RING-LIKE STRUCTURES DERIVED FROM λ-LATTICES WITH ANTITONE INVOLUTIONS

Ivan Chajda, Olomouc

(Received December 12, 2005)

Abstract. Using the concept of the λ-lattice introduced recently by V. Snášel we define λ-lattices with antitone involutions. For them we establish a correspondence to ring-like structures similarly as it was done for ortholattices and pseudorings, for Boolean algebras and Boolean rings or for lattices with an antitone involution and the so-called Boolean quasirings.

Keywords: λ-lattice, λ-semilattice, ortholattice, λ-ortholattice, antitone involution, Boolean quasiring

MSC 2000: 06C15, 16Y99, 06B99, 81P10, 06A12

The well-known correspondence between Boolean algebras and Boolean rings (see e.g. [1]) was extended to orthomodular lattices by H. Länger [12] and to ortholattices by the author in [3]. A general setting was described by G. Eigenthaler, H. Länger and the author in [4] and [5]. It was generalized to lattices with antitone involution by D. Dorninger, H. Länger and M. Mączyński [6], [7] and to generalized orthomodular lattices, see [5]. It was motivated by the use of these ring-like structures in certain logics of quantum mechanics, see e.g. [6], [7] for the description. However, in quantum mechanics it can happen that we cannot distinguish between two possibilities. It leads us to study more general structures than lattices where still ring-like structures can be induced.

Suitable tools for our investigation thus can be the so-called λ-lattices introduced by V. Snášel [13] and treated in [11], [13] and the so-called λ-semilattices known also under the name commutative directoids in [9]. On the other hand, this level of generality can make problems in computing formulas because some well-known

This work is supported by the Czech Goverment via the research project MSM 6198959214.

results for lattices fail for λ-lattices. For example, if $x \mapsto x'$ is an antitone involution on a lattice $\mathcal{L} = (L; \vee, \wedge)$ then the De Morgan laws

$$\text{(DM)} \qquad (x \vee y)' = x' \wedge y' \quad \text{and} \quad (x \wedge y)' = x' \vee y'$$

hold in $\mathcal{L}$, see e.g. [1], [8]. However, this is not the case for λ-lattices. Hence we must often require some additional properties.

1. λ-lattices with antitone involutions

First, we recall some well-known concepts. Let $(A; \leqslant)$ be an ordered set. For $a,\ b \in A$ denote

$$U(a,b) = \{x \in A;\ a \leqslant x \ \text{ and } \ b \leqslant x\},$$
$$L(a,b) = \{x \in A;\ x \leqslant a \ \text{ and } \ x \leqslant b\}.$$

An ordered set $(A; \leqslant)$ is called *up-directed* (*down-directed*) if $U(a,b) \neq \emptyset$ (or $L(a,b) \neq \emptyset$, respectively) for each $a, b \in A$. Further, $(A; \leqslant)$ is *directed* if it is both an up- and down-directed set.

Let $(A; \leqslant)$ be a down-directed set. Denote by $\operatorname{Exp} A$ the power set of A. Let λ be a mapping $\lambda\colon \operatorname{Exp} A \to A$ such that

(i) $\lambda(L(a,b)) \in L(a,b)$,
(ii) if $a \leqslant b$ then $\lambda(L(a,b)) = a$.

Define a binary operation $\wedge$ on A as follows:

$a \wedge b = \lambda(L(a,b))$.

The groupoid $(A; \wedge)$ will be called a λ-*semilattice* (or a commutative directoid in [9]). It is easy to verify that $(A; \wedge)$ satisfies the identities

(I) $x \wedge x = x$ (idempotency),
(C) $x \wedge y = y \wedge x$ (commutativity),
(SA) $x \wedge ((x \wedge y) \wedge z) = (x \wedge y) \wedge z$ (skew associativity).

Also conversely, if $(A; \wedge)$ is a groupoid satisfying (I), (C), (SA) and $\leqslant$ is defined by the rule

$a \leqslant b$ if and only if $a \wedge b = a$

then $(A; \leqslant)$ is a down-directed set and $\lambda(L(a,b)) = a \wedge b$ satisfies (i) and (ii) mentioned above.

