2} &= m + \mu\, D_t^\alpha m, \end{aligned} \tag{3.1'}$$

$0 < t,\ 0 < \xi < 1$, with the same boundary conditions (3.2), where $g \in \mathbf{C}([0, \infty))$ and without any growth condition. In case $g = 0$ system (3.1') becomes (3.1).

Let us remark that in system (3.1') we have a coefficient which is a discontinuous function with a discontinuity in $t = t_0 > 0$. Therefore, we shall construct a solution for the domain $D_1 = \{(\xi, t);\ 0 < \xi < 1,\ 0 < t < t_0\}$ with boundary conditions (3.2) and initial conditions in $t = 0$ and then for the domain $D_2 = \{(\xi, t);\ 0 < \xi < 1,\ t_0 < t\}$ using the method presented in part 2.

3.2. Separation of variables

Let us suppose that the solutions of the system (3.1'), (3.2) have the from

$$m(\xi, t) = M(\xi)V(t),\ u(\xi, t) = U(\xi)T(t). \tag{3.3}$$

It is easily seen that for M and U which satisfy boundary condition (3.2) we have for every $k \in \mathbb{N}$ a solution:

$$M_k(\xi) = C_k \sin\, k\pi\xi;\ \ U_k(\xi) = C_k \sin\, k\pi\xi,\ \ k \in \mathbb{N}. \tag{3.4}$$

To find the corresponding values T_k and V_k we have to solve the system:

$$\begin{aligned} T_k^{(2)}(t) - \lambda(k\pi)^2 T_k(t) - (k\pi)^2 V_k(t) &= g(t) \\ V_k(t) + V_k^{(\alpha)}(t) + (k\pi)^2 T_k(t) + + \mu_1 (k\pi)^2 T_k^{(\alpha)} + \mu_2 (k\pi)^2 T_k^{(\beta)}(t) &= 0,\ \ 0 < t. \end{aligned} \tag{3.5}$$

3.3. Localization of the solution

The defined domain D_1 calls for the analysis of system (3.5) in the interval $(0, t_0)$ with initial condition in $t = 0$. In this case $\lambda = B$, where B is a constant. By the results in [1], where system (3.5) has been treated by the same method, the solution to system (3.5) is

$$\begin{aligned} T_k(t) &= T_k(0)(\mu F_{\alpha+1}(t) + F_1(t)) \\ &\quad + T_k^{(1)}(0)(\mu F_\alpha(t) + F_0(t)) + ((\mu F_\alpha + F_0) * g)(t) \end{aligned} \tag{3.6}$$

$$\begin{aligned} V_k(t) &= -(k\pi)^2 \{T_k(0)[\mu_2 F_{1+\beta}(t) + \mu_1 F_{1+\alpha}(t) + F_1(t)] + T_k^{(1)}(0)[\mu_1 F_\alpha(t) \\ &\quad + \mu_2 F_\beta(t) + F_0(t)] + ((\mu_1 F_\alpha + \mu_2 F_\beta + F_0) * g)(t)\},\ \ 0 < t < t_0, \end{aligned} \tag{3.7}$$

where

$$F_p(t) = \mathcal{L}^{-1}(s^p / \Delta_{k0}(s))(t), \tag{3.8}$$

$$\begin{aligned} \Delta_{k0}(s) &= \mu s^{2+\alpha} + s^2 + (k\pi)^2(\mu_1 (k\pi)^2 - B\mu)s^\alpha + (k\pi)^4 \mu_2 s^\beta \\ &\quad + (k\pi)^2((k\pi)^2 - B) = \mu s^{2+\alpha} + s^2 + a s^\alpha + b s^\beta + d. \end{aligned} \tag{3.9}$$

With regard to domain D_2 we have to find a solution to system 3.(5) but in the interval (t_0, b) for any $b > t_0$, and $\lambda = B + A$. The mode of proceedings is the following: First we have to localize the supposed solution to (3.5) on the interval (t_0, b). Then we suppose that there exists a solution T_k, V_k to (3.5) such that $H_{t_0}T_k \in \mathbf{C}^1([t_0, b))$, $(H_{t_0}T_k)^{(2)} \in \mathbf{L}^1([t_0, b))$; $V_k \in \mathbf{C}((t_0, b)) \cap \mathbf{L}^1([t_0, b))$.

