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MIXED NORM SPACES OF DIFFERENCE SEQUENCES AND MATRIX TRANSFORMATIONS

A. M. Jarrah and E. Malkowsky

Abstract. In this paper, we generalise the definition of mixed norm spaces, define mixed norm spaces of difference sequences, determine their β-duals, and characterise matrix transformations on them. We obtain many known results as special cases.

1. Introduction

Let $1 \le p < \infty$. By ω we denote the set of all complex sequences $x = (x_k)_{k=1}^{\infty}$.

In 1968, *Maddox* [5] introduced and studied the sets

$$w_0^p = \left\{ x \in \omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0 \right\} \text{ and } w_\infty^p = \left\{ x \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k|^p < \infty \right\}$$

of sequences that are strongly summable and bounded, respectively, with index p by the Cesàro method of order 1. He also observed that the *sections* $1/n \sum_{k=1}^{n}$ can be replaced by the *blocks* $1/2^{\nu+1} \sum_{k=2^\nu}^{2^{\nu+1}-1}$, and that the *section* and *block norms*

$$\|x\| = \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \text{ and } \|x\|' = \sup_{\nu \ge 0} \left(\frac{1}{2^{\nu+1}} \sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p \right)^{1/p}$$

are equivalent.

In 1974, *Jagers* [3] studied the *Cesàro sequence spaces*

$$ces(p) = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}$$

which are Banach spaces with the norm

$$\|x\|_{ces(p)} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p}.$$

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It can be found in [1] that an equivalent norm on $ces(p)$ is

$$\|x\| = \left(\sum_{\nu=0}^{\infty} 2^{\nu(1-p)}\left(\sum_{k=2^\nu}^{2^{\nu+1}-1}|x_k|\right)^p\right)^{1/p}.$$

In 1969, *Hedlund* [2] introduced the *mixed norm spaces*

$$\ell(p,q) = \left\{x \in \omega : \sum_{\nu=0}^{\infty}\left(\sum_{k=2^\nu}^{2^{\nu+1}-1}|x_k|^p\right)^{q/p} < \infty\right\} \text{ see also } \textit{Kellogg } [4];$$

obviously the Cesàro sequence spaces $ces(p)$ are weighted $\ell(p,1)$ mixed norm spaces. Results on the equivalence of block and section norms on mixed norm spaces can also be found in [1].

In this paper, we generalise the definition of mixed norm spaces, define mixed norm spaces of difference sequences, determine their β-duals, and characterise matrix transformations on them. We obtain many known results as special cases.

2. Notations and Definitions

Let ℓ_∞, c, c_0 and ϕ be the sets of all bounded, convergent, null and finite sequences, cs and bs be the sets of all convergent and bounded series, and $\ell_p = \{x \in \omega : \sum_{k=1}^\infty |x_k|^p < \infty\}$ for $1 \le p < \infty$.

By e and $e^{(n)}$ $(n = 1, 2, \dots)$, we denote the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

An *FK space* X is a complete linear metric sequence space with continuous *coordinates* $P_k : X \to \mathbb{C}$ where $P_k(x) = x_k$ for all $x \in X$ and $k = 1, 2, \dots$; a *BK space* is a normed FK space. We say that an FK space $X \supset \phi$ has AK if $x^{[m]} = \sum_{k=1}^m x_k e^{(k)} \to x$ $(m \to \infty)$ for every sequence $x = (x_k)_{k=1}^\infty \in X$; $x^{[m]}$ is called the *m-section of the sequence x*.

If X and Y are subsets of ω, and z is a sequence, we write $z^{-1} * Y = \{x \in \omega : xz = (x_k z_k)_{k=1}^\infty \in Y\}$ and $M(X,Y) = \bigcap_{x\in X} x^{-1} * Y = \{z \in \omega : zx \in Y \text{ for all } x \in X\}$ for the *multiplier of X and Y*. In the special cases when $Y = \ell_1$ or $Y = cs$, we write $z^\alpha = z^{-1} * \ell_1$ or $z^\beta = z^{-1} * cs$, and the sets $X^\alpha = M(X, \ell_1)$ and $X^\beta = M(X, cs)$ are called the α- or *Köthe-Toeplitz-* and *β-duals of X*.

Let $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix of complex numbers, x be a sequence and X be a subset of ω. Then we write $A_n = (a_{nk})_{k=1}^\infty$ and $A^k = (a_{nk})_{n=1}^\infty$ for the sequences in the n-th row and the k-th column of A, respectively, A^T for the transpose of A, $A_n(x) = \sum_{k=1}^\infty a_{nk}x_k$ $(n = 1, 2, \dots)$ and $A(x) = (A_n(x))_{n=1}^\infty$, provided $A_n \in x^\beta$ for all n. The set $X_A = \{z \in \omega : A(z) \in X\}$ is called the *matrix domain of A in X*. Given any subsets X and Y of ω, then (X, Y) denotes the class of all matrices A that map X into Y, that is for which $A_n \in X^\beta$ for all n and $A(x) \in Y$ for all $x \in X$, or equivalently $A \in (X, Y)$ if and only if $X \subset Y_A$.

Throughout, let $(k(\nu))_{\nu=0}^\infty$ be a strictly increasing sequence of integers with $k(0) = 1$ and I_ν be the set of all integers k with $k(\nu) \le k \le k(\nu+1) - 1$ $(\nu = 0, 1, \dots)$. Given any sequence x, then, for each $\nu = 0, 1, \dots$, $x^{\langle\nu\rangle} = \sum_{k\in I_\nu} x_k e^{(k)}$ is the *ν-block of the sequence x*. Let $X, Y \supset \phi$ be sequence spaces, normed with $\|\cdot\|_X$ and $\|\cdot\|_Y$. We define the *generalised mixed norm spaces*

$$Z = [Y, X]^{\langle k(\nu)\rangle} = \left\{ z \in \omega : \left(\|z^{\langle \nu \rangle}\|_X \right)_{\nu=0}^{\infty} \in Y \right\}$$

and put

$$g(z) = \left\| \left(\|z^{\langle \nu \rangle}\|_X \right)_{\nu=0}^{\infty} \right\|_Y \ (z \in Z). \tag{2.1}$$

Since $\phi \subset X$, $\|z^{\langle \nu \rangle}\|_X$ is defined for every $z \in \omega$ and for all $\nu = 0, 1, \ldots$. Hence the sequence $y = (y_\nu)_{\nu=0}^{\infty}$ with $y_\nu = \|z^{\langle \nu \rangle}\|_X$ $(\nu = 0, 1, \ldots)$ is defined. Furthermore, since $\phi \subset X, Y$, we obviously have $\phi \subset Z$.

Finally, let $\Delta = (\delta_{nk})_{n,k=1}^{\infty}$ be the matrix with $\delta_{nn} = 1$, $\delta_{n,n-1} = -1$ and $\delta_{nk} = 0$ otherwise. Then we define the *mixed norm spaces of difference sequences*

$$Z_\Delta = \left([Y, X]^{\langle k(\nu)\rangle} \right)_\Delta .$$

We consider a few special cases.

Example 2.1. (a) Let $1 \le p < \infty$ and $1 \le r < \infty$. Then we obtain

$$[\ell_r, \ell_p]^{\langle k(\nu)\rangle} = \left\{ z \in \omega : \sum_{\nu=0}^{\infty} \left(\sum_{k \in I_\nu} |z_k|^p \right)^{r/p} < \infty \right\},$$

$$[l_r, \ell_\infty]^{\langle k(\nu)\rangle} = \left\{ z \in \omega : \sum_{\nu=0}^{\infty} \left(\max_{k \in I_\nu} |z_k| \right)^{r} < \infty \right\},$$

$$[c_0, \ell_p]^{\langle k(\nu)\rangle} = \left\{ z \in \omega : \lim_{\nu \to \infty} \sum_{k \in I_\nu} |z_k|^p = 0 \right\} \text{ and}$$

$$[\ell_\infty, \ell_p]^{\langle k(\nu)\rangle} = \left\{ z \in \omega : \sup_{\nu \ge 0} \sum_{k \in I_\nu} |z_k|^p < \infty \right\}.$$

In the special case of $r = p$ and $1 \le p \le \infty$, we have $[\ell_p, \ell_p]^{\langle k(\nu)\rangle} = \ell_p$.

If $k(\nu) = 2^\nu$ for $\nu = 0, 1, \ldots$, then $[\ell_r, \ell_p]^{\langle k(\nu)\rangle} = \ell(r, p)$, the mixed norm spaces in [2, 4].

If $k(\nu) = \nu + 1$ for $\nu = 0, 1, \ldots$, then we also obtain the classical sequence spaces $[\ell_r, \ell_1]^{\langle k(\nu)\rangle} = \ell_r$, $[c_0, \ell_1]^{\langle k(\nu)\rangle} = c_0$ and $[\ell_\infty, \ell_1]^{\langle k(\nu)\rangle} = \ell_\infty$.

(b) Let $1 \le p < \infty$ and $k(\nu) = 2^\nu$ for all ν. If $d_\nu = (1/k(\nu+1))^{1/p}$ for $\nu = 0, 1, \ldots$ then

$$[d^{-1} * c_0, \ell_p]^{\langle k(\nu)\rangle} = w_0^p \quad \text{and} \quad [d^{-1} * \ell_\infty, \ell_p]^{\langle k(\nu)\rangle} = w_\infty^p \ [5].$$

If $d_\nu = 2^{\nu(1/p-1)}$ for $\nu = 0, 1, \ldots$ then we obtain the Cesàro sequence spaces or weighted mixed norm spaces $[d^{-1} * \ell_p, \ell_1]^{\langle k(\nu)\rangle} = ces(p)$ [3].