Moreover, if $(A; \leqslant)$ is a directed set and $\lambda\colon \operatorname{Exp} A \to A$ satisfies also

(iii) $\lambda(U(a,b)) \in U(a,b)$,
(iv) if $a \leqslant b$ then $\lambda(U(a,b)) = b$

then we can introduce another operation $\vee$ by setting

$a \vee b = \lambda(U(a, b))$

and easily verify the following identities:

(I′) $x \vee x = x$,

(C′) $x \vee y = y \vee x$,

(SA′) $x \vee ((x \vee y) \vee z) = (x \vee y) \vee z$,

(Ab) $x \vee (x \wedge y) = x$, $x \wedge (x \vee y) = x$ (absorption).

Then the algebra $(A; \vee, \wedge)$ is called a λ-*lattice* (see [11], [13]). Also conversely, if $(A; \vee, \wedge)$ is an algebra of type $(2, 2)$ satisfying the identities (I), (I′), (C), (C′), (SA), (SA′), (Ab) then the relation defined by

$a \leqslant b$ if and only if $a \vee b = b$

coincides with the already introduced induced order on $(A; \wedge)$. If we put

$$\lambda(U(a, b)) = a \vee b$$

then (iii) and (iv) are satisfied.

Hence, λ-semilattices and λ-lattices can be viewed either as algebras satisfying certain identities or as ordered sets with constrains on upper and lower bounds. Contrary to the case of lattices, the choice of $\lambda(L(a, b))$ or $\lambda(U(a, b))$ need not be unique. Consider e.g. the directed sets drawn in Fig. 1 and Fig. 2.

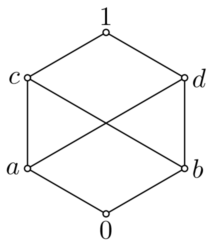

Fig. 1

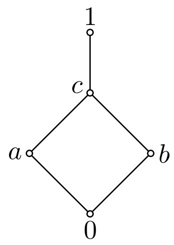

Fig. 2

In the first case, we have three choices for $a \vee b$, namely c or d or 1. Analogously, $c \wedge d$ can be a or b or 0 (for comparable elements the choice is unique due to the conditions (ii) and (iv)). In all 9 possible cases, the resulting algebra will be a λ-lattice. Analogously, in the second case (which is a lattice) we can pick up e.g. $a \vee b = c$ or $a \vee b = 1$. In the latter case, the resulting algebra will be a λ-lattice which is not a lattice.

Let $(A; \leqslant)$ be an ordered set. A mapping $x \mapsto x'$ on A is called an *antitone involution* if $x'' = x$ and $x \leqslant y$ implies $y' \leqslant x'$. It is well-known that if $(L; \vee, \wedge)$ is a lattice and $x \mapsto x'$ is an antitone involution on $(L; \leqslant)$ where $\leqslant$ is the induced order then the De Morgan laws (DM) hold.

Consider our λ-lattice $\mathcal{L} = (A; \vee, \wedge)$ visualized in Fig. 1. Define $x \mapsto x'$ on $A = \{0, a, b, c, d, 1\}$ by the table

x	0	a	b	c	d	1
x'	1	d	c	b	a	0

Then clearly it is an antitone involution on A. Let us pick up e.g. $a \vee b = c$ (for other elements the operation $\vee$ is determined as supremum). Then $(A; \vee)$ is a λ-semilattice (w.r.t. $\vee$). Now, if e.g. $c \wedge d = a$ (for other elements it is determined as infimum) then $\mathcal{L} = (A; \vee, \wedge)$ is a λ-lattice with an antitone involution but

$$b' \wedge a' = c \wedge d = a \neq b = c' = (b \vee a)',$$

thus in $\mathcal{L}$ the De Morgan laws do not hold. On the contrary, when choosing $c \wedge d = b$, De Morgan laws hold in $\mathcal{L}$. Hence, validity of (DM) depends not only on the induced order but also on our choice of operations.

