ISOMORPHISM OF COMMUTATIVE GROUP ALGEBRAS OF p-MIXED SPLITTING GROUPS OVER RINGS OF CHARACTERISTIC ZERO

Peter Danchev, Bulgaria

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Abstract. Suppose G is a p-mixed splitting abelian group and R is a commutative unitary ring of zero characteristic such that the prime number p satisfies $p \notin \mathrm{inv}(R) \cup \mathrm{zd}(R)$. Then $R(H)$ and $R(G)$ are canonically isomorphic R-group algebras for any group H precisely when H and G are isomorphic groups.

This statement strengthens results due to W. May published in J. Algebra (1976) and to W. Ullery published in Commun. Algebra (1986), Rocky Mt. J. Math. (1992) and Comment. Math. Univ. Carol. (1995).

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1. Introduction

Let $R(G)$ be the group algebra of G over R, where G is an arbitrary abelian group written multiplicatively as is customary when discussing group rings, and R is a commutative ring with identity 1_R and characteristic zero abbreviated as $\mathrm{char}(R) = 0$. For such a ring R, the set $\mathrm{inv}(R) = \{q\colon\ q \cdot 1_R \text{ is a unit in } R\}$ denotes the set of all invertible primes q in R whereas $\mathrm{zd}(R) = \{q\colon\ q \text{ is a zero divisor in } R\}$ denotes the set of all primes q that are zero divisors in R. As usual, in all that follows, R^+ will designate the additive group of R with torsion part $T(R^+)$; we emphasize that $T(R^+)$ is a proper ideal of R. For such a group G, the symbol $tG = \coprod_{\forall p} G_p$ denotes the maximal torsion subgroup of G with p-component G_p. All other undefined notions or unexplained notation as well as the terminology from the group algebra theory are standard and will follow the bibliography cited at the end of the paper, or more precisely the paper [2], in which the main result was previously announced, together with [11]–[13].

To make the present study more nearly self-contained and readers friendly, we shall give a brief introduction to some of the best principal achievements known in this area.

The more global investigation of group algebras over special sorts of commutative rings, mainly of zero characteristic, was started first by W. May in his remarkable works [9] and [10]. Later on, W. Ullery refined in ([11], [12] and [13]) upon the results of May. Actually, these explorations of Ullery are fundamental and established a natural connection between the Isomorphism Problems of modular and semi-simple group algebras of torsion abelian groups. We have studied in [4] some invariant properties of such group algebras over certain rings.

Our purpose in this modest work is to extend some of the aforementioned statements to the so-called p-mixed groups, whose only torsion is p-torsion. Particularly, let p be a prime and R a commutative ring with identity of characteristic 0 such that p is neither a unit nor a zero divisor in R. If G is a p-mixed abelian group that splits, and $R(G) \cong R(H)$ as R-algebras, then H splits as well; and, if in addition $R(G)$ and $R(H)$ are canonically R-isomorphic, in fact, $G \cong H$. In order to do this, we modify and develop the technique proposed by May-Ullery using moreover some new useful assertions pertaining to the topic explored.

2. The main result

Foremost, before formulating the main result, we need the following definition concerning certain properties of the isomorphisms between group algebras. It is well-known by virtue of [7], [8], [9] (see e.g. [3] too) that if L is a commutative unitary ring and if both A and B are abelian groups, then $L(A) \cong L(B)$ as L-algebras implies that $A/tA \cong B/tB$.

It is a simple matter to give an example so that not every L-isomorphism between $L(A)$ and $L(B)$ is canonical. Indeed, consider an example in which $A = B$. Let $A = C \times T$, where $C = \langle a \rangle$ is an infinite cyclic group with a generator a of infinite order $o(a) = \infty$ and T is a p-primary group, and let F be a field of characteristic p. Besides, let v be a p-torsion unit in $F(A)$ of augmentation 1 which is not a trivial unit, i.e. does not come from T. Consider the F-homomorphism of $F(A)$ to itself which sends a to av and is the identity on T. This is an F-isomorphism (= automorphism) since the reverse map sends a to av^{-1}, but clearly $A/tA \cong C$ is not taken to itself. By the same manner we can construct a non-canonical isomorphism of group algebras over a commutative ring R of zero characteristic.