Example 2.2. (a) Let $1 \le p < \infty$ and $1 \le r < \infty$. Then we obtain

$$\left([\ell_r, \ell_p]^{\langle k(\nu)\rangle} \right)_\Delta = \left\{ z \in \omega : \sum_{\nu=0}^{\infty} \left(\sum_{k \in I_\nu} |z_k - z_{k-1}|^p \right)^{r/p} < \infty \right\} \text{ etc.}$$

If $k(\nu) = \nu + 1$ for $\nu = 0, 1, \ldots$ then we obtain the *sets of sequences of bounded variation*

$$bv^p = \left([\ell_p, \ell_1]^{\langle k(\nu)\rangle} \right)_\Delta = \left\{ z \in \omega : \sum_{\nu=0}^{\infty} |z_{\nu+1} - z_\nu|^p < \infty \right\} \ [12],$$

and the sets of difference sequences that are convergent to zero or bounded $([c_0,\ell_1]^{\langle k(\nu)\rangle})_\Delta=(c_0)_\Delta=c_0(\Delta)$ and $([\ell_\infty,\ell_1]^{\langle k(\nu)\rangle})_\Delta=(\ell_\infty)_\Delta=\ell_\infty(\Delta)$ [9].

(b) Let $(\mu_k)_{k=0}^\infty$ be an increasing sequence of positive reals tending to infinity and $d_\nu=1/\mu_{k(\nu+1)}$ for $\nu=0,1,\dots$. Then we obtain the sets of sequences that are *μ-strongly convergent to zero* or *bounded*, respectively, *with index p*

$$c_0(\mu)=\mu^{-1}*\left([d^{-1}*c_0,\ell_p]^{<k(\nu)>}\right)_\Delta$$
$$=\left\{z\in\omega:\lim_{\nu\to\infty}\frac{1}{\mu^p_{k(\nu+1)}}\sum_{k\in I_\nu}|\mu_kz_k-\mu_{k-1}z_{k-1}|^p=0\right\}$$

and $c_\infty(\mu)=\mu^{-1}*([d^{-1}*\ell_\infty,\ell_p]^{\langle k(\nu)\rangle})_\Delta$ [10].

3. The topological properties of the spaces Z and Z_Δ

Here we study the topological properties of $Z=[Y,X]^{\langle k(\nu)\rangle}$ and $Z_\Delta=([Y,X]^{\langle k(\nu)\rangle})_\Delta$.

A norm $\|\cdot\|$ on a sequence space X is said to be *monotonous* if $|x_k|\le|\tilde{x}_k|$ $(k=1,2,\dots)$ for $x,\tilde{x}\in X$ implies $\|x\|\le\|\tilde{x}\|$. A subset X of ω is called *normal* if $x\in X$ and $|y_k|\le|x_k|$ $(k=1,2,\dots)$ for a sequence y together imply $y\in X$.

Given $z\in\omega$, we write $y=(y_\nu)_{\nu=0}^\infty$ for the sequence with $y_\nu=\|z^{\langle\nu\rangle}\|_X$ $(\nu=0,1,\dots)$.

Proposition 3.1. *Let $X\supset\phi$ and $Y\supset\phi$ be normed sequence spaces and $Z=[Y,X]^{\langle k(\nu)\rangle}$.*

(a) If Y is normal and $\|\cdot\|_X$ is monotonous then Z is normal.

(b) If $\|\cdot\|_Y$ is monotonous then Z is normed with respect to g defined in (2.1). If, however, $\|\cdot\|_Y$ is not monotonous, then g does not satisfy the triangle inequality in general.

Proof. (a) If $z\in Z$ and $\tilde{z}\in\omega$ with $|\tilde{z}_k|\le|z_k|$ for all k, then the monotony of $\|\cdot\|_X$ implies $|\tilde{y}_\nu|\le|y_\nu|$ for all ν. Since Y is normal, it follows that $\tilde{z}\in Z$.

(b) We show that g satisfies the triangle inequality, since it obviously satisfies the other properties of a norm. Let $z,\tilde{z}\in Z$. Then $\|(z+\tilde{z})^{<\nu>}\|_X=\|z^{\langle\nu\rangle}+\tilde{z}^{<\nu>}\|_X\le\|z^{\langle\nu\rangle}\|_X+\|\tilde{z}^{<\nu>}\|_X=y_\nu+\tilde{y}_\nu$ $(\nu=0,1,\dots)$, so $z+\tilde{z}\in Z$, since Y is normal. Furthermore, by the monotony of $\|\cdot\|_Y$, we have $g(z+\tilde{z})\le\|y+\tilde{y}\|_Y\le\|y\|_Y+\|\tilde{y}\|_Y=g(z)+g(\tilde{z})$.

To prove the last part, we choose $Y=(\ell_1)_\Delta$, $\|y\|_{bv}=\|\Delta(y)\|_1$, $k(\nu)=\nu+1$ $(\nu=0,1,\dots)$ and $X=\ell_1$ with its natural norm. Then obviously $\|\cdot\|_Y$ is not monotonous. If we choose $z=e^{(1)}+e^{(2)}+e^{(3)}$ and $\tilde{z}=e^{(1)}-e^{(2)}+e^{(3)}$ then $g(z+\tilde{z})=2g(e^{(1)}+e^{(3)})=8>4=g(z)+g(\tilde{z})$. ∎

Theorem 3.2. *Let $X\supset\phi$ be a normed sequence space, $Y\supset\phi$ be a normal BK space and $\|\cdot\|_Y$ be monotonous. Then Z is a BK space with $\|\cdot\|_Z=g$ where*

g is defined in (2.1). Furthermore, if Y has AK and $\|\cdot\|_X$ is monotonous then Z also has AK.

Proof. By Proposition 3.1, $\|\cdot\|_Z = g$ is a norm. We write $\|\cdot\| = \|\cdot\|_Z$ for short. First, since Y is a BK space, $\|z^{(m)} - z\| \to 0$ $(m \to \infty)$ implies $\|(z^{(m)})^{<\nu>} - z^{\langle\nu\rangle}\|_X \to 0$ $(m \to \infty)$ for each ν, and it follows that $|z_k^{(m)} - z_k| \to 0$ $(m \to \infty)$ for each $k \in I_\nu$ $(\nu = 0, 1, \dots)$, since for each ν there are only finitely many $k \in I_\nu$. Thus the norm $\|\cdot\|$ is stronger than the metric of ω on Z.

To show that Z is complete with $\|\cdot\|$, let $(z^{(m)})_{m=1}^\infty$ be a Cauchy sequence in Z, hence in ω by what we have just shown. Thus there exists $z \in \omega$ such that

$$z^{(m)} \to z \ (m \to \infty) \text{ in } \omega. \tag{3.1}$$

Furthermore, by the completeness of Y, there is $y \in Y$ such that

$$y^{(m)} = \left(\|(z^{(m)})^{<\nu>}\|_X\right)_{\nu=0}^\infty \to y \ (m \to \infty) \text{ in } Y. \tag{3.2}$$

From (3.1), we conclude $z_k^{(m)} \to z_k$ $(m \to \infty)$ for each k, hence $(z^{(m)})^{<\nu>} \to z^{\langle\nu\rangle}$ $(m \to \infty)$ for each ν, and so

$$y_\nu^{(m)} = \|(z^{(m)})^{<\nu>}\|_X \to \|z^{\langle\nu\rangle}\|_X \ (m \to \infty) \text{ or each } \nu. \tag{3.3}$$

Since Y is a BK space, (3.2) implies $y_\nu^{(m)} \to y_\nu$ $(m \to \infty)$ for each ν, and so, by (3.3), $y_\nu = \|z^{\langle\nu\rangle}\|_X$ for each ν and $y = (\|z^{<\nu>}\|_X)_{\nu=0}^\infty \in Y$, hence $z \in Z$. This shows that Z is complete.

Finally, let Y have AK and $\|\cdot\|_X$ be monotonous. We show that Z as AK. Let $z = (z_k)_{k=1}^\infty \in Z$ and $\varepsilon > 0$ be given. For each $m \in \mathbb{N}$ let ν_m be the uniquely defined integer for which $m \in I_{\nu_m}$. We define the sequence $y = (y_\nu)_{\nu=0}^\infty$ by $y_\nu = \|z^{\langle\nu\rangle}\|_X$ for $\nu = 0, 1, \dots$, and write $y^{[\mu]} = \sum_{\nu=0}^\mu y_\nu e^{(\nu)}$ for $\mu = 0, 1, \dots$. Since Y has AK, there exists an integer μ_0 such that $\|y - y^{[\mu]}\|_Y < \varepsilon$ for all $\mu \geq \mu_0$.We choose $m_0 = k(\mu_0 + 1)$. Let $m \geq m_0$ be given. Then $\nu_m \geq \mu_0 + 1$ and

$$\tilde{y}_\nu = \|(z - z^{[m]})^{\langle\nu\rangle}\|_X = 0 = y_\nu - y_\nu^{[\nu_m - 1]} \text{ for } 0 \leq \nu \leq \nu_m - 1$$

$$\tilde{y}_{\nu_m} = \|(z - z^{[m]})^{\langle\nu_m\rangle}\|_X = \|(0, \dots, 0, z_{m+1}, \dots)^{\langle\nu_m\rangle}\|_X \leq \|z^{\langle\nu_m\rangle}\|_X = y_{\nu_m}$$

since $\|\cdot\|_X$ is monotonous, and

$$\tilde{y}_\nu = \|(z - z^{[m]})^{\langle\nu\rangle}\|_X = \|z^{\langle\nu\rangle}\|_X = y_\nu \text{ for all } \nu \geq \nu_m + 1.$$

Thus $|\tilde{y}_\nu| \leq |y_\nu - y_\nu^{[\nu_m - 1]}|$ for all ν, and so

$$\|\tilde{y}\|_Y = \left\| \left(\left\| (z - z^{[m]})^{<\nu>} \right\|_X \right)_{\nu=0}^\infty \right\|_Y \leq \|y - y^{[\nu_m - 1]}\|_Y < \varepsilon,$$

since $\|\cdot\|_Y$ is monotonous. Therefore $z^{[m]} \to z$ $(m \to \infty)$. ∎

As an immediate consequence of Theorem 3.2 and [14, Theorem 4.3.12, p. 63], we obtain

Corollary 3.3. *Let $X \supset \phi$ be a normed sequence space, $Y \supset \phi$ be a normal BK space and $\|\cdot\|_Y$ be monotonous. Then Z_Δ is a BK space with $\|z\|_\Delta = g(\Delta(z))$ $(z \in Z_\Delta)$ where g is defined in (2.1).*

We close this section with a few examples.

