ON THE ALGEBRA OF A^k-FUNCTIONS

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Abstract. For a domain $\Omega \subset \mathbb{C}^n$ let $H(\Omega)$ be the holomorphic functions on Ω and for any $k \in \mathbb{N}$ let $A^k(\Omega) = H(\Omega) \cap C^k(\overline{\Omega})$. Denote by $\mathscr{A}_D^k(\Omega)$ the set of functions $f\colon \Omega \to [0,\infty)$ with the property that there exists a sequence of functions $f_j \in A^k(\Omega)$ such that $\{|f_j|\}$ is a nonincreasing sequence and such that $f(z) = \lim\limits_{j\to\infty} |f_j(z)|$. By $\mathscr{A}_I^k(\Omega)$ denote the set of functions $f\colon \Omega \to (0,\infty)$ with the property that there exists a sequence of functions $f_j \in A^k(\Omega)$ such that $\{|f_j|\}$ is a nondecreasing sequence and such that $f(z) = \lim\limits_{j\to\infty} |f_j(z)|$. Let $k \in \mathbb{N}$ and let Ω_1 and Ω_2 be bounded A^k-domains of holomorphy in $\mathbb{C}^{m_1}$ and $\mathbb{C}^{m_2}$ respectively. Let $g_1 \in \mathscr{A}_D^k(\Omega_1)$, $g_2 \in \mathscr{A}_I^k(\Omega_1)$ and $h \in \mathscr{A}_D^k(\Omega_2) \cap \mathscr{A}_I^k(\Omega_2)$. We prove that the domains $\Omega = \{(z,w) \in \Omega_1 \times \Omega_2\colon\ g_1(z) < h(w) < g_2(z)\}$ are A^k-domains of holomorphy if $\operatorname{int}\overline{\Omega} = \Omega$. We also prove that under certain assumptions they have a Stein neighbourhood basis and are convex with respect to the class of A^k-functions. If these domains in addition have C^1-boundary, then we prove that the A^k-corona problem can be solved. Furthermore we prove two general theorems concerning the projection on $\mathbb{C}^n$ of the spectrum of the algebra A^k.

Keywords: A^k-domains of holomorphy, A^k-convexity

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1. Introduction

For a domain Ω in $\mathbb{C}^n$ let $H(\Omega)$ denote the holomorphic functions on Ω and for any natural number $k \in \mathbb{N} = \{0,1,2,\ldots\}$ let $A^k(\Omega)$ denote the set $H(\Omega) \cap C^k(\overline{\Omega})$. According to the Cartan-Thullen theorem ([3]) a domain Ω in $\mathbb{C}^n$ is a domain of holomorphy if and only if it is convex with respect to the holomorphic functions on Ω. This means that domains of holomorphy (which are defined using the ambient

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space $\mathbb{C}^n$) can be characterized by an intrinsic property in terms of convexity conditions with respect to $H(\Omega)$. Furthermore the solution of the Levi Problem ([8], [1], [7]) shows that a domain in $\mathbb{C}^n$ is a domain of holomorphy if and only if it is locally a domain of holomorphy.

For $A^k(\Omega)$ the situation is different. It is not known whether a domain that is locally an A^k-domain of holomorphy is an A^k-domain of holomorphy. This makes it much more difficult to analyse the A^k-situation and few results have been obtained. In general an A^k-domain of holomorphy does not have to be convex with respect to the class of A^k-functions and there are also examples of A^k-convex domains which are not A^k-domains of holomorphy. (See section 3.)

M. Jarnicki and P. Pflug ([6]) have shown that any bounded Reinhardt domain Ω in $\mathbb{C}^n$ such that $\operatorname{int}\overline{\Omega} = \Omega$ is an A^k-domain of holomorphy for any $k \in \mathbb{N}$. Moreover it follows from work of D. Catlin ([2]) and M. Hakim and N. Sibony ([5]) that a bounded pseudoconvex domain with C^∞-boundary is an A^k-convex A^k-domain of holomorphy for any $0 \leqslant k \leqslant \infty$.