Nevertheless, we conjecture that for any commutative unitary ring L and for any abelian groups A and B it is true that $L(A) \cong L(B)$ are isomorphic over L if and only if $L(A) \cong L(B)$ are canonically isomorphic over L. Because we may presume with no loss of generality that $L(A) = L(B)$, it would be necessary to find an L-automorphism of the group algebra $L(A)$ taking A/tA to itself. However, it is not at all clear how this can be done.

That is why, the following is of some actuality.

Definition 1. Under the above conditions for L, A and B, given that $\Phi\colon L(A) \to L(B)$ is an L-isomorphism of group algebras, then Φ is called a canonical L-isomorphism, and so $L(A)$ and $L(B)$ are called canonically L-isomorphic or canonically isomorphic as L-algebras, provided that $\Phi(A/tA) = B/tB$ that is Φ isomorphically maps A/tA to B/tB.

This notion is important and plays a crucial role in our future study.

The major goal here is to proceed by proving the following assertion, which motivated the present paper.

Central Theorem (Isomorphism). *Suppose G is a p-mixed splitting abelian group and R is a commutative ring with unity 1_R of $\mathrm{char}(R) = 0$ so that $p \notin \mathrm{inv}(R) \cup \mathrm{zd}(R)$. Then $R(H) \cong R(G)$ are canonically isomorphic as R-algebras for some other group $H \iff H \cong G$.*

Before proceeding to our proof, we need a few technical conventions quoted in several steps.

It is well-known and elementary to verify that R^+ is torsion-free $\iff$ $\mathrm{zd}(R) = \emptyset$. We now concentrate on another helpful property of R^+ concerning its p-divisibility.

Before doing this, it is desirable to recall certain notions and characters from the abelian group theory.

Definition 2. An additive abelian group Y is said to be p-divisible whenever $Y = pY$.

Lemma 1. *For any commutative ring R with identity,*

$$R^+ \text{ is } p\text{-divisible} \iff p \in \mathrm{inv}(R).$$

P r o o f. First, take R^+ to be p-divisible, i.e., by definition, $R^+ = pR^+$. Since $1_R \in R^+$, there exists an element $r \in R^+$ with $1_R = pr$. Thereby, $p \in \mathrm{inv}(R)$ and the first implication follows.

Next, we choose $p \in \mathrm{inv}(R)$ whence $pr = 1_R$ for some $r \in R$. Furthermore, for any $f \in R$ we infer that $f = p(rf) \in pR$, hence $R = pR$, thus ending the proof. □

R e m a r k. In view of the foregoing assertion $\mathrm{inv}(R) = \emptyset \iff R^+$ is not p-divisible over all prime numbers $p \iff R^+$ is not divisible. So, in terms of numerical divisions, that are structural invariants for R, we conclude that R^+ is torsion-free and not p-divisible $\iff \mathrm{zd}(R) = \emptyset$ and $p \notin \mathrm{inv}(R)$.

Lemma 2. *If $p \notin \mathrm{zd}(R)$, then R^+ is p-divisible $\iff R^+/T(R^+)$ is p-divisible.*

P r o o f. The necessity is self-evident.

Next, we deal with the sufficiency. Let $r \in R^+$, hence $r = pr' + f$ for some $r' \in R^+$ and $f \in T(R^+)$ since by hypothesis $R^+ = pR^+ + T(R^+)$. Therefore, there exists $n > 0$ with $n(r - pr') = 0$. Further, we distinguish two cases:

1) p/n. Thus $n = p^k m$; $k, m \in \mathbb{N}$ and $(p, m) = 1$. Henceforth $p^k(mr - mpr') = 0$. But $p \notin \mathrm{zd}(R)$, and so $m(r - pr') = 0$. That is why we may consider

2) $(p, n) = 1$. Consequently, there exist integers s and t such that $sn + tp = 1$. Moreover, $0 = sn(r-pr') = (1-tp)(r-pr') = r-p(r'+tr-ptr')$, which is equivalent to $r = p(tr + r' - ptr') \in pR^+$. Thereby, $R^+ = pR^+$ and everything we wanted is successfully proved. □

Actually, we have demonstrated that $T(R^+) \subseteq pR^+$, i.e. $T(R^+)$ is ever p-divisible, whenever $p \notin \mathrm{zd}(R)$.