EXAMPLE 3.4. (a) Let $1 \le p < \infty$ and $1 \le r < \infty$. Then $[\ell_r, \ell_p]^{\langle k(\nu)\rangle}$ and $[c_0, \ell_p]^{\langle k(\nu)\rangle}$ are BK spaces with AK with

$$\|z\|_{(r,p)} = \left(\sum_{\nu=0}^{\infty}\left(\sum_{k\in I_\nu}|z_k|^p\right)^{r/p}\right)^{1/r} \text{ and } \|z\|_{(\infty,p)} = \sup_{\nu\ge 0}\left(\sum_{k\in I_\nu}|z_k|^p\right)^{1/p},$$

and $[\ell_\infty, \ell_p]^{\langle k(\nu)\rangle}$ is a BK space with $\|\cdot\|_{(\infty,p)}$; moreover, $[c_0, \ell_p]^{\langle k(\nu)\rangle}$ is a closed subspace of $[\ell_\infty, \ell_p]^{\langle k(\nu)\rangle}$ by [14, Corollary 4.2.4, p. 56]. The spaces $[l_r, \ell_\infty]^{\langle k(\nu)\rangle}$ are BK spaces with AK with

$$\|z\|_{(r,\infty)} = \left(\sum_{\nu=0}^{\infty}\left(\max_{k\in I_\nu}|z_k|\right)^r\right)^{1/r}.$$

(b) Let $1 \le p < \infty$ and the sequences $(k(\nu))_{\nu=0}^\infty$ and $d = (d_\nu)_{\nu=0}^\infty$ be defined as in Example 2.1(b). Since c_0 and ℓ_∞ are BK spaces and c_0 has AK, and since $d_\nu \neq 0$ for all ν, the sets $Y_0 = d^{-1} * c_0$ and $Y_\infty = d^{-1} * \ell_\infty$ are BK spaces with $\|y\|_{Y_\infty} = \|yd\|_\infty$, and Y_0 has AK (cf. [14, Theorems 4.3.6 and 4.3.12, pp. 62 and 63]. Furthermore, obviously $\|\cdot\|_{Y_\infty}$ and $\|\cdot\|_p$ are monotonous. Therefore w_0^p and w_∞^p are BK spaces with

$$\|z\|' = \sup_{\nu=0}\left(\frac{1}{2^{\nu+1}}\sum_{k=2^\nu}^{2^{\nu+1}-1}|x_k|^p\right)^{1/p},$$

and w_0^p has AK; moreover w_0^p is a closed subspace of w_∞^p by [14, Corollary 4.2.4, p. 56].

EXAMPLE 3.5. (a) Let $1 \le p < \infty$ and $1 \le r < \infty$. Then $([\ell_r, \ell_p]^{\langle k(\nu)\rangle})_\Delta$, $([c_0, \ell_p]^{\langle k(\nu)\rangle})_\Delta$ and $([\ell_\infty, \ell_p]^{\langle k(\nu)\rangle})_\Delta$ are BK spaces with

$$\|z\|_{(r,p)_\Delta} = \left(\sum_{\nu=0}^{\infty}\left(\sum_{k\in I_\nu}|z_k - z_{k-1}|^p\right)^{r/p}\right)^{1/r} \qquad \text{and}$$

$$\|z\|_{(\infty,p)_\Delta} = \sup_{\nu\ge 0}\left(\sum_{k\in I_\nu}|z_k - z_{k-1}|^p\right)^{1/p},$$

and $([c_0, \ell_p]^{\langle k(\nu)\rangle})_\Delta$ is a closed subspace of $([\ell_\infty, \ell_p]^{\langle k(\nu)\rangle})_\Delta$ by Example 3.4(a) and [14, Theorem 4.3.14, p. 64].

(b) Let the sequences $(\mu_k)_{k=0}^\infty$ and $d = (d_\nu)_{\nu=0}^\infty$ be as in Example 2.2(b). Then, by a similar argument as that used in Example 3.4(b), $c_0(\mu)$ and $c_\infty(\mu)$ are BK spaces with

$$\|z\|_{c_\infty(\mu)} = \sup_{\nu\ge 0}\frac{1}{\mu_{k(\nu+1)}}\left(\sum_{k\in I_\nu}|\mu_k x_k - \mu_{k-1}x_{k-1}|^p\right)^{1/p},$$

and $c_0^p(\mu)$ is a closed subspace of $c_\infty^p(\mu)$ by Example 3.4 and [14, Theorem 4.3.14, p. 64].

4. The β-duals of the spaces Z and matrix transformations

In this section, we determine the β-duals of the spaces Z and characterise some classes of matrix transformations between them.

We denote the closed unit ball in a normed space X by $B_X = \{x \in X : \|x\| \leq 1\}$. If X is a normed sequence space and $a \in \omega$, we write $\|a\|_{X,\alpha} = \sup_{x\in B_X} \sum_{k=0}^{\infty}|a_k x_k|$ and $\|a\|_{X,\beta} = \sup_{x\in B_X} |\sum_{k=0}^{\infty} a_k x_k|$ provided the expressions exist and are finite which is the case whenever X is a BK space and $a \in X^{\alpha}$ or $a \in X^{\beta}$ (cf. [14, Theorems 4.3.15 and 7.2.9, pp. 64 and 107].

A norm on a sequence space X is said to be KB if the set $\mathcal{P} = \{P_k : X \to \mathbb{C} : P_k(x) = x_k\ (x \in X)\ k = 1, 2, \dots\}$ of coordinates is equicontinuous, that is if there is a constant K such that $|x_k| \leq K\|x\|$ $(k = 1, 2, \dots)$ for all $x \in X$. If X is a Banach sequence space with a norm which is KB then it is obviously a BK space. Conversely the norm of a BK space need not be KB in general. To see this, we choose $X = (\ell_\infty)_\Delta$ with $\|x\| = \sup_k |x_k - x_{k-1}|$, a BK space, and the sequence x with $x_k = k$ for $k = 1, 2, \dots$.

If X is a normed sequence space then we write $X^{\delta} = \{a \in \omega : \|a\|_{X,\alpha} < \infty\}$.

THEOREM 4.1. *Let X and Y be normed sequence spaces with $X, Y \supset \phi$ and $\|\cdot\|_Y$ be monotonous.*

(a) Then $[Y^{\delta}, X^{\delta}]^{\langle k(\nu)\rangle} \subset ([Y, X]^{\langle k(\nu)\rangle})^{\delta}$.

(b) If, in addition, the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are both KB, $\|\cdot\|_X$ is monotonous and Y is normal then $([Y, X]^{\langle k(\nu)\rangle})^{\delta} \subset [Y^{\delta}, X^{\delta}]^{\langle k(\nu)\rangle}$.

Proof. We write $Z = [Y, X]^{\langle k(\nu)\rangle}$ and $W = [Y^{\delta}, X^{\delta}]^{\langle k(\nu)\rangle}$. Since $\|\cdot\|_{X,\alpha}$ and $\|\cdot\|_{Y,\alpha}$ are norms on X^{δ} and Y^{δ}, respectively, and $\phi \subset X, Y$, the set $W = \{w \in \omega : (\|w^{\langle\nu\rangle}\|_{X,\alpha})_{\nu=0}^{\infty} \in Y^{\delta}\}$ is defined.

(a) First we observe that Z is a normed space with $\|\cdot\| = g$ by Proposition 3.1. Let $a \in W$ and $z \in B_Z$. Then $z^{\langle\nu\rangle} \in X$ for $\nu = 0, 1, \dots$, and, by the definition of the norm $\|\cdot\|_{X,\alpha}$, we have

$$\sum_{k=1}^{\infty} |a_k^{<\nu>} z_k^{<\nu>}| \leq \|a^{\langle\nu\rangle}\|_{X,\alpha}\, \|z^{\langle\nu\rangle}\|_X \text{ for all } \nu = 0, 1, \dots. \tag{4.1}$$

We define the sequences y and b by $y_\nu = \|z^{\langle\nu\rangle}\|_X$ and $b_\nu = \|a^{\langle\nu\rangle}\|_{X,\alpha}$ $(\nu = 0, 1, \dots)$. Then $y \in B_Y$ and $b \in Y^{\delta}$, and it follows from (4.1) that $\sum_{k=1}^{\infty}|a_k z_k| = \sum_{\nu=0}^{\infty}\sum_{k=1}^{\infty}|a_k^{<\nu>} z_k^{<\nu>}| \leq \sum_{\nu=0}^{\infty}|b_\nu y_\nu| \leq \|b\|_{Y,\alpha}$ by the definition of the norm $\|\cdot\|_{Y,\alpha}$. Therefore $\|a\|_{Z,\alpha} = \sup_{z\in B_Z} \sum_{k=1}^{\infty}|a_k z_k| \leq \|b\|_{Y,\alpha} < \infty$, that is $a \in Z^{\delta}$.