2. λ-Boolean quasirings

By a *Boolean quasiring* (see [4], [6], [7]) we mean an algebra $\mathcal{R} = (R; +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the identities

(R1) $$x + y = y + x,$$

(R2) $$x + 0 = x,$$

(R3) $$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

(R4) $$x \cdot y = y \cdot x,$$

(R5) $$x \cdot x = x,$$

(R6) $$x \cdot 0 = 0,$$

(R7) $$x \cdot 1 = x,$$

(R8) $$1 + (1 + x \cdot y) \cdot (1 + y) = y.$$

For our purposes, we modify this definition as follows. An algebra $\mathcal{R} = (R; +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ is called a λ-*Boolean quasiring* if it satisfies the identities (R1), (R2), (R4)–(R8) and

(R3*) $$x \cdot ((x \cdot y) \cdot z) = (x \cdot y) \cdot z.$$

One can immediately verify that every Boolean quasiring is a λ-Boolean quasiring since (R3*) follows easily by (R3), (R4) and (R5). The mutual correspondence between Boolean quasirings and lattices with an antitone involution was established in [6]. We are going to extend this correspondence to λ-Boolean quasirings.

Theorem 1. *Let $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ be a bounded λ-lattice with an antitone involution. Define*

$$x + y = (x \vee y) \wedge (x \wedge y)' \quad \text{and} \quad x \cdot y = x \wedge y.$$

Then $\mathcal{R}(L) = (L; +, \cdot, 0, 1)$ is a λ-Boolean quasiring. If, moreover, the De Morgan laws hold in $\mathcal{L}$ then $\mathcal{R}(L)$ satisfies the correspondence identity
(Cor) $(1 + (1 + x) \cdot (1 + y)) \cdot (1 + x \cdot y) = x + y.$

Proof. Since $(L; \wedge)$ is a bounded λ-semilattice, the identities (R3*), (R4)–(R7) are immediate consequences of (I), (C), (SA) and the properties of the induced order. The identity (R1) is a trivial consequence of the definition and (R2) is evident. It remains to prove (R8) and (Cor).

For (R8) we use (Ab) to compute

$$1 + (1 + x \cdot y) \cdot (1 + y) = ((x \wedge y)' \wedge y')' = (y')' = y'' = y.$$

For (Cor), we apply De Morgan laws to derive

$$(1 + (1 + x) \cdot (1 + y)) \cdot (1 + x \cdot y) = (x' \wedge y')' \wedge (x \wedge y)' = (x \vee y) \wedge (x \wedge y)' = x + y.$$

□

We can prove also the converse.

Theorem 2. *Let $\mathcal{R} = (R; +, \cdot, 0, 1)$ be a λ-Boolean quasiring. Define*

$$x \wedge y = x \cdot y, \quad x' = 1 + x \quad \text{and} \quad x \vee y = 1 + (1 + x) \cdot (1 + y).$$

Then $\mathcal{L}(R) = (R; \vee, \wedge, ', 0, 1)$ is a bounded λ-lattice with an antitone involution in which the De Morgan laws hold.

Proof. By (R3*), (R4)–(R7), $(R; \wedge)$ is a bounded λ-semilattice. If we put $y = x$ in (R8) and apply (R5), we obtain the identity

$$(*) \qquad 1 + (1 + x) = x$$

proving that the unary operation $x' = 1 + x$ is an involution on R. Suppose $x \leqslant y$. Then $x \wedge y = x \cdot y = x$ and, by (R8),

$$1 + x' \cdot y' = 1 + (1 + x) \cdot (1 + y) = 1 + (1 + x \cdot y) \cdot (1 + y) = y.$$

Thus, applying $(*)$, we arrive at

$$x' \wedge y' = x' \cdot y' = 1 + (1 + x' \cdot y') = 1 + y = y'$$

whence $y' \leqslant x'$, i.e. this operation is an antitone involution on R.

Further, using $(*)$, we obtain

$$x' \vee y' = 1 + (1 + x') \cdot (1 + y') = 1 + x \cdot y = (x \wedge y)'$$

and

$$x' \wedge y' = (1 + x) \cdot (1 + y) = 1 + (1 + (1 + x) \cdot (1 + y)) = (x \vee y)'$$

thus the De Morgan laws hold in $\mathcal{L}(R)$. Due to this fact, $(R; \vee)$ is also a λ-semilattice and, by (R8),

$$(x \wedge y) \vee y = ((x \cdot y)' \cdot y')' = 1 + (1 + x \cdot y) \cdot (1 + y) = y.$$

The dual absorption law can be established by the De Morgan laws and involutorness. Hence, $\mathcal{L}(R) = (R; \vee, \wedge, ', 0, 1)$ is a bounded λ-lattice with an antitone involution.