Lemma 3. *Given that L is a commutative unitary ring of arbitrary characteristic and A is an abelian group, then*

(i) *$L(A)$ is unitary;*

(ii) *$\mathrm{char}(L(A)) = \mathrm{char}(L)$;*

(iii) *$p \in \mathrm{inv}(L(A)) \iff p \in \mathrm{inv}(L)$;*

(iv) *$p \in \mathrm{zd}(L(A)) \iff p \in \mathrm{zd}(L)$.*

P r o o f. The first two points are straightforward.

Next, we concern with the last two relationships.

First of all, it is apparent that $\mathrm{inv}(L) \subseteq \mathrm{inv}(L(A))$ because $L \subseteq L(A)$ with $1_{L(A)} = 1_L \cdot 1_A$ and thereby the second relation concerning the "sufficiency" is obvious. Now, to deal with the remaining part named "necessity", choose $p \in \mathrm{inv}(L(A))$. Hence there exists $\sum\limits_k r_k a_k \in L(A)$ with the property that $p\Big(\sum\limits_k r_k a_k\Big) = \sum\limits_k p r_k a_k = 1$. Therefore there is $r \in L$ such that $pr = 1$, whence $p \in \mathrm{inv}(L)$ and so we are done.

Further, we observe that $\mathrm{zd}(L) \subseteq \mathrm{zd}(L(A))$ holds since $L \subseteq L(A)$, so "sufficiency" is fulfilled. To treat the other half, termed "necessity", take $p \in \mathrm{zd}(L(A))$. Henceforth, there exists a non-trivial element $\sum\limits_k f_k c_k \in L(A)$ such that $p\Big(\sum\limits_k f_k c_k\Big) = \sum\limits_k p f_k c_k = 0$. Consequently, there is $f \in L$ with $pf = 0$, whence $p \in \mathrm{zd}(L)$, completing the proof. □

Proposition 1 ([5]). *Let G be a multiplicative p-mixed abelian group and let C be a multiplicative torsion-free abelian group which is not p-divisible. Then G splits if and only if $G \otimes_{\mathbb{Z}} C$ splits, where $\mathbb{Z}$ is regarded as the additive group of all integers.*

Proposition 2 ([9], [6], [1]). *For any commutative unitary ring L and any abelian group G, the following implication holds:*

$$L(G) \cong L(H) \Rightarrow L^+ \otimes_{\mathbb{Z}} G \cong L^+ \otimes_{\mathbb{Z}} H.$$

Remark. Since L^+ is an additive group while G and H are multiplicative ones, for our further application, to be more precise, we may interpret the last isomorphism relationship as $C \otimes_{\mathbb{Z}} G \cong C \otimes_{\mathbb{Z}} H$, where C is a multiplicative group isomorphic to L^+.

Proposition 3 ([13], [12]). *Assume that R is a commutative ring with unity of $\operatorname{char}(R) = 0$ such that $p \notin \operatorname{inv}(R) \cup \operatorname{zd}(R)$. If $R(A) \cong R(B)$ are R-isomorphic for some p-group A and an arbitrary group B, then $A \cong B$.*

Proposition 4. *For R a commutative ring with identity of characteristic zero such that $p \notin \operatorname{inv}(R) \cup \operatorname{zd}(R)$ and for abelian groups G and H, the R-isomorphism $R(G) \cong R(H)$ yields $P(G) \cong P(H)$ as P-algebras for some commutative ring P with identity and with $\operatorname{char}(P) = 0$ so that P^+ is not p-divisible and is torsion-free.*