(b) First we observe that Z is a BK space by Theorem 3.2. Let $a \in Z^{\delta}$ be given. Then

$$\sum_{k=1}^{\infty} |a_k z_k| \leq \|a\|_{Z,\alpha} = K_1 < \infty \text{ for all } z \in B_Z. \tag{4.2}$$

We have to show $a \in W$, that is

$$\sup_{y\in B_Y} \sum_{\nu=0}^{\infty} \|a^{\langle\nu\rangle}\|_{X,\alpha}\, |y_\nu| < \infty. \tag{4.3}$$

We note that $\|a^{\langle\nu\rangle}\|_{X,\alpha}$ is defined for each ν. For if $x \in B_X$ is given then $\sum_{k=1}^{\infty}|a_k^{<\nu>}x_k| = \sum_{k\in I_\nu}|a_k^{<\nu>}x_k|$, and since $\|\cdot\|_X$ is KB, there is a constant K_2 such that

$$\sum_{k=1}^{\infty}|a^{<\nu>}x_k| \le K_2\sum_{k\in I_\nu}|a_k^{<\nu>}|\,\|x\|_X \le K_2\sum_{k\in I_\nu}|a_k^{<\nu>}|,$$

hence $\|a^{\langle\nu\rangle}\|_{X,\alpha} = \sup_{x\in B_X}\sum_{k=1}^{\infty}|a_k^{<\nu>}x_k| \le K_2\sum_{k\in I_\nu}|a_k^{<\nu>}| < \infty$ for all ν. Now let $y \in B_Y$ be given. By the definition of $\|\cdot\|_{X,\alpha}$, for every ν, we can choose a sequence $x(\nu) = (x_k(\nu))_{k=1}^{\infty} \in B_X$ such that $\|a^{\langle\nu\rangle}\|_{X,\alpha} \le \sum_{k=1}^{\infty}|a_k^{<\nu>}x_k(\nu)| + 2^{-(\nu+1)}$, whence

$$\sum_{\nu=0}^{\infty}\|a^{\langle\nu\rangle}\|_{X,\alpha}\,|y_\nu| \le \sum_{\nu=0}^{\infty}\left(\sum_{k=1}^{\infty}|a_k^{<\nu>}x_k(\nu)y_\nu| + \frac{1}{2^{\nu+1}}|y_\nu|\right). \tag{4.4}$$

Since $\|\cdot\|_Y$ is KB, there is a constant K_3 such that $|y_\nu| \le K_3\|y\|_Y \le K_3$ for all $\nu = 0,1,\dots$, and it follows from (4.4) that

$$\sum_{\nu=0}^{\infty}\|a^{\langle\nu\rangle}\|_{X,\alpha}\,|y_\nu| \le \sum_{\nu=0}^{\infty}\left(\sum_{k=1}^{\infty}|a_k^{<\nu>}x_k(\nu)y_\nu|\right) + K_3. \tag{4.5}$$

We define the sequence z by $z_k = x_k(\nu)y_\nu$ $(k \in I_\nu;\ \nu = 0,1,\dots)$. Then $\|z^{\langle\nu\rangle}\|_X = |y_\nu|\|(x(\nu))^{<\nu>}\|_X$ for all $\nu = 0,1,\dots$. Since, for each ν, we have $|(x_k(\nu))^{<\nu>}| \le |x_k(\nu)|$ $(k = 1,2,\dots)$, the monotony of $\|\cdot\|_X$ implies $\|(x(\nu))^{<\nu>}\|_X \le \|x(\nu)\|_X = 1$ $(\nu = 0,1,\dots)$, hence $\|z^{\langle\nu\rangle}\|_X \le |y_\nu|$ $(\nu = 0,1,\dots)$. Since Y is normal, this implies $(\|z^{\langle\nu\rangle}\|_X)_{\nu=0}^{\infty} \in Y$, that is $z \in Z$. Furthermore, $|\,|y_\nu|\,| \le |y_\nu|$ for $\nu = 0,1,\dots$ implies $(|y_\nu|)_{\nu=0}^{\infty} \in Y$, since Y is normal, and the monotony of $\|\cdot\|_Y$ yields $\|z\|_Z \le \|(|y_\nu|)_{\nu=0}^{\infty}\|_Y \le \|y\|_Y$. Now (4.5) and (4.2) together imply

$$\sum_{\nu=0}^{\infty}\|a^{\langle\nu\rangle}\|_{X,\alpha}\,|y_\nu| \le \sum_{\nu=0}^{\infty}\sum_{k\in I_\nu}|a_k^{<\nu>}z_k| + K_3 = \sum_{k=1}^{\infty}|a_kz_k| \le K_1\|z\|_Z + K_3 \le K_1 + K_3.$$

Since $y \in B_Y$ was arbitrary, condition (4.3) follows. ∎

If X is a BK space then $X^\alpha = X^\delta$ by [14, Theorem 4.3.15, p. 64], and if X is normal then $X^\alpha = X^\beta$. Therefore we obtain from Proposition 3.1 and Theorems 3.2 and 4.1

COROLLARY 4.2. *Let X be a normed sequence space, Y be a normal BK space and the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ be monotonous and KB. Then $Z^\alpha = ([Y,X]^{\langle k(\nu)\rangle})^\alpha = [Y^\alpha, X^\alpha]^{\langle k(\nu)\rangle}$. If, in addition, X is normal then $Z^\beta = [Y^\beta, X^\beta]^{\langle k(\nu)\rangle}$.*

EXAMPLE 4.3. (a) Let $1 \le p < \infty$, $1 \le r < \infty$, q and s be the *conjugate numbers of* p and r, that is $q = \infty$ for $p = 1$ and $q = p/(p-1)$ for $1 < p < \infty$ and s defined similarly. Since the norms $\|\cdot\|_{\ell_p,\beta}$ and $\|\cdot\|_q$ and $\|\cdot\|_{\ell_\infty,\alpha}$ and $\|\cdot\|_1$ are equivalent on ℓ_p^β and on $\ell_\infty^\beta = c_0^\beta$, we have $([\ell_r,\ell_p]^{\langle k(\nu)\rangle})^\beta = [\ell_s,\ell_q]^{\langle k(\nu)\rangle}$ and $([c_0,\ell_p]^{\langle k(\nu)\rangle})^\beta = ([\ell_\infty,\ell_p]^{\langle k(\nu)\rangle})^\beta = [\ell_1,\ell_q]^{\langle k(\nu)\rangle}$.

(b) Let $\mathcal{U}$ denote the set of all sequences u with $u_k \neq 0$ for all k. If $u \in \mathcal{U}$ then we write $1/u = (1/u_k)_{k=1}^\infty$, and it is obvious that $(u^{-1} * X)^\beta = (1/u)^{-1} * X^\beta$ for arbitrary subsets X of ω. Let the sequences $k(\nu)$ and d be defined as in Example 2.1(b). Then

$$(w_0^p)^\beta = (w_\infty^p)^\beta = \mathcal{M}_p = \begin{cases} \left\{a \in \omega : \sum_{\nu=0}^\infty 2^{\nu+1} \max_{k \in I_\nu} |a_k| < \infty\right\} & (p=1) \\ \left\{a \in \omega : \sum_{\nu=0}^\infty 2^{\nu+1} \Big(\sum_{k \in I_\nu} |a_k|^q\Big)^{1/q} < \infty\right\} & (1 < p < \infty). \end{cases}$$

Now we characterise some classes of matrix transformations between mixed norm spaces.

Let $(m(\mu))_{\mu=0}^\infty$ be a strictly increasing sequence of integers with $m(0) = 1$ and $M_\mu = \{m \in \mathbb{N} : m(\mu) \leq m \leq m(\mu+1) - 1\}$ $(\mu = 0, 1, \ldots)$. Furthermore, let T denote the set of all sequences $(t_\mu)_{\mu=0}^\infty$ of integers such that for each μ there is one and only one $t_\mu \in M_\mu$.

First we give a result that characterises the classes (X, Y) where X is any BK space and Y is any of the spaces ℓ_∞, c_0, ℓ_1, $[\ell_\infty, \ell_1]^{\langle m(\mu)\rangle}$, $[\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$ or $[c_0, \ell_1]^{\langle m(\mu)\rangle}$.

Theorem 4.4. *Let X be a BK space, or a BK space with AK in the cases marked $*$. We write $\sup_N$ for the supremum taken over all finite subsets N of $\mathbb{N}_0$. Then the conditions for $A \in (X, Y)$ when Y is any of the spaces ℓ_∞, c_0, ℓ_1, $[\ell_\infty, \ell_1]^{\langle m(\mu)\rangle}$, $[\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$ or $[c_0, \ell_1]^{\langle m(\mu)\rangle}$ can be read from the table*

From \ *To*	ℓ_∞	c_0	ℓ_1	$[\ell_\infty, \ell_1]^{\langle m(\mu)\rangle}$	$[\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$	$[c_0, \ell_1]^{\langle m(\mu)\rangle}$
X	*(1.)*	**(2.)*	*(3.)*	*(4.)*	*(5.)*	**(6.)*

where

(1.) *(1.1)* *where (1.1)* $\sup_n \|A_n\|_{X,\beta} < \infty$

(2.) *(1.1) and (2.1)* *where (2.1)* $\lim_{n\to\infty} a_{nk} = 0$ *for each* k

(3.) *(3.1)* *where (3.1)* $\sup_N \Big\| \sum_{n \in N} A_n \Big\|_{X,\beta} < \infty$

(4.) *(4.1)* *where (4.1)* $\sup_\mu \Big(\max_{M(\mu) \subset M_\mu} \Big\| \sum_{m \in M(\mu)} A_m \Big\|_{X,\beta}\Big) < \infty$

(5.) *(5.1)* *where (5.1)* $\sup_N \Big(\sup_{t \in T} \Big\| \sum_{\mu \in N} A_{t_\mu} \Big\|_{X,\beta}\Big) < \infty$

(6.) *(4.1) and (6.1)* *where (6.1)* $\lim_{\mu\to\infty} \sum_{n \in M_\mu} |a_{nk}| = 0$ *for each* k.