□

Theorem 3. *Let $\mathcal{R} = (R; +, \cdot, 0, 1)$ be a λ-Boolean quasiring satisfying the correspondence indentity* (Cor). *Then $\mathcal{R}(\mathcal{L}(R)) = \mathcal{R}$.*

Let $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ be a bounded λ-lattice with an antitone involution in which De Morgan laws hold. Then $\mathcal{L}(\mathcal{R}(L)) = \mathcal{L}$.

Proof. Evidently, the multiplicative operations coincide in $\mathcal{R}(\mathcal{L}(R))$ and $\mathcal{R}$. To prove $\mathcal{R}(\mathcal{L}(R)) = \mathcal{R}$ we need only to show that also $\oplus = +$ where $\oplus$ is the additive operation in $\mathcal{R}(\mathcal{L}(R))$. Applying (Cor) we compute

$$x \oplus y = (x \vee y) \wedge (x \wedge y)' = (1 + (1 + x) \cdot (1 + y)) \cdot (1 + x \cdot y) = x + y.$$

Analogously, the operation meet clearly coincides in $\mathcal{L}(\mathcal{R}(L))$ and $\mathcal{L}$. Hence, it remains to prove $\sqcup = \vee$ and $x^* = x'$ where $\sqcup$ is the join and * is the antitone involution in $\mathcal{L}(\mathcal{R}(L))$. We have

$$x^* = 1 + x = (1 \vee x) \wedge (1 \wedge x)' = 1 \wedge x' = x'$$

and

$$x \sqcup y = 1 + (1 + x) \cdot (1 + y) = (x' \wedge y')' = x \vee y$$

due to the De Morgan laws. □

By an *ortholattice* (see e.g. [2], [10]) we mean a bounded lattice $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ with an antitone involution which is a complementation, i.e. $x \wedge x' = 0$ and $x \vee x' = 1$ for each $x \in L$. Of course, De Morgan laws hold in $\mathcal{L}$ and hence $x \wedge x' = 0$ is equivalent to $x \vee x' = 1$.

We can extend this concept to λ-lattices.

By a λ-*ortholattice* we mean a bounded λ-lattice $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ with an antitone involution such that $x \wedge x' = 0$ and $x \vee x' = 1$.

Formally, the definition is the same as for ortholattices, but we must be careful. For example, the λ-lattice depicted in Fig. 3 is an λ-ortholattice.

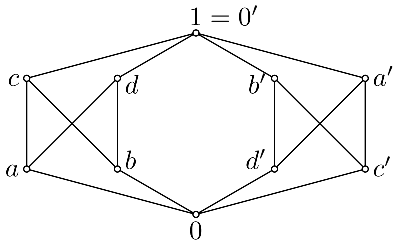

Fig. 3

Since it is a λ-lattice, we must specify joins and meets of non-comparable elements. If e.g. $a \vee b = c$ and $a' \wedge b' = d'$ then $a' \wedge b' = d' \neq c' = (a \vee b)'$ and hence the De Morgan laws do not hold. It means that for $a \vee b = c$ we must set $a' \wedge b' = c'$ etc.

The mutual correspondence between ortholattices and the so-called pseudosemirings was settled in [3]. However, it is easy to establish such a correspondence for λ-ortholattices and certain λ-Boolean quasirings.

Theorem 4. *Let $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ be a λ-ortholattice in which the De Morgan laws hold. Define*

$$x + y = (x \vee y) \wedge (x \wedge y)' \quad \text{and} \quad x \cdot y = x \wedge y.$$

Then $\mathcal{R}(L) = (L; +, \cdot, 0, 1)$ is a λ-Boolean quasiring of characteristic 2 (i.e. satisfying the identity $x + x = 0$) satisfying the correspondence identity (Cor).

Let $\mathcal{R} = (R; +, \cdot, 0, 1)$ be a λ-Boolean quasiring of characteristic 2. Define

$$x \vee y = 1 + (1 + x) \cdot (1 + y), \quad x \wedge y = x \cdot y, \quad x' = 1 + x.$$

Then $\mathcal{L}(R) = (R; \vee, \wedge, ', 0, 1)$ is a λ-ortholattice in which the De Morgan laws hold.