Proof. Because of the ring homomorphism $R \to R/T(R^+) = P$, which is actually the natural map and endows P with the structure of an R-algebra, we conclude that $P(G) \cong P \otimes_R R(G) \cong P \otimes_R R(H) \cong P(H)$, i.e. $P(G) \cong P(H)$, as P-algebras. Referring to Lemmas 1 and 2, we find that P^+ is not a p-divisible group and is torsion-free, i.e. $p \notin \operatorname{inv}(P)$ and $\operatorname{zd}(P) = \emptyset$. That $\operatorname{char}(P) = 0$ follows trivially, but for completeness of the exposition we give arguments like these (see also [13, Proposition 2]). If the contrary, i.e. $\operatorname{char}(P) \neq 0$ holds, there is $m \in \mathbb{N}$: $m \cdot 1_R \in T(R^+)$, hence there exists $n \in \mathbb{N}$ with $nm \cdot 1_R = 0$. But $\operatorname{char}(R) = 0$ implies the desired contradiction, so the proof is over. $\square$

Now, we are ready to attack the

Proof of the Central Theorem. Let $\varphi\colon G \to H$ be an isomorphism. Clearly $\varphi(tG) = tH$ and thus $\varphi\colon G/tG \to H/tH$ is also an isomorphism. On the other hand, φ may be linearly extended to the isomorphism $\Phi\colon R(G) \to R(H)$ so that $\Phi_G = \varphi$. This allows us to deduce that Φ would carry G/tG to H/tH, i.e. in terms of our definition Φ is taken to be canonical.

Turning to the necessity, since $p \notin \operatorname{inv}(R)$, it is long-known that (see, for instance, [7]) there is a maximal ideal $J \lhd R$ such that $p \in J$. Hence $F = R/J$ is a field of $\mathrm{char}(F) = p$ and by tensor multiplication with F over R, similarly to the above, we derive $F(G) \cong F \otimes_R R(G) \cong F \otimes_R R(H) \cong F(G)$, that is $F(G) \cong F(H)$. Furthermore, it is easily seen that H must be p-mixed, too (see, for example, [3]), hence $tH = H_p$.

Besides, the application of [7] or [3] leads to $G/tG \cong H/tH$.

Further, by combining Propositions 1, 2 and 4, we deduce that H is splitting as well. Thus we write $G \cong tG \times G/tG$ and $H \cong tH \times H/tH$. Consequently, $R(G) \cong R(G/tG)(tG) = K(tG)$ and by symmetry $R(H) \cong R(H/tH)(tH) \cong R(G/tG)(tH) = K(tH)$ whence, under our additional circumstances that the R-isomorphism carries G/tG onto H/tH, we detect that $R(G) \cong R(H)$ as R-algebras insures $K(tG) \cong K(tH)$ as K-algebras via putting $K = R(G/tG)$. Employing now Lemma 3, one may conclude that K is a commutative ring with 1 of $\mathrm{char}(K) = 0$ such that $p \notin \mathrm{inv}(K) \cup \mathrm{zd}(K)$. That is why Proposition 3 applies yielding $tG \cong tH$.

Finally, $G \cong tG \times G/tG \cong tH \times H/tH \cong H$, i.e. $G \cong H$, thus the proof is complete in all generality. □

As is the tradition, $J(R)$ denotes the Jacobson radical of a ring R.

Central Corollary. *Suppose G is a splitting p-mixed abelian group and R is a commutative ring with 1 of $\mathrm{char}(R) = 0$ such that $p \in J(R) \setminus \mathrm{inv}(R)$. Then $R(H) \cong R(G)$ being canonically isomorphic as R-algebras for another group H implies that $H \cong G$.*

P r o o f. The assertion follows by combining Central Theorem and the proof of Proposition 3 from [13]. This ends our proof. □

E x a m p l e. The restriction on G to be p-mixed cannot be dropped. Indeed, let $p_1, p_2, \ldots, p_n$ $(n \geqslant 2)$ be distinct primes, and consider the direct product $R = \mathbb{Z}[1/p_1] \times \ldots \times \mathbb{Z}[1/p_n]$. Plainly $\mathrm{inv}(R) \cup \mathrm{zd}(R) = \emptyset$, hence R is a torsion-free commutative ring with 1 of $\mathrm{char}(R) = 0$ for which $p \notin \mathrm{inv}(R) \cup \mathrm{zd}(R)$ over each prime number p. Even much more, R is not an ND-ring (= nicely decomposing) in the sense of Ullery ([11], [12]) since $\mathrm{inv}(R) = \emptyset$ while each direct factor of the decomposition of R contains an invertible prime. Consulting ([11], Example 1) and ([12], Example 3.3) there exist two mixed groups G and H such that $G \ncong H$ but $R(G) \cong R(H)$ as R-algebras; it is not directly seen whether or not this R-isomorphism is canonical or can be chosen by extending to be canonical. Note that, via the construction of Ullery, G and H are not p-mixed for any prime p because $\mathrm{supp}\,(G) = \mathrm{supp}\,(H)$ possesses at least two prime numbers, specifically $\{p_1, \ldots, p_n\}$. Therefore R does not satisfy the Isomorphism Theorem in the terminology of Ullery ([11], [12]), namely that $R(G) \cong R(H)$ as R-algebras does not always imply that $G/\coprod_{\forall q \in \mathrm{inv}(R)} G_q \cong H/\coprod_{\forall q \in \mathrm{inv}(R)} H_q$ for all groups G and H.