By (2.8) and (2.9) to (3.5), on the interval $[t_0, b)$, it corresponds in $\mathcal{D}'_*([t_0, b))$

$$\begin{aligned} &D^2[H_{t_0}T_k] - (A+B)(k\pi)^2[H_{t_0}T_k] - (k\pi)^2[H_{t_0}V_k] \\ &\qquad = T_k(t_0)D^1\delta(t-t_0) + T_k^{(1)}(t_0)\delta(t-t_0) + [H_{t_0}g] \\ &[H_{t_0}V_k] + \mu D^\alpha[H_{t_0}V_k] + (k\pi)^2[H_{t_0}T_k] \\ &\qquad + \mu_1(k\pi)^2D^\alpha[H_{t_0}T_k] + \mu_2(k\pi)^2D^\beta[H_{t_0}T_k] = 0. \end{aligned} \tag{3.10}$$

Let $T_k^0 \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}_+ + [t_0, b))$ such that $T_k^0|_{(-\infty,b)} = H_{t_0}T_k$, $V_k^0 \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}_+ + [t_0, b))$ such that $V_k^0|_{(-\infty,b)} = H_{t_0}V_k$ and $g^0 \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}_+ + [t_0, b))$, $g_0|_{(-\infty,b)} = H_{t_0}g$. Applying to (3.10) the defined LT, we get

$$\begin{aligned} &(s^2 - (A+B)(k\pi)^2)\widehat{T}_k^0(s) - (k\pi)^2\widehat{V}_k^0(s) \\ &\qquad = T_k(t_0)se^{-t_0 s} + T_k^{(1)}(t_0)e^{-t_0 s} + \widehat{g}^0(s) + \widehat{r}_1(s), \\ &(1+\mu s^\alpha)\widehat{V}_k^0(s) + (k\pi)^2(1 + \mu_1 s^\alpha + \mu_2 s^\beta)\widehat{T}_k^0(s) = \widehat{r}_2(s). \end{aligned} \tag{3.11}$$

where r_1 and $r_2 \in \mathcal{A}$. By $\widehat{T}_k^0$ is denoted the LT of T_k^0.

When we solve this system in $\widehat{T}_k^0$, $\widehat{V}_k^0$ and use the inverse LT, we get

$$\begin{aligned} &(H_{t_0}T_k)(t) = T_k(t_0)\theta(t-t_0)(\mu G_{\alpha+1}(t-t_0) + G_1(t-t_0)) + \\ &+ T_k^{(1)}(t_0)\theta(t-t_0)(\mu G_\alpha(t-t_0) + G_0(t-t_0)) + +((\mu G_\alpha + G_0) * H_{t_0}g)(t), \end{aligned} \tag{3.12}$$

$$\begin{aligned} (H_{t_0}V_k) = -(k\pi)^2\{&T_k(t_0)\theta(t-t_0)[\mu_2 G_{1+\beta}(t-t_0) + \mu_1 G_{\alpha+1}(t-t_0) + G_1(t-t_0)] + \\ &+ T_k^{(1)}(t_0)\theta(t-t_0)[\mu_1 G_\alpha(t-t_0) + \mu_2 G_\beta(t-t_0) + G_0(t-t_0)] \\ &+ [(\mu_1 G_\alpha + \mu_2 G_\beta + G_0) * (H_{t_0}g)](t)\}, \quad t_0 < t < b, \end{aligned} \tag{3.13}$$

where $G_p(t) = \mathcal{L}^{-1}(s^p/\Delta'_{k0})$, Δ'_{k0} equals Δ_{k0} in which instead of B we have $A+B$. Therefore, we can use the properties of solution (3.6), (3.7) to system (3.5) (cf. [1]) taking into account that we have $A + B$ instead of B.

3.4. Global solution

Now we have a solution for the domain D_1, given by (3.6), (3.7) and a solution for the domain D_2, given by (3.12), (3.13). It is easily observe that not only solutions (3.6), (3.7) and (3.12), (3.13) have the same structure but also the solution in [1] for $\lambda = B$ which has the form of solution (3.6), (3.7) but for $0 < t < b$, where b is any positive number. Therefore, we can use the results contained in [1]. In this way we know that T_k and V_k, given by (3.6), (3.7) and (3.12), (3.13) but in case $g = 0$, have the following properties which do not depend on the value of B:

1. $T_k \in \mathbf{C}^2((0, t_0]) \cap \mathbf{C}^1([0, t_0])$, $T_k^{(2)} \in \mathbf{L}^1([0, t_0]) \cap \mathbf{C}((0, t_0])$, $T_k^{(2)}(t)$ is not bounded at $t = 0$ and $V_k \in \mathbf{L}^1([0, t_0]) \cap \mathbf{C}((0, t_0])$, $V_k(t)$ is not bounded at $t = 0$. If

additionally $T_k(0)=0$, then $T_k \in \mathbf{C}^2([0,t_0])$ and $V_k \in \mathbf{C}([0,t_0]), V_k(0)=0, V_k^{(1)} \in \mathbf{L}^1([0,t_0]) \cap \mathbf{C}((0,t_0])$ and $V_k^{(1)}(t)$ is not bounded at $t=0$.