In this paper we study the algebra of A^k-functions on domains in $\mathbb{C}^n$. First we treat the notion of sequential A^k-convexity. We then introduce a class of domains and we prove in Theorem 4.2, using properties of the spectrum of A^k, that these domains are A^k-domains of holomorphy for every $k \in \mathbb{N}$. In section 5 we prove two general theorems (Theorem 5.1 and Theorem 5.3), which are of independent interest, concerning the projection on $\mathbb{C}^n$ of the spectrum of A^k. Under certain assumptions we then prove, in Theorem 5.4, that the domains considered in the statement of Theorem 4.2 have a Stein neighbourhood basis and if in addition they have C^1-boundary we use the results obtained to prove that the A^k-corona problem can be solved. In the last section we prove that the domains considered in the statement of Theorem 5.4 are A^k-convex.

2. Preliminaries

We study properties of A^k-domains of holomorphy and A^k-convex domains for $k \in \mathbb{N}$. With the norm

$$\|f\|_{k,\Omega} = \sum_{\alpha\in\mathbb{N}^n, |\alpha|\leqslant k} \sup_{z\in\Omega} |D^\alpha f(z)|$$

$A^k(\Omega) = H(\Omega) \cap C^k(\overline{\Omega})$ is a Banach algebra. The set of nonzero multiplicative complex homomorphisms on $A^k(\Omega)$ is called the spectrum of $A^k(\Omega)$, when it is equipped with the weak*-topology. We denote the spectrum by $\mathfrak{M}^{A^k(\Omega)}$. For $z \in \overline{\Omega}$ the point evalutaion m_z is defined by $m_z(f) = f(z)$ for every $f \in A^k(\Omega)$. The closure of the domain Ω can then be embedded as a subset $\overline{\Omega}_e = \{m_z\colon\ z \in \overline{\Omega}\}$ of $\mathfrak{M}^{A^k(\Omega)}$.

Definition 2.1. A domain $\Omega \subset \mathbb{C}^n$ is said to be A^k-convex (or convex with respect to the class of A^k-functions) if for every compact subset K of Ω the set

$$\widehat{K}_{A^k} = \Big\{ z \in \Omega\colon\ |f(z)| \leqslant \sup_{\zeta \in K} |f(\zeta)| \quad \forall f \in A^k(\Omega) \Big\}$$

is a compact subset of Ω. The set $\widehat{K}_{A^k}$ is called the A^k-convex hull of K in Ω.

Definition 2.2. A domain $\Omega \subset \mathbb{C}^n$ is said to be an A^k-domain of holomorphy (or a domain of existence for A^k) if there do not exist nonempty open sets Ω_1 and Ω_2 such that

(1) $\Omega_1 \subset \Omega_2 \cap \Omega$
(2) Ω_2 is connected and not contained in Ω
(3) for every function $u \in A^k(\Omega)$ there is a function u_2 holomorphic on Ω_2 such that $u = u_2$ on Ω_1.

For every domain $\Omega \subset \mathbb{C}^n$ there exists a unique A^k-envelope of holomorphy $(\Omega, \Pi, \mathbb{C}^n)$ which is a Riemann domain spread over $\mathbb{C}^n$ ([11]).

It is easy to see that the interior of the intersection of any family of A^k-domains of holomorphy is an A^k-domain of holomorphy and that the interior of the intersection of any family of A^k-convex domains is an A^k-convex domain. A bounded pseudoconvex domain with C^∞-boundary is an A^k-convex A^k-domain of holomorphy. This implies that the increasing union of A^k-domains of holomorphy (respectively, A^k-convex domains) does not have to be an A^k-domain of holomorphy (respectively, A^k-convex domain) since an arbitrary pseudoconvex domain can be exhausted by an increasing sequence of bounded pseudoconvex domains with C^∞-boundary.

The following proposition will be used later on:

Proposition 2.3. *Let D_1 and D_2 be A^k-domains of holomorphy in $\mathbb{C}^{m_1}$ and $\mathbb{C}^{m_2}$ respectively. Then $\Omega = D_1 \times D_2 \subset \mathbb{C}^{m_1+m_2}$ is an A^k-domain of holomorphy.*

Proof. Suppose that Ω is not an A^k-domain of holomorphy. Then there exist open sets Ω_1 and Ω_2 as in Definition 2.2 and since Ω_2 intersects the boundary of Ω it intersects either $\partial D_1 \times D_2$ or $D_1 \times \partial D_2$. In either case there is a function in $A^k(\Omega)$ which cannot be continued to Ω_2. $\square$

