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THE RADIUS OF CONVEXITY FOR THE CLASS OF JANOWSKI CONVEX FUNCTIONS OF COMPLEX ORDER

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Abstract. In the present paper we shall give the sharp bound of radius of convexity for the class of Janowski convex functions of complex order.

1. Introduction

Let $\mathcal{A}$ be the family of functions $\omega(z)$ regular in the unit disc $D = \{z : |z| < 1\}$ and satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in D$.

Next, for arbitrary fixed numbers A, B, $-1 < B \leqslant A \leqslant 1$ denote by $p(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{1.1}$$

regular in D and such that $p(z) = \dfrac{1 + A\omega(z)}{1 + B\omega(z)}$ for some function $\omega(z) \in \mathcal{A}$ and every $z \in D$. The class $p(A, B)$ was introduced by W. Janowski [3].

Moreover let $\mathcal{C}(A, B, b)$ ($b \neq 0$ complex) denote the family of functions

$$f(z) = z + a_2 z^2 + \cdots \tag{1.2}$$

regular in D such that $1 + \dfrac{1}{b} z \dfrac{f''(z)}{f'(z)} = p(z)$ for some $p(z) \in p(A, B)$ and all $z \in D$. The class $\mathcal{C}(A, B, b)$ is the class of Janowski convex functions of complex order b.

It can be noticed that by giving specific values to A, B and b, we obtain the following important subclasses studied by various authors in earlier works [2]:

(i) $\mathcal{C}(1, -1, 1)$ is the well known class of convex functions [2].

(ii) $\mathcal{C}(1, -1, b)$ is the class of convex functions of complex order b. This class was introduced by P. Wiatrowski [11].

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(iii) $\mathcal{C}(1,-1,1-\alpha)$, $0 \leqslant \alpha < 1$ is the class of convex functions of order α, introduced by M.S. Robertson [7].

(iv) $\mathcal{C}(1,-1,e^{-i\lambda}\cos\lambda)$, $|\lambda| < \pi/2$ is the class of functions for which $zf'(z)$ is λ-spirallike, introduced by M.S. Robertson [8,9].

(v) $\mathcal{C}(1,-1,(1-\alpha)e^{-i\lambda}\cos\lambda)$ is the class of functions for which $zf'(z)$ is λ-spirallike of order α $(0 \leqslant \alpha < 1,\ |\lambda| < \pi/2)$, studied by B. Pinchuk [5] and Sizuk [10].

(vi) $\mathcal{C}(1-2\beta,-1,b)$, $0 \leqslant \beta < 1$ is the set defined by $\Re\left(1+\dfrac{1}{b}z\dfrac{f''(z)}{f'(z)}\right) > \beta$.

(vii) $\mathcal{C}(1,0,b)$ is the set defined by $\left|\left(1+\dfrac{1}{b}z\dfrac{f''(z)}{f'(z)}\right)-1\right| < 1$.

(viii) $\mathcal{C}(\beta,0,b)$ is the set defined by $\left|\left(1+\dfrac{1}{b}z\dfrac{f''(z)}{f'(z)}\right)-1\right| < \beta$.

(ix) $\mathcal{C}(1,-1+\frac{1}{M},b)$, $M > \frac{1}{2}$ is the set defined by $\left|\left(1+\dfrac{1}{b}z\dfrac{f''(z)}{f'(z)}\right)-M\right| < M$.

(x) $\mathcal{C}(\beta,-\beta,b)$, $0 \leqslant \beta < 1$ is the set defined by $\left|\dfrac{(1+\frac{1}{b}z\frac{f''(z)}{f'(z)})-1}{(1+\frac{1}{b}z\frac{f''(z)}{f'(z)})+1}\right| < \beta$.

2. Radius of convexity for the class $\mathcal{C}(A,B,b)$

In this section we shall give sharp bound of the radius of convexity for the class of Janowski convex functions of complex order.