2. $T_k \in \mathbf{C}^2((t_0,b)) \cap \mathbf{C}^1([t_0,b)), T_k^{(2)} \in \mathbf{L}^1([t_0,b)) \cap \mathbf{C}((t_0,b))$ and $V_k \in \mathbf{L}^1([t_0,b)) \cap \mathbf{C}((t_0,b))$. V_k is not bounded at $t=t_0$. If $T_k(0)=0$, then $T_k \in \mathbf{C}^{(2)}([t_0,b))$ and $V_k^{(1)} \in \mathbf{L}^1([t_0,b)) \cap \mathbf{C}((t_0,b))$ for every $b>0$; $V_k^{(1)}(t)$ is not bounded at $t=t_0$.

3. If $t \to 0^+$, then: $T_k(t) \to T_{k0}, T_k^{(1)}(t) \to T_{k_0}^1$, $V_k(t) \sim t^{-(\beta-\alpha)}$ and $T_k^{(2)}(t)$ is also not bounded.

We add a property more:

4. If $T_{k0}=0$, then $\lim_{t\to 0^+} T_k^{(2)}(t)=0$.

Let us prove it.

In this case T_k^0 and V_k^0 which correspond to $T_k = T_k^0|_{(-\infty,b)}$ and $V_k = V_k^0|_{(-\infty,b)}$, respectively, are

$$T_k^0(t) = T_{k0}^1(\mu F_\alpha(t) + F_0(t)) \tag{3.14}$$

$$V_k^0(t) = T_{k0}^1(\mu_1 F_\alpha(t) + \mu_2 F_\beta(t) + F_0(t)). \tag{3.15}$$

Hence, by LT we get

$$\widehat{T}_k^0(s) = T_{k0}^1(\mu f_\alpha(s) + f_0(s)), \qquad \widehat{V}_k^0(s) = T_{k0}^1(\mu_1 f_\alpha(s) + \mu_2 f_\beta(s) + f_0(s)).$$

Let us consider the function

$$\begin{aligned} f(s) &= s^2\widehat{T}_k^0(s) - T_{k0}^1 = s^2(\widehat{T}_k^0(s) - T_{k0}^1/s^2) \\ &= T_{k0}^1 s^2(\mu s^\alpha/\Delta_{k0}(s) + 1/\Delta_{k0}(s) - 1/s^2) \\ &= -T_{k0}^1(af_\alpha(s) + bf_\beta(s) + df_0(s)) = -T_{k0}^1\mathcal{L}(aF_\alpha(t) + bF_\beta(t) + dF_0(t))(s). \end{aligned}$$

It follows that there exists continuous function $F(t),\ t \ge 0$,

$$F(t) = -T_{k0}^1(aF_\alpha(t) + bF_\beta(t) + dF_0(t)),\ \ t \ge 0,$$

such that $\mathcal{L}(F)(s) = s^2(\mathcal{L}(T_k^0)(s) - T_{k0}^1/s^2)$.

Let $G(t)$ denote the function $G(t) = \int_0^t d\tau \int_0^\tau F(u)du, \quad t \ge 0$. Then by [3, I, Kap. 2, §12, Satz 5] $\mathcal{L}(G)(s) = \widehat{T}_k^0 - T_{k0}^1/s^2$. Since $G \in \mathbf{C}^2([0,\infty))$, $G(0)=0$, $G^{(1)}(0)=0$ and since by (9) $T_k^0(t) = G(t) + T_{k0}^1 t, t \ge 0$, it is easily seen that $T_k^0 \in \mathbf{C}^2([0,\infty))$, as well and $T_k^0(0)=0$, $(T_k^0)^{(1)}(0) = T_{k0}^1$, $(T_k^0)^{(2)}(t) = G^{(2)}(t) = F(t)$. By the properties of $F_p(t)$ (cf. [1]) $\lim_{t\to 0^+}(T_k^0)^{(2)}(t) = 0$. $T_k = T_k^0|_{[0,b)}$ has the same properties as T_k^0. Namely, we proved that $T_k \in \mathbf{C}^2([0,b))$.

We need some propertes more in case $g \ne 0$.

5. The function

$$Z(t) = ((\mu F_\alpha + F_0) * g)(t) = \int_0^t (\mu F_\alpha(t-\tau) + F_0(t-\tau))g(\tau)\,d\tau$$

has the following properties which follows from the properties of F_p:

$$Z'(t) = \int_0^t (\mu F_{\alpha+1}(t-\tau) + F_0(t-\tau)g(\tau)\,d\tau \in \mathbf{C}([0,b))$$

for every $b > 0$. Also $Z'(t) \to 0, t \to 0$. Consequently $Z(t) \in \mathbf{C}^1([0,b))$ and $Z(t) \to 0, Z'(t) \to 0, t \to 0$.

