A REMARK ON SUPRA-ADDITIVE AND SUPRA-MULTIPLICATIVE OPERATORS ON $C(X)$

Z. Ercan, Ankara

(Received September 9, 2005)

Abstract. M. Radulescu proved the following result: Let X be a compact Hausdorff topological space and $\pi\colon C(X)\to C(X)$ a supra-additive and supra-multiplicative operator. Then π is linear and multiplicative. We generalize this result to arbitrary topological spaces.

Keywords: $C(X)$-space, supra-additive, supra-multiplicative operator, realcompact

MSC 2000: 46J10, 46E25

1. The result

We follow the terminology of [1]. As usual for a topological space X, the space of real valued continuous (bounded) functions on K is denoted by $C(X)$ ($C_b(X)$). For each $x\in X$, $\delta_x\colon C(X)\to\mathbb{R}$ is defined by $\delta_x(f)=f(x)$. For $B\subset X$, χ_B denotes the characteristic function of B. For each $n\in\mathbb{R}$, $\mathbf{n}$ denotes the constant function with value $\mathbf{n}$. A map $\pi\colon C(X)\to C(Y)$ is called

(i) *supra-additive* if $\pi(f+g)\geqslant\pi(f)+\pi(g)$ for each $f,g\in C(X)$,

(ii) *supra-multiplicative* if $\pi(fg)\geqslant\pi(f)\pi(g)$ for each $f,g\in C(X)$.

The following theorem is the main result of [4].

Theorem 1. *Let X be a compact Hausdorff space and $\pi\colon C(X)\to C(X)$ a supra-additive and supra-multiplicative map. Then π is multiplicative and linear.*

The main result of this note is to generalize the above theorem as follows.

Theorem 2. *Let X and Y be topological spaces and $\pi\colon C(X) \to C(Y)$ a supra-additive and supra-multiplicative map. Then the following statements are equivalent.*

(i) $\pi(f^+ \wedge \mathbf{n} - f^- \wedge \mathbf{n})(y) \to \pi(f)(y)$ *for each* $f \in C(X)$ *and* $y \in Y$.

(ii) π *is linear and multiplicative.*

Proof. (ii) $\Longrightarrow$ (i): For each $y \in T$, $\delta_y \circ \pi$ is a Riesz homomorphism, so

$$\pi(f \wedge \mathbf{n})(y) = \delta_y \circ \pi(f \wedge \mathbf{n}) = \delta_y \circ \pi(f) \wedge n \to \delta_y \circ \pi(f) = \pi(f)(y)$$

(i) $\Longrightarrow$ (ii):

Claim 1. Let K be a compact Hausdorff space and let $T\colon C(K) \to \mathbb{R}$ be supra-additive and supra-multiplicative. Then T is linear and multiplicative.

Indeed, let $T^\sim\colon C(K) \to C(K)$ be defined by $T^\sim(f) = T(f)\mathbf{1}$. Then $T^\sim$ is supra-additive and supra-multiplicative, so by Theorem 1, $T^\sim$ is linear and multiplicative, so T is linear and multiplicative.

Claim 2. For each topological space M there exists a compact Hausdorff space K_M such that $C(K_M)$ and $C_b(M)$ are Riesz and algebraic isomorphic spaces.

As $C_b(M)$ is an AM-space with order unit $\mathbf{1}$, this follows from the Kakutani-Krein Representation Theorem (see [1]).

Claim 3. Let $\pi^\sim = \pi|_{C_b(X)}$. Then for each $y \in Y$, $\delta_y \circ \pi^\sim\colon C_b(X) \to \mathbb{R}$ is linear and multiplicative.

This follows from Theorem 1 and from the above claims.

Claim 4. π is linear.

To see this we use the linearity of $\delta_y \circ \pi^\sim$ as follows. Let $f, g \geqslant 0$ be given. Then

$$\pi(f+g)(y) = \lim \delta_y \circ \pi^\sim((f+g) \wedge \mathbf{n}) \leqslant \lim \delta_y \circ \pi^\sim(f \wedge \mathbf{n} + g \wedge \mathbf{n}).$$

Since $\delta_y \circ \pi^\sim$ is linear and π is supra-additive we have

$$\pi(f+g) \leqslant \pi(f) + \pi(g) \leqslant \pi(f+g),$$

so π is additive on $C(X)^+$. Now by the Kantorovic Theorem (see Theorem 1.7. [1]), $\varphi\colon C(X) \to C(Y)$ defined by $\varphi(f) = \pi(f^+) - \pi(f^-)$ is linear and from the second assumption it is clear that $\varphi = \pi$, so π is linear.

Claim 5. π is multiplicative.