Proof. **(1.)** is [11, Theorem 1.23, p. 155], **(2.)** follows from **(1.)** and [14, 8.3.6, p. 123], since c_0 is a closed subspace of ℓ_∞, and **(3.)** is [8, Satz 1].

(4.) We have $A \in (X, [\ell_\infty, \ell_1]^{\langle m(\mu)\rangle})$ if and only if $A_n \in X^\beta$ for all n and

$$(\|(A(x))^{\langle m(\mu)\rangle}\|_1)_{\mu=0}^\infty \in \ell_\infty \text{ for all } x \in X. \tag{4.6}$$

Since by a well-known inequality [13]

$$\max_{M(\mu)\subset M_\mu}\Big|\sum_{m\in M(\mu)} A_m(x)\Big| \le \sum_{m\in M_\mu}|A_m(x)| = \|(A(x))^{\langle\mu\rangle}\|_1 \le$$

$$\le 4\cdot \max_{M(\mu)\subset M_\mu}\Big|\sum_{m\in M(\mu)} A_m(x)\Big| \text{ for all } \mu \text{ and all } x\in X,$$

it follows by condition (1.1) that (4.6) holds if and only if condition (4.1) is satisfied.

(5.) First we assume that condition (5.1) holds. Then obviously $A_n \in X^\beta$ for all n. Let $x \in X$ be given. For each $\mu = 0, 1, \dots$, let $m_\mu \in M_\mu$ be such that $|A_{m_\mu}(x)| = \max_{m\in M_\mu}|A_m(x)|$. Let μ_0 be an arbitrary nonnegative integer. Then we have by the definition of the norm $\|\cdot\|_{X,\beta}$

$$\sum_{\mu=0}^{\mu_0}\|(A(x))^{\langle m(\mu)\rangle}\|_\infty = \sum_{\mu=0}^{\mu_0}|A_{m_\mu}(x)| \le 4\cdot \sup_{\substack{N\subset\mathbb{N}_0\\ N\text{finite}}}\Big|\sum_{\mu\in N}A_{m_\mu}(x)\Big|$$

$$\le 4\cdot\left(\sup_{\substack{N\subset\mathbb{N}_0\\ N\text{finite}}}\Big\|\sum_{\mu\in N}A_{m_\mu}\Big\|_{X,\beta}\right)\|x\| \le 4\cdot \sup_{\substack{N\subset\mathbb{N}_0\\ N\text{finite}}}\left(\sup_{t\in T}\Big\|\sum_{\mu\in N}A_{t_\mu}\Big\|_{X,\beta}\right)\|x\| < \infty.$$

Since μ_0 was arbitrary, it follows that $(\|(A(x))^{\langle m(\mu)\rangle}\|_\infty)_{\mu=0}^\infty \in \ell_1$, that is $A(x) \in [\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$.

Conversely we assume $A \in [\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$. Since X and $[\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$ are BK spaces, the map $f_A : X \to [\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$ with $f_A(x) = A(x)$ $(x \in X)$ is continuous (cf. [14, Theorem 4.2.8, p. 57]. Hence there is a constant K such that

$$\|f_A(x)\|_{(1,\infty)} = \|A(x)\|_{(1,\infty)} \le K\|x\| \text{ for all } x \in X. \tag{4.7}$$

We observe that $A_m \in X^\beta$ for all m implies $\sum_{\mu\in N} A_{t_\mu} \in X^\beta$ for all finite subsets N of $\mathbb{N}_0$ and for all sequences $t \in T$, and so by (4.7), $|\sum_{\mu\in N} A_{t_\mu}(x)| \le \sum_{\mu=0}^\infty |A_{t_\mu}(x)| \le \|f_A(x)\|_{(1,\infty)} \le K\|x\|$. Now condition (5.1) follows from the definition of the norm $\|\cdot\|_{X,\beta}$.

(6.) By Example 3.4(a), $[c_0, \ell_1]^{\langle m(\mu)\rangle}$ is a closed subspace of $[\ell_\infty, \ell_1]^{\langle m(\mu)\rangle}$. Thus **(6.)** is an immediate consequence of **(4.)** and [14, 8.3.6, p. 123]. ∎

We obtain as an immediate consequence of Example 4.3 and Theorem 4.4

COROLLARY 4.5. *Let $1 < r < \infty$ and $1 < p \le \infty$ and s and q be the conjugate numbers of r and p. Then the conditions for $A \in ([\ell_r, \ell_p]^{\langle k(\nu)\rangle}, Y)$ where Y is any of the spaces in Theorem 4.4 can be read from the table*

To / *From*	ℓ_∞	c_0	ℓ_1	$[\ell_\infty, \ell_1]^{\langle m(\mu)\rangle}$	$[\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$	$[c_0, \ell_1]^{\langle m(\mu)\rangle}$
$[\ell_r, \ell_p]^{\langle k(\nu)\rangle}$	*(1.)*	*(2.)*	*(3.)*	*(4.)*	*(5.)*	*(6.)*

where

(1.) *(1.1)* *where (1.1)* $\sup_n \sum_{\nu=0}^{\infty} \Big(\sum_{k\in I_\nu} |a_{nk}|^q\Big)^{s/q} < \infty$

(2.) *(1.1) and (2.1)* *where (2.1) is (2.1) in Theorem 4.4*

(3.) *(3.1)* *where (3.1)* $\sup_N \sum_{\nu=0}^{\infty} \Big(\sum_{k\in I_\nu} \Big|\sum_{n\in N} a_{nk}\Big|^q\Big)^{s/q} < \infty$

(4.) *(4.1)* *where (4.1)*

$$\sup_\mu \Big(\max_{M(\mu)\subset M_\mu} \sum_{\nu=0}^{\infty} \Big(\sum_{k\in I_\nu} \Big|\sum_{m\in M(\mu)} a_{mk}\Big|^q\Big)^{s/q}\Big) < \infty$$

(5.) *(5.1)* *where (5.1)* $\sup_N \Big(\sup_{t\in T} \sum_{\nu=0}^{\infty} \Big(\sum_{k\in I_\nu} \Big|\sum_{\mu\in N} a_{t_\mu,k}\Big|^q\Big)^{s/q}\Big) < \infty$

(6.) *(4.1) and (6.1)* *where (6.1) is (6.1) in Theorem 4.4.*

If $r = 1$ or $p = 1$ replace $\sum_{\nu=0}^{\infty}$ or $\sum_{k\in I_\nu}$ by $\sup_{\nu\geq 0}$ or $\max_{k\in I_\nu}$ in conditions (1.1), (3.1), (4.1) and (5.1) in **(1.)**–**(6.)**. *The conditions for $A \in ([c_0, \ell_p]^{\langle k(\nu)\rangle}), Y)$ are those in* **(1.)**–**(6.)** *with $s = 1$ in (1.1), (3.1), (4.1) and (5.1). Finally $([\ell_\infty, \ell_p]^{\langle k(\nu)\rangle}, Y) = ([c_0, \ell_p]^{\langle k(\nu)\rangle}, Y)$ for $Y \neq c_0, [c_0, \ell_1]^{\langle m(\mu)\rangle}$.*

Now we give the *dual result* of Theorem 4.4. We write T' for the set of all strictly increasing sequences $t = (t_\nu)_{\nu=0}^{\infty}$ of integers such that for each ν there is one and only one $t_\nu \in I_\nu$.

THEOREM 4.6. *Let W be a BK space with AK and $Y = W^\beta$. Then the conditions for $A \in (X, Y)$ where X is any of the spaces ℓ_∞, c_0, ℓ_1, $[\ell_1, \ell_\infty]^{\langle k(\nu)\rangle}$, $[\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}$ or $[c_0, \ell_1]^{\langle k(\nu)\rangle}$ can be read from the table*

To \ From	ℓ_∞	c_0	ℓ_1	$[\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}$	$[\ell_1, \ell_\infty]^{\langle k(\nu)\rangle}$	$[c_0, \ell_1]^{\langle k(\nu)\rangle}$
Y	*(1.)*	*(2.)*	*(3.)*	*(4.)*	*(5.)*	*(6.)*

where

(1.) *(1.1)* *where (1.1)* $\sup_N \|\sum_{n\in N} A^n\|_Y < \infty$

(2.) *(1.1)*

(3.) *(3.1)* *where (3.1)* $\sup_n \|A^n\|_Y < \infty$

(4.) *(4.1)* *where (4.1)* $\sup_N \Big(\sup_{t\in T'} \|\sum_{\nu\in N} A^{t_\nu}\|_Y\Big) < \infty$

(5.) *(5.1)* *where (5.1)* $\sup_N \Big(\max_{K(\nu)\subset K_\nu} \|\sum_{m\in K(\nu)} A^m\|_Y\Big) < \infty$

(6.) *(4.1).*

Proof. Since X is a BK space with AK when X is any of the spaces c_0, ℓ_1, $[\ell_1, \ell_\infty]^{\langle k(\nu)\rangle}$ and $[c_0, \ell_1]^{\langle k(\nu)\rangle}$, we have $A \in (X, Y)$ if and only if $A^T \in (W, X^\beta)$ by [14, Theorem 8.3.9, p. 124], and **(2.)**, **(3.)**, **(5.)** and **(6.)** are immediate consequences of Theorem 4.4 **(3.)**, **(1.)**, **(4.)** and **(5.)**. Furthermore, since $c_0^{\beta\beta} = \ell_\infty$ and $([c_0, \ell_1]^{\langle k(\nu)\rangle})^{\beta\beta} = [\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}$, and $(X, Y) = (X^{\beta\beta}, Y)$ by [14, Theorem 8.3.9, p. 124], **(1.)** and **(4.)** follow from **(2.)** and **(6.)**. ■