Theorem 2.1 *The radius of convexity for the class $\mathcal{C}(A,B,b)$ is*

$$r_c = \frac{2}{|b|(A-B)+\sqrt{|b|^2(A-B)^2-4[(B^2-AB)\Re b-B^2]}}. \qquad (*)$$

This radius is sharp because the extremal function is

$$f_*(z) = \begin{cases} \dfrac{B}{b(A-B)+B}(1+Bz)^{\frac{b(A-B)+B}{B}}, & \text{if } B \neq 0,\\ \dfrac{1}{bA}e^{bAz}, & \text{if } B = 0. \end{cases}$$

Really, if we take $w = \dfrac{r(r-\sqrt{\bar{b}/b})}{1-r\sqrt{\bar{b}/b}}$ we obtain $1+w\dfrac{f''(w)}{f'(w)} = r_c$.

Proof. In [3] Janowski proved that, if $p(z) \in p(A,B)$, then

$$\left|p(z)-\frac{1-ABr^2}{1-B^2r^2}\right| \leqslant \frac{(A-B)r}{1-B^2r^2}. \qquad (2.1)$$

On the other hand from the definition of the class $\mathcal{C}(A,B,b)$ we write

$$1+\frac{1}{b}z\frac{f''(z)}{f'(z)} = p(z). \qquad (2.2)$$

From the relations (2.1) and (2.2) we can write

$$\left|\left[1+\frac{1}{b}z\frac{f''(z)}{f'(z)}\right]-\frac{1-ABr^2}{1-B^2r^2}\right|\leqslant\frac{(A-B)r}{1-B^2r^2}. \tag{2.3}$$

After simple calculations from the relation (2.3) we obtain

$$\Re\left(1+z\frac{f''(z)}{f'(z)}\right)\geqslant\frac{[(B^2-AB)\Re b-B^2]r^2-|b|(A-B)r+1}{1-B^2r^2}. \tag{2.4}$$

Hence for $r<r_c$ the left-hand side of preceding inequality is positive which implies the assertion $(*)$.

Also note that the inequality (2.4) becomes an equality for the function $f_*(z)$. It follows that the radius of convexity for the class $\mathcal{C}(A,B,b)$ is r_c. If we take $w=\dfrac{r(r-\sqrt{\bar{b}/b})}{1-r\sqrt{\bar{b}/b}}$ we see that $1+w\dfrac{f_*''(w)}{f_*'(w)}=\dfrac{[(B^2-AB)\Re b-B^2]r^2+|b|(A-B)r-1}{1-r^2B^2}$. ∎

(i) For $A=1$, $B=-1$, $r_c=\dfrac{1}{|b|+\sqrt{|b|^2-[2\Re b]+1}}$. This is the radius of convexity for the class of convex functions of complex order. This result was obtained by M.A. Nasr and M.K. Aouf [4].

(ii) For $A=1-2\beta$, $B=-1$, $0\leqslant\beta<1$,

$$r_c=\frac{1}{|b|(1-\beta)+\sqrt{|b|^2(1-\beta)^2-2(1-\beta)\Re b+1}}.$$

This is the radius of convexity for the class $\mathcal{C}(1-\beta,-1,b)$.

(iii) For $A=1$, $B=0$, $r_c=1/|b|$. This is the radius of convexity for the class $\mathcal{C}(1,0,b)$.

(iv) For $A=\beta$, $B=-\beta$, $0\leqslant\beta<1$, $r_c=\dfrac{1}{\beta[|b|+\sqrt{|b|^2-[2\Re b]+1}]}$. This is the radius of convexity for the class $\mathcal{C}(\beta,-\beta,b)$.

(v) For $A=\beta$, $B=0$, $0\leqslant\beta<1$, $r_c=1/(\beta|b|)$. This is the radius of convexity for the class $\mathcal{C}(\beta,0,b)$.

(vi) For $A=1$, $B=\frac{1}{M}-1$, $M>1/2$,

$$r_c=\frac{2}{|b|(2-\frac{1}{M})+\sqrt{|b|^2(2-\frac{1}{M})^2-4[(2-\frac{3}{M}+\frac{1}{M^2})\Re b-(\frac{1}{M}-1)^2]}}.$$

This is the radius of convexity for the class $\mathcal{C}(1,\frac{1}{M}-1,b)$.

We note that by giving specific values to b we obtain the radius of convexity for the corresponding classes.

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