Indeed, let $0 \leqslant f \in C(X)$ be given. As for each $y \in Y$, $\delta_y \circ \pi^\sim$ is multiplicative, we have

$$\begin{aligned}\pi(f^2)(y) = \delta_y \circ \pi(f^2) &= \lim \delta_y \circ \pi^\sim(f^2 \wedge \mathbf{n}) = \lim \delta_y \circ \pi^\sim((f \wedge \mathbf{n}^{\frac{1}{2}})^2) \\ &= (\lim \delta_y \circ \pi^\sim(f \wedge \mathbf{n}^{\frac{1}{2}}))^2 = \pi(f)^2(y),\end{aligned}$$

so $\pi(f^2) = \pi(f)^2$. Let $f \in C(X)$ be given. As $\pi(f^+)\pi(f^-) = 0$, due to the linearity of π we have $\pi(f^2) = \pi(f)^2$. Now the multiplicativity follows from the equality

$$fg = \tfrac{1}{4}((f+g)^2 - (f-g)^2).$$

□

Recall that a topological space X is called pseudocompact if $C(X) = C_b(X)$ ([3]). It is clear that any countable compact space is pseudocompact. Now the following corollary immediately follows from the above theorem.

Corollary 3. *Let X be a pseudocompact space and Y a topological space. A map $\pi\colon\ C(X) \to C(Y)$ is supra-additive and supra-multiplicative if and only if it is linear and multiplicative.*

Recall that a topological space is called *realcompact* if it is homeomorphic to a closed subspace of the product space of $\mathbb{R}$. It is well known that a Hausdorff space is compact if and only if it is realcompact and pseudocompact (see [3]). If K is a realcomapct space and $T\colon\ C(K) \to \mathbb{R}$ is nonzero linear and multiplicative then there exists $k \in K$ such that $T(f) = f(k)$ for each $f \in C(K)$ (see [2] for a simple proof). By using this fact we have the following theorem.

Theorem 4. *Let X be a realcompact space and let Y be an arbitrary topological space. Let $\pi\colon\ C(X) \to C(Y)$ be a supra-additive and supra-multiplicative map. Then the following assertions are equivalent.*

(i) $\pi(f^+ \wedge \mathbf{n} - f^- \wedge \mathbf{n})(y) \to \pi(f)(y)$ *for each* $f \in C(X)$ *and* $y \in Y$

(ii) *There exists a clopen subset $B \subset Y$ and a continuous function $\sigma\colon\ Y \to X$ such that*

$$\pi(f)(y) = \chi_B(y) f(\sigma(y))$$

for each $y \in Y$, $f \in C(X)$.

P r o o f. It is clear that (ii) $\Longrightarrow$ (i). Suppose that (i) holds. Then from Theorem 2, π is linear and multiplicative. The fact that $\pi(\mathbf{1})^2 = \pi(\mathbf{1})$ for each $y \in Y$ implies that either $\pi(\mathbf{1})(y) = 0$ or $\pi(\mathbf{1})(y) = 1$, so $B = \{y \in Y\colon\ \pi(\mathbf{1})(y) = 1\}$ is clopen in Y. Let $y \in Y$ be given. As X is realcompact and $\delta_y \circ \pi\colon\ C(X) \to \mathbb{R}$ is linear and multiplicative there exists $\alpha(y)$ such that

$$\pi(f)(y) = \pi(\mathbf{1})(y) f(\alpha(y)) = \chi_B(y) f(\alpha(y)).$$

Since X is completely regular Hausdorff space, $\alpha(y)$ must be unique for each $y \in B$. Let $x_0 \in Y$ be fixed and let $\sigma\colon\ Y \to X$ be defined by $\sigma(y) = \alpha(y)$ when $y \in B$ and $\sigma(y) = x_0$ otherwise. It is clear that $\sigma|_B\colon\ B \to X$ is continuous. Since B is clopen, actually σ itself is continuous. This completes the proof. □

References

[1] *C. D. Aliprantis, O. Burkinshaw*: Positive Operators. Academic Press, New York, 1985.
[2] *Z. Ercan, S. Önal*: A remark on the homomorphism on $C(X)$. Proc. Amer. Math. Soc. *133* (2005), 3609–3611.
[3] *K. P. Hart, J. Nagata, J. E. Vaughan*: Encyclopedia of General Topology. Elsevier, Amsterdam, 2004.
[4] *M. Radulescu*: On a supra-additive and supra-multiplicative operator of $C(X)$. Bull. Math. Soc. Sci. Math. Républ. Soc. Roum., Nouv. Sér. *24* (1980), 303–305.

Author's address: *Z. Ercan*, Middle East Technical University, Department of Mathematics, 065 31 Ankara, Turkey, e-mail: `zercan@metu.edu.tr`.