We obtain as an immediate consequence of Theorem 4.6

COROLLARY 4.7. *Let $1 < r < \infty$ and $1 < p < \infty$. Then the conditions for $A \in (X, [\ell_r, \ell_p]^{\langle m(\mu)\rangle})$ where X is any of the spaces in Theorem 4.4 can be read from the table*

From / *To*	ℓ_∞	c_0	ℓ_1	$[\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}$	$[\ell_1, \ell_\infty]^{\langle k(\nu)\rangle}$	$[c_0, \ell_1]^{\langle k(\nu)\rangle}$
$[\ell_r, \ell_p]^{\langle m(\mu)\rangle}$	*(1.)*	*(2.)*	*(3.)*	*(4.)*	*(5.)*	*(6.)*

where

(1.) *(1.1) where (1.1)* $\sup_N \sum_{\mu=0}^{\infty} \left(\sum_{k \in M_\mu} \left| \sum_{n \in N} a_{kn} \right|^p \right)^{r/p} < \infty$

(2.) *(1.1)*

(3.) *(3.1) where (3.1)* $\sup_n \sum_{\mu=0}^{\infty} \left(\sum_{k \in M_\mu} |a_{kn}|^p \right)^{r/p} < \infty$

(4.) *(4.1) where (4.1)* $\sup_N \left(\sup_{t \in T'} \sum_{\mu=0}^{\infty} \left(\sum_{k \in M_\mu} \left| \sum_{\nu \in N} a_{k,t_\nu} \right|^p \right)^{r/p} \right) < \infty$

(5.) *(5.1) where (5.1)* $\sup_N \left(\max_{k(\nu) \in K_\nu} \sum_{\nu=0}^{\infty} \left(\sum_{k \in M_\mu} \left| \sum_{m \in K(\nu)} a_{km} \right|^p \right)^{r/p} \right) < \infty$

(6.) *(4.1).*

5. The β-duals of the spaces Z_Δ and matrix transformations

In this section, we determine the β-duals of the sets Z_Δ and characterise some matrix transformations between them.

First we prove a general result which reduces the determination of $(X_\Delta)^\beta$ for arbitrary BK spaces with AK to that of X^β and the characterisation of the class (X, c_0).

If X is a normed space, we write X^* its continuous dual, that is the set of all continuous linear functionals f on X with the norm $\|f\| = \sup_{x \in B_X} |f(x)|$.

Let $\Sigma = (\sigma_{nk})_{n,k=1}^{\infty}$ be the matrix with $\sigma_{nk} = 1$ for $1 \leq k \leq n$ and $\sigma_{nk} = 0$ for $k > n$ $(n = 1, 2, \ldots)$. Then $x = \Delta(\Sigma(x)) = \Sigma(\Delta(x))$ for all $x \in \omega$. Let $X \subset \omega$

and $Y = X_\Delta$. Then $x \in X$ if and only if $y = \Sigma(x) \in Y$, and $y \in Y$ if and only if $x = \Delta(y) \in X$. If X is a BK space then so is Y and $B_X = B_Y$ by [14, Theorem 4.3.12, p. 63].

Given any sequence a, we write B^a for the matrix with the rows $B_n^a = a_n e^{[n]}$ $(n = 1, 2, \dots)$. Then $B_n^a(x) = a_n \Sigma_n(x) = a_n y_n$ for all $x \in X$, $y = \Sigma(x)$ and all n, that is

$$a \in M(X_\Delta, W) \text{ if and only if } B^a \in (X, W) \text{ for arbitrary subsets } X \text{ and } W \text{ of } \omega. \tag{5.1}$$

THEOREM 5.1. *Let $E = \Sigma^T$. If X is a BK space with AK then $a \in (X_\Delta)^\beta$ if and only if $a \in (X^\beta)_E$ and $V^a \in (X, c_0)$ where V^a is the matrix with the rows $V_n^a = E_n(a)e^{[n]}$ $(n = 1, 2, \dots)$. Furthermore if $a \in (X_\Delta)^\beta$ then*

$$\sum_{k=1}^\infty a_k y_k = \sum_{k=1}^\infty E_k(a)\Delta_k(y) \text{ for all } y \in X_\Delta. \tag{5.2}$$

Proof. We write $Y = X_\Delta$ and $V = V^a$ for short.

First we assume $a \in Y^\beta$. Then $B^a \in (X, cs)$ by (5.1), and so $C = \Sigma B^a \in (X, c)$ by [11, Theorem 3.8, p. 180]. Since c is a closed subspace of ℓ_∞, we have by [14, 8.3.6, p. 123]

$$\lim_{n\to\infty} c_{nk} = \sum_{j=k}^\infty a_j = E_k(a) \text{ exists for all } k \tag{5.3}$$

and

$$C \in (X, \ell_\infty). \tag{5.4}$$

From (5.3), we obtain that the matrix V is defined and

$$\lim_{n\to\infty} v_{nk} = \lim_{n\to\infty} \sum_{j=n}^\infty a_j = 0. \tag{5.5}$$

We also have

$$\sum_{k=1}^{m-1} a_k y_k = \sum_{k=1}^m E_k(a)\Delta_k(y) - \sum_{k=1}^m v_{mk}\Delta_k(y) \text{ for all } m \text{ and all } y. \tag{5.6}$$

Since X is a BK space with AK, condition (5.4) implies $C^T \in (\ell_1, X^\beta)$ by [14, Theorem 8.3.9, p. 124]. Now X^β is a BK space with

$$\|b\|^\beta = \sup_m \sup_{x \in B_X} \left| \sum_{k=1}^m b_k x_k \right| = \sup_m \|b^{[m]}\|_{X,\beta} \ (b \in X^\beta)$$

by [14, Example 4.3.16, p. 65]. Therefore, by [14, Example 8.4.1, p. 126], the columns of the matrix C^T, that is the rows of C are a bounded set in X^β. Thus there is a constant K_1 such that

$$\left| \sum_{k=1}^m c_{nk} x_k \right| \le K_1 \text{ for all } m \text{ and } n \text{ and for all } x \in B_X. \tag{5.7}$$

Now (5.3) implies $|\sum_{k=1}^m E_k(a)x_k| \le K_1$ for all m and all $x \in B_X$. It follows from this and (5.6) that

$$|V_m(x)| \le K_1 + \left| \sum_{k=1}^{m-1} a_k y_k \right| \text{ for all } x \in B_X,\ y \in B_Y \text{ and all } m. \tag{5.8}$$

We define the linear functionals f_m $(m=1,2,\dots)$ on Y by $f_m(y)=\sum_{k=1}^{m-1}a_ky_k$ $(y\in Y)$. We note that $f_m\in Y^*$ for all m, since Y is a BK space. Furthermore $a\in Y^\beta$ implies that $f(y)=\lim_{m\to\infty}f_m(y)$ exists for every $y\in Y$, that is the sequence $(f_m)_{m=1}^\infty$ is pointwise convergent, hence pointwise bounded, and so uniformly bounded by the uniform boundedness principle. Thus there exists a constant K_2 such that $|f_m(y)|=|\sum_{k=1}^{m-1}a_ky_k|\le K_2$ for all $y\in B_Y$ and all m, and it follows from (5.8) that $|V_m(x)|\le K_1+K_2$ for all m and for all $x\in B_X$, hence $\sup_m\|V_m\|_{X,\beta}<\infty$. This and (5.5) imply $V\in(X,c_0)$ by Theorem 4.4**(2.)**; and then (5.6) implies $E(a)\in X^\beta$, that is $a\in(X^\beta)_E$.

If $a\in Y^\beta$ then $E(a)\in X^\beta$ and $V\in(X,c_0)$, as we have just shown, and so (5.2) follows from (5.6).

Conversely, if $a\in(X^\beta)$ and $V\in(X,c_0)$ then $a\in Y^\beta$ by (5.6). ■

Now we give the $(Z_\Delta)^\beta$ in some special cases.

Example 5.2. (a) Let $1\le p\le\infty$, $1\le r\le\infty$ and q and s be the conjugate numbers of p and r. The conditions for $E(a)\in([\ell_r,\ell_p]^{\langle k(\nu)\rangle})^\beta$ and $E(a)\in([c_0,\ell_p]^{\langle k(\nu)\rangle})^\beta$ are given in Example 4.3(a). Corollary 4.5 yields the conditions for $V^a\in([\ell_r,\ell_p]^{\langle k(\nu)\rangle},c_0)$ and $V^a\in([c_0,\ell_p]^{\langle k(\nu)\rangle},c_0)$, the condition $\lim_{n\to\infty}v_{nk}^a=0$ for each k being redundant. For each positive integer n, let $\nu(n)$ denote the uniquely defined integer such that $n\in I_{\nu(n)}$. We define the sequence $b^{s,q}$ by

$$b_n^{s,q}=\begin{cases}\left(\sum\limits_{\nu=0}^{\nu(n)-1}((k(\nu+1)-k(\nu))^{s/q}+(n+1-k(\nu(n))^{s/q}\right)^{1/s} & (1<r\le\infty,1<p\le\infty)\\ (\nu(n)+1)^{1/s} & (1<r\le\infty,p=1)\\ \max\left\{\max\limits_{0\le\nu\le\nu(n)-1}(k(\nu+1)-k(\nu))^{1/q},(n+1-k(\nu(n)))^{1/q}\right\} & (r=1,1<p\le\infty).\end{cases}$$

It is easy to see that condition (1.1) for $A=V^a$ in Corollary 4.5 is equivalent to $E(a)\in(b^{s,q})^{-1}*\ell_\infty$; in the case of $[c_0,\ell_p]^{\langle k(\nu)\rangle}$ $(1\le p<\infty)$, we use the sequence $b^{1,q}$.