The properties of the function $U(t) = ((\mu F_\alpha + F_0) * H_{t_0} g)(t)$ can be deduced from the properties of $Z(t)$. Let us do it.

$$\begin{aligned}((\mu F_\alpha + F_0) * H_{t_0} g)(t) &= \int_{t_0}^{t} (\mu F_\alpha(t-\tau) + F_0(t-\tau))(H_{t_0} g)(\tau)\, d\tau \\ &= \int_0^{t-t_0} (\mu F_\alpha(t-u-\tau_0) + F_0(t-u-t_0)(H_{t_0} g)(u+t_0)\, du \\ &= \int_0^{x} \mu F_\alpha(x-u) + F_0(x-u)(H_{t_0} g)(u+t_0)\, du,\end{aligned}$$

where $x = t - t_0$.

It follows from the properties of $Z(t)$ that $u(t) = ((\mu F_\alpha + F_0) * H_{t_0} g) \in \mathbf{C}^1([t_0, b))$, and that $U(t) \to 0$, $U^{(1)}(t) \to 0$, $t \to t_{0+}$. Taking into account these cited properties we extend solution (3.6), (3.7) continuously to the whole $(0,b), b > t_0$, and also its derivatives, if it is possible. Let us start with T_k. We know that there exist $\lim_{t\to t_0-} T_k(t) = T_{kt_0}$ and $\lim_{t\to t_0-} T_k^{(1)} = T_{kt_0}^1$. We define $T_k(t_0)$ and $T_k^{(1)}(t_0)$ so that $T_k(t_0) = T_{kt_0}$ and $T_k^{(1)}(t_0) = T_{kt_0}^1$. Then by (3.12) it follows $\lim_{t\to t_{0+}} T_k(t) = T_k(t_0)$ and $\lim_{t\to t_0+} T_k^{(1)}(t) = T_k^{(1)}(t_0)$. Consequently $T_k \in \mathbf{C}^1([0,b)), b > t_0$. Since $\lim_{t\to 0^+} T_k^{(2)}(t)$ is not bounded, $\lim_{t\to t_0+} T_k^{(2)}(t)$ is not bounded, as well. Thus $T_k^{(2)}(t)$ is not bounded at $t = t_0$.

Consequently, the extended solution (3.6) to $(0,b), b > t_0$, *is a solution in the generalized sense.*

If $T_k(0) = 0$, *and* $g = 0$ *then* $\lim_{t\to 0^+} T_k^{(2)}(t) = 0$ *and the solution (3.6) can be extended as a classical solution to* $(0,b), b > t_0$.

With regard to $V_k(t), V_k(t) \sim T_k(t)(t_0)\mu_2 t^{-(\beta-\alpha)}, t \to 0^+$. *Then there exists* $V_k^{(\alpha)}(t), t \geq 0$ *and* $V_k(t) \in \mathbf{L}^1([0,b)), b > t_0$, *is a classical solution.*

4. Some remarks in the case $g = 0$

4.1. Stability of solutions

We know (cf. [1]) that system (3.5), for $\lambda = B$, $k_0 \in \mathbb{N}$ and $B > (k_0\pi)^2$ has solutions which are not stable. But if $B = (k_0\pi)^2$, these solutions are stable. We can use this fact in the following way: If the rod at its end is loaded by the constant compressive force $F = B$ and the state is not stable, we can always, in a moment $t = t_0$ change the force F_2 by an A such that for $t \geq t_0$ the rod is loaded by $A + B = (k\pi)^2$ which leads to a stable state.

4.2. A new solution to (3.1), (3.2)

We can construct a new solution to (3.1), (3.2) by the sinus series (cf. [6]):

$$u(\xi,t) = \sum_{k=1}^{\infty} c_k T_k(t) \sin k\pi\xi, \qquad m(\xi,t) = \sum_{k=1}^{\infty} c_k T_k(t) \sin k\pi\xi \tag{4.1}$$

Since $c_k, k \in \mathbb{N}$, are arbitrary constants, we can always give to $c_k, k \in \mathbb{N}$, such values that the first series (4.1) converges in $[a,b]$, $0 < a < b < \infty$ and if $T_k(0) = 0, k \in \mathbb{N}$, then it converges in $[0,b], 0 < b < \infty$. With regard to the second series, $c_k, k \in \mathbb{N}$, can be defined such that the series and its first derivative converge in [0,b], $0 < b < \infty$ and the second derivative converge in $[a,b]$, $0 < a < b < \infty$. But if $T_k(0) = 0$, then a can be sero, as well.

4.3. More complex force *F*

If the force F is more complex,

$$F(t) = B + \sum_{i=1}^{n} A_i \theta(t - t_i), 0 < t_1 < \cdots < t_n,$$

the procedure can be repeated for every interval $[t_i, t_{i+1})$.

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