Moreover, $\mathcal{L}(\mathcal{R}(L)) = \mathcal{L}$ *and if* $\mathcal{R}$ *satisfies* (Cor) *then also* $\mathcal{R}(\mathcal{L}(R)) = \mathcal{R}$.

P r o o f. Of course, if $x \wedge x' = 0$ then $x + x = x \wedge x' = 0$, thus the induced λ-Boolean quasiring $\mathcal{R}(L)$ is of characteristic 2 for each λ-ortholattice $\mathcal{L}$. The rest of the proof follows by Theorem 1.

Conversely, if $\mathcal{R}$ satisfies $x + x = 0$ then $x \wedge x' = (x \vee x) \wedge (x \wedge x)' = x + x = 0$ and, due to the De Morgan laws, also $x \vee x' = 1$ and thus $\mathcal{L}(R)$ is really a λ-ortholattice.

The remaining statements follow directly by Theorems 2 and 3. □

In what follows we concentrate on a special case of a λ-ortholattice, the so-called *orthomodular* λ*-lattice* which satisfies the *orthomodular law*

(OML) $$x \leqslant y \Rightarrow x \vee (y \wedge x') = y$$

which is equivalent to the identity

(**) $$(x \wedge y) \vee (x \wedge (x \wedge y)') = x.$$

An example of an orthomodular λ-lattice can be constructed from the lattice drawn in Fig. 4.

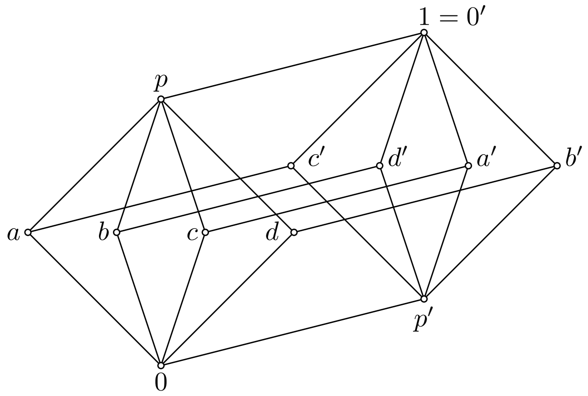

Fig. 4

One can easily verify that $\mathcal{L}$ is an orthomodular lattice. To change it into a λ-lattice which is not a lattice, we define $a \vee b = 1$ and $a' \wedge b' = 0$ and leave all other joins and meets as they were in $\mathcal{L}$. Of course, the new algebra is a λ-lattice but not a lattice, but the orthomodular law remains valid.

We are wonder what a λ-Boolean quasiring corresponds to an orthomodular λ-lattice.

Theorem 5. *Let $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ an orthomodular λ-lattice in which the De Morgan laws hold. Define*

$$x + y = (x \vee y) \wedge (x \wedge y)' \quad \text{and} \quad x \cdot y = x \wedge y.$$

Then $\mathcal{R}(L)$ is a λ-Boolean quasiring of characteristic 2 satisfying the identity

$$\text{(OM)} \qquad (1 + x \cdot y) + (x + x \cdot y) = 1 + x.$$

Let $\mathcal{R} = (R; +, \cdot, 0, 1)$ be a λ-Boolean quasiring of characteristic 2 satisfying the identity (OM). *Define*

$$x \vee y = 1 + (1 + x) \cdot (1 + y), \quad x \wedge y = x \cdot y \quad \text{and} \quad x' = 1 + x.$$

Then $\mathcal{L}(R) = (R; \vee, \wedge, ', 0, 1)$ is an orthomodular λ-lattice in which the De Morgan laws hold.

Moreover, $\mathcal{L}(\mathcal{R}(L)) = \mathcal{L}$ and if $\mathcal{R}$ satisfies also (Cor) *then $\mathcal{R}(\mathcal{L}(R)) = \mathcal{R}$.*

P r o o f. In virtue of Theorem 4, we need only to verify the orthomodular law for $\mathcal{L}(R)$ and the identity (OM) for $\mathcal{R}(L)$.