Generally, returning to the whole mixed case, we will formulate below a universal algorithm, analogous to that of Ullery ([11]–[13]), about the isomorphism of group algebras of mixed abelian groups.

Theorem 1 (Equivalence)**.** *The following two implications are equivalent:*

(a) *For each field F of* $\mathrm{char}(F) = p \neq 0$ *and for each p-mixed abelian group G, the F-isomorphism $F(H) \cong F(G)$ for some group H implies $H \cong G$.*

(b) *For every commutative unitary ring R of* $\mathrm{char}(R) = 0$ *and for every p-mixed abelian group G such that $p \notin \mathrm{inv}(R)$, the R-isomorphism $R(H) \cong R(G)$ for any group H yields $H \cong G$.*

P r o o f. What we need to prove is that $F(G) \cong F(H) \iff R(G) \cong R(H)$ for appropriate objects F and R described above.

(a) $\Rightarrow$ (b). Given that $R(G) \cong R(H)$ as R-algebras, then, since $p \notin \mathrm{inv}(R)$, there is a maximal ideal J of R such that $p \in J$. Henceforth, $F = R/J$ is a field of $\mathrm{char}(F) = p > 0$ and $F(G) \cong F \otimes_R R(G) \cong F \otimes_R R(H) \cong F(H)$ as F-algebras. Therefore, under our hypothesis, $G \cong H$, as expected.

(b) $\Rightarrow$ (a). Assume now that $F(G) \cong F(H)$ as F-algebras. It is then clear that H is p-mixed too. Hence by [7] and an assertion of Karpilovsky [6] we derive $G/tG \cong H/tH$ and $|tG| = |tH|$. Constructing $R = F \times E$, where E is an algebraically closed field of $\mathrm{char}(E) = 0$, we deduce appealing to [8] that $E(G) \cong E(H)$ which along with $F(G) \cong F(H)$ implies $R(G) \cong F(G) \times E(G) \cong F(H) \times E(H) \cong R(H)$. Moreover, it is simply checked that $\mathrm{char}(R) = 0$ since $\mathrm{char}(E) = 0$ and that $p \notin \mathrm{inv}(R)$ since $p \notin \mathrm{inv}(F)$. Thereby, the hypothesis enables us that $G \cong H$, as asserted. $\square$

In what follows, we assume that the *Splitting Group Bases Problem*, namely that the abelian group G being p-mixed splitting and $F(G) \cong F(H)$ as F-algebras for some field F of non-zero characteristic p force that H is also a splitting p-mixed abelian group and, as a consequence, that $F(G_p) \cong F(H_p)$, both hold in the affirmative.

Theorem 2 (Equivalence)**.** *The following two points are equivalent:*

(a′) *For every field F of* $\mathrm{char}(F) = p \neq 0$ *and for every abelian p-group G, the F-isomorphism $F(H) \cong F(G)$ for some group H ensures $H \cong G$.*

(b′) *For each commutative unitary ring R of* $\mathrm{char}(R) = 0$ *and for each p-mixed splitting abelian group G such that $p \notin \mathrm{inv}(R)$, the R-isomorphism $R(H) \cong R(G)$ for any group H ensures $H \cong G$.*

P r o o f. (a′) $\Rightarrow$ (b′). Given that $R(G) \cong R(H)$, then, as in the preceding Theorem 1, we infer $F(G) \cong F(H)$ for some field F of $\mathrm{char}(F) = p > 0$ that depends on R and, under our circumstances, that $G \cong tG \times G/tG$ as well as $H \cong tH \times H/tH$

with $tG = G_p$ and $tH = H_p$. By virtue of our above noted assumption, which refines the Ullery's one for torsion groups ([12]), we have $F(tG) \cong F(tH)$. Consequently, our assumption leads to $tG \cong tH$. Besides, by the same argument as in the previous point, $G/tG \cong H/tH$. Finally, $G \cong H$ and we are done.