Let us mention that the condition $E(a)\in(b^{s,q})^{-1}*\ell_\infty$ becomes redundant is some cases. As in Example 2.2, let $k(\nu)=\nu+1$ $(\nu=0,1,\dots)$. Then, for $1<p<\infty$, we have $bv^p=([\ell_p,\ell_1]^{\langle k(\nu)\rangle})_\Delta$, and $a\in(bv^p)^\beta$ if and only if $\sum_{\nu=0}^\infty|\sum_{k=\nu}^\infty a_k|^q<\infty$ and $\sup_n(n+1)^{1/q}|\sum_{k=n}^\infty a_k|<\infty$, and it is easy to see that, in general neither condition implies the other. If, however, $p=1$, then $bv=([\ell_1,\ell_1]^{\langle k(\nu)\rangle})_\Delta=([\ell_1,\ell_\infty]^{\langle k(\nu)\rangle})_\Delta$, and the conditions $E(a)\in[\ell_\infty,\ell_1]^{\langle k(\nu)\rangle}$ and $E(a)\in b^{\infty,1}*\ell_\infty$ are the same, namely $\sup_n|\sum_{k=n}^\infty a_k|<\infty$ that is $a\in cs$.

(b) Let the sequences μ and d be defined as in Example 2.2(b). First we observe that $a\in(c_0^p(\mu))^\beta$ if and only if $a/u=(a_k/u_k)_{k=1}^\infty\in(([d^{-1}*c_0,\ell_p]^{\langle k(\nu)\rangle})_\Delta)^\beta$. Also $E(a/u)\in[(1/d)^{-1}*\ell_1,\ell_q]^{\langle k(\nu)\rangle}$ if and only if

$$c\in[\ell_1,\ell_q]^{\langle k(\nu)\rangle}\text{ where }c_k=1/d_\nu E_k(a/\mu)=\mu_{k(\nu+1)}\sum_{j=k}^\infty\frac{a_j}{\mu_j}\ (k\in I_\nu;\nu=0,1,\dots).\tag{5.10}$$

Since obviously, for all $u \in \mathcal{U}$ and for all $X, Y \subset \omega$, we have $A \in (u^{-1} * X, Y)$ if and only if $B \in (X, Y)$ where $b_{nk} = a_{nk}/u_k$ for all n and k, it follows that $V^a \in (\mu^{-1} * [d^{-1} * c_0, \ell_p]^{\langle k(\nu)\rangle}, c_0)$ if and only if $\tilde{V}^a \in ([d^{-1} * c_0, \ell_p]^{\langle k(\nu)\rangle}, c_0)$, where $\tilde{v}^a_{nk} = E_n(a/u)$ for $1 \le k \le n$ and $\tilde{v}^a_{nk} = 0$ for $k > n$ $(n = 1, 2, \dots)$. Finally, since $z \in [d^{-1} * c_0, \ell_p]^{\langle k(\nu)\rangle}$ if and only if $y \in [c_0, \ell_p]^{\langle k(\nu)\rangle}$ where $y_k = d_\nu z_k$ $(k \in I_\nu; \nu = 0, 1, \dots)$, we have $\tilde{V}^a \in ([d^{-1} * c_0, \ell_p]^{\langle k(\nu)\rangle}, c_0)$ if and only if $W^a \in ([c_0, \ell_p]^{\langle k(\nu)\rangle}, c_0)$ where $w^a_{nk} = \tilde{v}^a_{nk} 1/d_\nu$ $(k \in I_\nu; \nu = 0, 1, \dots)$ for all $n = 1, 2, \dots$. Again, the condition $\lim_{n\to\infty} w_{nk} = 0$ is redundant, and we need

$$\sup_n \sum_{\nu=0}^{\infty} \|(W^a_n)^{\langle\nu\rangle}\|_q < \infty. \tag{5.11}$$

We define the sequence $b^{1,q}(\mu)$ by

$$b^{1,q}_n(\mu) = \begin{cases} \sum\limits_{\nu=0}^{\nu(n)-1} \mu_{k(\nu+1)}(k(\nu+1) - k(\nu))^{1/q} - \mu_{k(\nu(n)+1)}(n + 1 - k(\nu(n)))^{1/q} & (1 < p < \infty) \\ \sum\limits_{\nu=0}^{\nu(n)} \mu_{k(\nu+1)} & (p = 1). \end{cases}$$

Condition (5.10) is equivalent to

$$\sum_{\nu=0}^{\infty} \mu_{k(\nu+1)} \left(\sum_{k \in I_\nu} \left| \sum_{j=k}^{\infty} \frac{a_j}{\mu_j} \right|^q \right)^{1/q} < \infty \ (1 < p < \infty),$$

$$\sum_{\nu=0}^{\infty} \mu_{k(\nu+1)} \max_{k \in I_\nu} \left| \sum_{j=k} \frac{a_j}{\mu_j} \right| < \infty \ (p = 1),$$

and it is easy to see that condition (5.11) is equivalent to $E(a/\mu) \in (b^{1,q})^{-1} * \ell_\infty$ for $1 < p < \infty$ and $E(a/u) \in (b^{1,\infty}(\mu))^{-1} * \ell_\infty$ for $p = 1$, this condition being redundant, if there are reals s and t with $0 < s \le \mu_{k(\nu)}/\mu_{k(\nu+1)} \le t < 1$ for all ν.

The next result reduces the the characterisation of (X_Δ, Y) to that of (X, Y) and (X, c_0).

THEOREM 5.3. *Let $X \supset \phi$ be a BK space with AK and Y be a subset of ω. Then $A \in (X_\Delta, Y)$ if and only if*

$$E^A \in (X, Y) \ \textit{where} \ e^A_{nk} = \sum_{j=k}^{\infty} a_{nj} \ \textit{for all } n \textit{ and } k \tag{5.12}$$

and

$$V^{A_n} \in (X, c_0) \ \textit{for all } n \tag{5.13}$$

where V^{A_n} is the matrix with the rows $V^{A_n}_m = E_m(A_n)e^{[m]}$ $(m = 1, 2, \dots)$.

Proof. First we assume $A \in (X_\Delta, Y)$. Then $A_n \in (X_\Delta)^\beta$ for all n, hence condition (5.13) holds and

$$E(A_n) \in X^\beta \text{ for all } n \tag{5.14}$$

by Theorem 5.1. Let $x \in X$ be given. Then $A_n \in (X_\Delta)^\beta$ implies

$$(E^A)_n(x) = A_n(\Sigma(x)) \text{ for all } n \tag{5.15}$$

by (5.2). Since $\Sigma(x) \in X_\Delta$, it follows that $A(\Sigma(x)) \in Y$, hence $E^A(x) \in Y$. Thus (5.12) also holds.

Conversely we assume that conditions (5.12) and (5.13) are satisfied. Then (5.14) holds, and this and (5.13) imply $A_n \in (X_\Delta)^\beta$ for all n by Theorem 5.1. Again (5.15) holds and then $A \in (X_\Delta, Y)$. ■

Now we give some characterisations of matrix transformations between Z and Z_Δ.

We obtain as an immediate consequence of Theorems 5.3 and 4.4 and of [11, Theorem 3.8, p. 180]

THEOREM 5.4. *Let X be a BK space with AK and Y be any of the spaces ℓ_∞, c_0, ℓ_1, $[\ell_\infty, \ell_1]^{\langle m(\mu)\rangle}$, $[\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$ or $[c_0, \ell_1]^{\langle m(\mu)\rangle}$.*

(a) Then $A \in (X_\Delta, Y)$ holds if and only if condition (5.13) holds in addition to the respective conditions in Theorem 4.4 with the A replaced by E^A.

(b) Let $C = \Delta A$, that is $c_{nk} = a_{nk} - a_{n-1,k}$ for all n and k. Then $A \in (X_\Delta, Y_\Delta)$ if and only if condition (5.13) with V^{A_n} replaced by V^{C_n} holds in addition to the respective conditions of Theorem 4.4 with A replaced by E^C.

In particular, we have, applying Corollary 4.5

COROLLARY 5.5. *Let $1 \le r < \infty$ and $1 \le p \le \infty$, s and q be the conjugate numbers of r and p, and Y be any of the spaces in Theorem 5.4. Finally, let the sequences $b^{s,q}$ be defined as in Example 5.2(a).*

(a) Then $(A \in ([\ell_r, \ell_p]^{\langle k(\nu)\rangle})_\Delta, Y)$ if and only if $E(A_n) \in (b^{(s,q)})^{-1} * \ell_\infty$ for all n, and the respective conditions in Corollary 4.5 hold with A replaced by E^A. Furthermore, $A \in (([c_0, \ell_p]^{\langle k(\nu)\rangle})_\Delta, Y)$ for $1 \le p < \infty$ if and only if $E(A_n) \in (b^{1,q})^{-1} * \ell_\infty$ for all n, and the respective conditions in Corollary 4.5 hold with A replaced by E^A.

(b) The conditions for $A \in (([\ell_r, \ell_p]^{\langle k(\nu)\rangle})_\Delta, Y_\Delta)$ and $(([c_0, \ell_p]^{\langle k(\nu)\rangle})_\Delta, Y_\Delta)$ are obtained from the respective ones in Part (a) by replacing A by C throughout.

For the next result, we need to know the β-duals of ℓ_∞ and $[\ell_\infty, \ell_1]$ which cannot be determined by Theorem 5.1, since they do not have AK.