Let $\mathcal{L}$ be an orthomodular λ-lattice in which the De Morgan laws hold. We can easily derive $x' \vee (x' \vee y') = x' \vee y'$ since $x' \leqslant x' \vee y'$ and thus

$$\begin{aligned} x \cdot (1 + x \cdot y) &= x \wedge (x \wedge y)' = x \wedge (x' \vee y') = x \wedge (x' \vee (x' \vee y')) \\ &= x \wedge (x \wedge (x \wedge y))' = (x \vee (x \wedge y)) \wedge (x \wedge (x \wedge y))' = x + x \cdot y. \end{aligned}$$

Since $1 + x \cdot y = (x \wedge y)' \geqslant x \wedge (x \wedge y)' = x \cdot (1 + x \cdot y)$ and for $a \geqslant b$ we have

$$a + b = (a \vee b) \wedge (a \wedge b)' = a \wedge b' = a \cdot b',$$

then also $(1 + x \cdot y) + x \cdot (1 + x \cdot y) = (1 + x \cdot y) \cdot (1 + x \cdot (1 + x \cdot y))$. Now we apply $(**)$ to compute

$$\begin{aligned} (1 + x \cdot y) + (x + x \cdot y) &= (1 + x \cdot y) + x \cdot (1 + x \cdot y) = (1 + x \cdot y) \cdot (1 + x \cdot (1 + x \cdot y)) \\ &= [(x \wedge y) \vee (x \wedge (x \wedge y)')]' = x' = 1 + x. \end{aligned}$$

Thus $\mathcal{R}(L)$ satisfies (OM).

Conversely, let $\mathcal{R}$ be a λ-Boolean quasiring of characteristic 2 satisfying (OM). Let $x \leqslant y$ in $\mathcal{L}(R)$. Then

$$\begin{aligned} x \vee (y \wedge x') &= 1 + (1 + x) \cdot (1 + y \cdot (1 + x)) \\ &= 1 + ((1 + x) + y \cdot (1 + x)) = 1 + ((1 + x \cdot y) + y \cdot (1 + x \cdot y)) \\ &= 1 + ((1 + x \cdot y) + (y + x \cdot y)) = 1 + (1 + y) = y \end{aligned}$$

(by $(*)$ of the proof of Theorem 2). Hence, $\mathcal{L}(R)$ is an orthomodular λ-lattice. $\square$

References

[1] *G. Birkhof*: Lattice Theory. (Third edition.) Amer. Math. Soc. Colloq. Publ. 25, Providence, RI, 1979.
[2] *L. Beran*: Orthomodular Lattices. Reidel Publ., Dordrecht, 1985.
[3] *I. Chajda*: Pseudorings induced by ortholattices. Czechoslovak Math. J. *46* (1996), 405–411.
[4] *I. Chajda, G. Eigenthaler*: A note on orthopseudorings and Boolean quasirings. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II *207* (1998), 83–94.
[5] *I. Chajda, H. Länger, M. Mączyński*: Ring-like structures corresponding to generalized orthomodular lattices. Math. Slovaca *54* (2004), 143–150.
[6] *D. Dorminger, H. Länger, M. Mączyński*: The logic induced by a system of homomorphisms and its algebraic characterizations. Demonstratio Math. *30* (1997), 215–232.
[7] *D. Dorminger, H. Länger, M. Mączyński*: On ring-like structures occuring in axiomatic quantum mechanics. Österr. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II *206* (1997), 279–289.
[8] *G. Grätzer*: General Lattice Theory. Birkhäuser, Basel, 1978.
[9] *J. Ježek, R. Quackenbush*: Directoids: algebraic models of up-directed sets. Algebra Univers. *27* (1990), 49–69.
[10] *G. Kalmbach*: Orthomodular Lattices. Academic Press, London, 1983.
[11] *J. Karásek*: Rotations of λ-lattices. Math. Bohem. *121* (1996), 293–300.
[12] *H. Länger*: Generalizations of the correspondence between Boolean algebras and Boolean rings to orthomodular lattices. Tatra Mt. Math. Publ. *15* (1998), 97–105.
[13] *V. Snášel*: λ-Lattices. Math. Bohem. *122* (1997), 267–272.

Author's address: *Ivan Chajda*, Department of Algebra and Geometry, Palacký University Olomouc, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: `chajda@inf.upol.cz`.