(b′) ⇒ (a′). Let $F(G) \cong F(H)$ and let G be an abelian p-group, whence H is also p-torsion abelian. Same as above, we obtain $R(G) \cong R(H)$ for a suitable ring R of $\mathrm{char}(R) = 0$ such that $p \notin \mathrm{inv}(R)$. Since both G and H being p-primary are obviously p-mixed splitting, our assumption is applicable to get that $G \cong H$, as required. □

Theorem 3 (Equivalence)**.** *The following two conditions are equivalent:*

(a″) *For each field F of $\mathrm{char}(F) = p \neq 0$ and for each p-mixed splitting abelian group G, the group algebras $F(H)$ and $F(G)$ are canonically F-isomorphic over another group H precisely when H and G are isomorphic.*

(b″) *For every commutative unitary ring R of $\mathrm{char}(R) = 0$ and for every abelian p-group G such that $p \notin \mathrm{inv}(R)$, the group algebras $R(H)$ and $R(G)$ are canonically R-isomorphic over any group H only when H and G are isomorphic.*

P r o o f. (a″) ⇒ (b″). Suppose that $R(G) \cong R(H)$ and that G is an abelian p-group. In the same manner as in the foregoing theorem, we establish $F(G) \cong F(H)$ for a field F of $\mathrm{char}(F) = p > 0$. But because H is obviously a p-group as well, hence both G and H are p-mixed splitting, which is ensured via the hypothesis cited above, the assumption from the text means that $G \cong H$, as wanted.

(b″) ⇒ (a″) Let $F(G) \cong F(H)$ and let G as well as H be both p-mixed splitting abelian groups. Hence we write $G \cong tG \times G/tG$ and $H \cong tH \times H/tH$ so that $tG = G_p$ and $tH = H_p$. Invoking [7], $G/tG \cong H/tH$. As we have observed above, $R(G) \cong R(H)$ as R-algebras for a ring R of $\mathrm{char}(R) = 0$ such that $p \notin \mathrm{inv}(R)$. Since $R(G/tG) \cong R(H/tH)$, we infer at once that $R(G) \cong [R(G/tG)](tG) \cong [R(H/tH)](tG)$ and that $[R(H/tH)](tH) \cong R(H)$, whence by defining $K = R(H/tH)$ and taking into account the extra requirements on the isomorphisms over F and R we have $K(tG) \cong K(tH)$ as K-algebras. Employing Lemma 3, $\mathrm{char}(K) = 0$ and $p \notin \mathrm{inv}(K)$. That is why the hypothesis from the text assures that $tG \cong tH$. Finally, $G \cong tG \times G/tG \cong tH \times H/tH \cong H$, as claimed. □

3. Concluding discussion and remarks

In closing, we comment some more specific aspects of the direction presented. In fact, we first conjecture that both the restrictions $p \notin \mathrm{zd}(R)$ and G is splitting can be ignored, by stating the following

Conjecture. *Suppose G is a p-mixed abelian group and R is a commutative unitary ring of* $\mathrm{char}(R) = 0$ *such that* $p \notin \mathrm{inv}(R)$. *Then $R(H)$ and $R(G)$ are R-isomorphic group algebras for some arbitrary group H if and only if H and G are isomorphic groups.*

Notice that we have demonstrated above in three independent variants that the last problem is equivalent to its modular analogue, by generalizing the corresponding ones due to Ullery ([11]–[13]). Nevertheless, the full solution to the conjecture seems to be in the distant future.

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Author's address: *Peter Danchev*, 13, General Kutuzov Street, block 7, floor 2, apartment 4, 4003 Plovdiv, Bulgaria-BGR, e-mail: `pvdanchev@yahoo.com`.