LEMMA 5.6. *Let $E = \Sigma^T$. Then*

(a) $a \in ((\ell_\infty)_\Delta)^\beta$ if and only if $a \in (\ell_1 \cap ((n)_{n=1}^\infty)^{-1} * c_0)_E$;

(b) $a \in (([\ell_\infty, \ell_1]^{\langle k(\nu)\rangle})_\Delta)^\beta$ if and only if $a \in ([\ell_1, \ell_\infty]^{\langle k(\nu)\rangle} \cap (b^{1,\infty})^{-1} * c_0)_E$ where the sequence $b^{1,\infty}$ is defined as in Example 5.2(a).

In both parts, if $a \in ((\ell_\infty)_\Delta)^\beta$ or $a \in (([\ell_\infty, \ell_1]^{\langle k(\nu)\rangle})_\Delta)^\beta$ then (5.2) holds.

Proof. (a) This follows from [9, Theorem 2, Corollary 2].

(b) We write $X_\infty = (([\ell_\infty, \ell_1]^{\langle k(\nu)\rangle})_\Delta$ and $X_0 = ([c_0, \ell_1]^{\langle k(\nu)\rangle})_\Delta$, for short.

First $X_0 \subset X_\infty$ implies $X_\infty^\beta \subset X_0^\beta$, hence $X_\infty^\beta \subset ([\ell_1, \ell_\infty]^{\langle k(\nu)\rangle})_E$ by Example 4.3(a). Now we assume $a \in X_\infty^\beta$. Since $e \in X_\infty$, the sequence $E(a)$ is defined. Let $y \in X_\infty$ be given. Then, by (5.6), $a \in X_\infty^\beta$ and $E(a) \in [\ell_1, \ell_\infty]^{\langle k(\nu)\rangle}$ together yield $V^a \in ([\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}, c)$, that is $E(a) \in (X_\infty, c)$ by (5.1). Conversely, if $a \in ([\ell_1, \ell_\infty]^{\langle k(\nu)\rangle} \cap M(X_\infty, c))_E$, then $E(a) \in [\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}$ and $V^a \in (X_\infty, c)$, hence $a \in X_\infty^\beta$ by (5.6). Thus we have shown $X_\infty^\beta = ([\ell_1, \ell_\infty]^{\langle k(\nu)\rangle} \cap M(X_\infty, c))_E$. We will prove

$$M(X_\infty, c) = M(X_\infty, c_0) = (b^{1,\infty})^{-1} * c_0. \tag{5.16}$$

We write $b = b^{1,\infty}$ and observe that $a \in M(X_\infty, c)$ if and only if $B^a \in ([\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}, c)$ by (5.1).

First we assume $B^a \in ([\ell_\infty, c]^{\langle k(\nu)\rangle}, c)$. Then, by [7, Satz 4.8],

$$\sum_{\nu=0}^{\infty} \max_{k \in I_\nu} |b_{nk}^a| \text{ converges uniformly in } n \tag{5.17}$$

and $\lim_{n\to\infty} b_{nk}^a = \alpha_k$ exists for each k. Since $[c_0, \ell_1]^{\langle k(\nu)\rangle} \subset [\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}$ implies $([\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}, c) \subset ([c_0, \ell_1]^{\langle k(\nu)\rangle}, c)$, we have $\sup_n \|B_n^a\|_{1,\infty} < \infty$ by Corollary 4.5(**2.**), and this is equivalent to $a \in b^{-1} * \ell_\infty$, by Example 5.2(a). Thus there is a constant K such that $\sup_n |a_n| b_n \le K$, whence $|a_n| \le K/b_n \to 0$ $(n \to \infty)$, that is $a \in c_0$. By (5.17), given $\varepsilon > 0$ there is $\nu_0 \in \mathbb{N}_0$ such that

$$\sum_{\nu=\nu_0}^{\infty} \max_{k \in I_\nu} |b_{nk}^a| \le |a_n| \left(b_n - b_{k(\nu_0)-1}\right) < \varepsilon/2 \text{ for all } n.$$

Furthermore, since $a \in c_0$, we can choose $n_0 \in \mathbb{N}$ such that $|a_n| b_{k(\nu_0)-1} < \varepsilon/2$ for all $n \ge n_0$. Then $|a_n| b_n < \varepsilon$ for all $n \ge n_0$, that is $a \in b^{-1} * c_0$.

Conversely we assume $a \in b^{-1} * c_0$. Then obviously $a \in c_0$. Furthermore $a \in b^{-1} * c_0$ implies

$$\|B_n^a\|_{(1,\infty)} = \sum_{\nu=0}^{\infty} \max_{k \in I_\nu} |b_{nk}^a| \to 0 \ (n \to \infty) \text{ and } \sup \|B_n^a\|_{(1,\infty)} < \infty$$

By [6, Lemma, p. 168], these two conditions together imply (5.17). From this and $\lim_{n\to\infty} b_{nk}^a = \lim_{n\to\infty} a_n = 0$, we conclude $B^a \in ([\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}, c_0)$ by [7, Satz 4.8], hence $a \in M(X_\infty, c_0)$. ∎

We obtain as an immediate consequence of Theorems 5.3, 4.6, Example 5.2(a) and Lemma 5.6

Theorem 5.7. *Let W be a BK space with AK and $Y = W^\beta$ and X be any of the spaces ℓ_∞, c_0, ℓ_1, $[\ell_1, \ell_\infty]^{\langle k(\nu)\rangle}$, $[\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}$ or $[c_0, \ell_1]^{\langle k(\nu)\rangle}$.*

(a) Then $A \in (X_\Delta, Y)$ if and only if the respective conditions in Theorem 4.6 hold with A replaced by E^A and, in addition for all m, $E(A_m) \in ((n)_{n=1}^\infty)^{-1} * c_0$ when $X = \ell_\infty$, $E(A_m) \in ((n)_{n=1}^\infty)^{-1} * \ell_\infty$ when $X = c_0$, $E(A_m) \in (b^{1,\infty})^{-1} * c_0$ when $X = [\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}$, $E(A_m) \in (b^{\infty,1})^{-1} * \ell_\infty$ when $X = [\ell_1, \ell_\infty]^{\langle k(\nu)\rangle}$, and $E(A_m) \in (b^{1,\infty})^{-1} * \ell_\infty$ when $X = [c_0, \ell_1]^{\langle k(\nu)\rangle}$; no additional condition is needed when $X = \ell_1$ by Example 5.2(a).

(b) Then $A \in (X_\Delta, Y_\Delta)$ if and only if the respective conditions in Part (a) hold with A replaced by $C = \Delta A$.

We obtain from Corollary 4.7

COROLLARY 5.8. *Let $1 < r < \infty$ and $1 < p < \infty$ ad X be any of the spaces in Theorem 5.7.*

(a) Then $A \in (X_\Delta, [\ell_r, \ell_p]^{\langle m(\mu)\rangle})$ if and only if the conditions in Corollary 4.7 with A replaced by E^A and the additional conditions of Theorem 5.7(a) hold.

(b) Then $A \in (X_\Delta, ([\ell_r, \ell_p]^{\langle m(\mu)\rangle})_\Delta)$ if and only if the conditions of Part (a) hold with A replaced by $C = \Delta A$.

REFERENCES

[**1**] K.-G. Grosse-Erdmann, *The blocking technique, weighted mean operators and Hardy's inequality*, Lecture Notes in Mathematics, No. **1679**, Springer Verlag, 1998.

[**2**] J. H. Hedlund, *Multipliers of H^p spaces*, J. Math. Mech. **18** (1968/1969), 1067–1074.

[**3**] A. A. Jagers, *A note on the Cesàro sequence spaces*, Nieuw Arch. Wisk. **22** (1974), 113–124.

[**4**] C. N. Kellog, *An extension of the Hausdorff-Young theorem*, Michigan Math. J. **18** (1971), 121–127.

[**5**] I. J. Maddox, *On Kuttner's theorem*, J. London Math. Soc. **43** (1968), 282–290.

[**6**] I. J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, 1972.

[**7**] E. Malkowsky, *Toeplitz–Kriterien für Matrizenklassen bei Räumen stark limitierbarer Folgen*, Acta Math. Sci. (Szeged) **48** (1985), 297–313.

[**8**] E. Malkowsky, *Klassen von Matrixabbildungen in paranormierten FK–Räumen*, Analysis **7** (1987), 275–292.

[**9**] E. Malkowsky, *A note on the Köthe–Toeplitz duals of generalized sets of bounded and convergent difference sequences*, J. Analysis **4**(1996), 81–91.

[**10**] E. Malkowsky, *On Λ-strong convergence and boundedness with index $p \geq 1$*, Proceedings of the 10^{th} Congress of Yugoslav Mathematicians, (Belgrade 2001), 251–260.

[**11**] E. Malkowsky, V. Rakočević, *An introduction into the theory of sequence spaces and measures of noncompactness*, Zbornik radova, Matematički institut SANU **9(17)** (2000), 143–234.

[**12**] E. Malkowsky , V. Rakočević, S. Živković-Zlatanović, *Matrix transformations between some sequence spaces and their measures of noncompactness*, to appear in Bulletin Academie Serbe des Sciences et des Arts.

[**13**] A. Peyerimhoff, *Über ein Lemma von Herrn Chow*, J. London Math. Soc. **32** (1957), 33–37.

[**14**] A. Wilansky, *Summability through Functional Analysis*, North-Holland Mathematics studies, No. **85**, 1984.

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A. Jarrah, Department of Mathematics, Qatar University Doha, Qatar

E-mail: jarrahahed@hotmail.com

E. Malkowsky, Mathematisches Institut, Universität Giessen, Arndtstrasse 2, D-35392 Giessen, Germany
Department of Mathematics, Faculty of Science and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Yugoslavia

E-mail: eberhard.malkowsky@math.uni-giessen.de ema@bankerinter.net