3. Sequential A^k-convexity

One way of proving that a domain is A^k-convex or an A^k-domain of holomorphy is to show that it is sequentially A^k-convex. We recall that a domain in $\mathbb{C}^n$ is a domain of holomorphy if and only if for every discrete sequence $\{p_j\}_{j=0}^{\infty}$ in Ω there exists a function $f \in H(\Omega)$ such that $\sup\limits_{j\in\mathbb{N}} |f(p_j)| = +\infty$. We will see that a corresponding notion for A^k is a sufficient condition for a domain to be an A^k-domain of holomorphy as well as an A^k-convex domain. It is however not a necessary condition.

Definition 3.1. A domain $\Omega \subset \mathbb{C}^n$ is said to be sequentially A^k-convex if for every discrete sequence $\{p_j\}_{j=0}^{\infty}$ in Ω there exists a function $f \in A^k(\Omega)$, not identically constant, such that $\sup\limits_{j\in\mathbb{N}} |f(p_j)| = \|f\|_{L^\infty(\Omega)}$.

Proposition 3.2. *A sequentially A^k-convex domain $\Omega \subset \mathbb{C}^n$ is A^k-convex.*

Proof. Suppose $\Omega \subset \mathbb{C}^n$ is not A^k-convex. Then there exists a compact set K in Ω such that $\widehat{K}_{A^k}$ is not a compact subset of Ω and hence we can find a discrete sequence $\{p_j\}_{j=0}^{\infty} \subset \widehat{K}_{A^k}$ such that $|f(p_j)| \leqslant \|f\|_{L^\infty(K)}$ for every $f \in A^k(\Omega)$. It follows from the maximum principle for holomorphic functions that $\sup\limits_{j\in\mathbb{N}} |f(p_j)| < \|f\|_{L^\infty(\Omega)}$ for every non-constant function $f \in A^k(\Omega)$ and this means that Ω is not sequentially A^k-convex. $\square$

Proposition 3.3. *A sequentially A^k-convex domain $\Omega \subset \mathbb{C}^n$ is an A^k-domain of holomorphy.*

Proof. Suppose Ω is not an A^k-domain of holomorphy. Then there exist open sets Ω_1 and Ω_2 as in Definition 2.2. In particular, Ω_2 is not a subset of Ω, but $\Omega \cap \Omega_2 \neq \emptyset$. Let K be a compact set in $\Omega \cup \Omega_2$ such that $K \setminus \Omega_2$ is a compact subset in Ω. We choose a discrete sequence $\{p_j\}_{j=0}^{\infty} \subset \Omega \cap K$. It follows from the maximum principle for holomorphic functions and the fact that holomorphic functions cannot increase in norm when extended, that $\sup\limits_{j\in\mathbb{N}} |f(p_j)| < \|f\|_{L^\infty(\Omega)}$ for every non-constant function f in $A^k(\Omega)$. $\square$

In [10] N. Sibony constructed a pseudoconvex Runge domain Ω contained in the bidisk $\Delta^2 \subset \mathbb{C}^2$ such that $\operatorname{int}\overline{\Omega} = \Omega$, $\Delta^2 \setminus \Omega \neq \emptyset$ and so that all bounded holomorphic functions on Ω can be holomorphically continued to Δ^2. Hence Ω is not an A^k-domain of holomorphy for any $k \in \mathbb{N}$ but since it is Runge, it follows that it is also convex with respect to the class of A^k-functions. By Proposition 3.3 the domain Ω cannot be sequentially A^k-convex. Moreover the Hartogs triangle $\{(z_1, z_2) \in \mathbb{C}^2 \colon\ |z_1| < |z_2| < 1\}$ is not a sequentially A^k-convex domain since it is not an A^k-convex domain for any $k \in \mathbb{N}$. It is however an A^k-domain of holomorphy for every $k \in \mathbb{N}$ and therefore the following corollary can be established.

Corollary 3.4. *There exists a bounded domain $D_1 \subset \mathbb{C}^2$ with $\operatorname{int} \overline{D}_1 = D_1$ which for every $k \in \mathbb{N}$ is an A^k-convex domain but not a sequentially A^k-convex domain.*

There also exists a bounded domain $D_2 \subset \mathbb{C}^2$ with $\operatorname{int} \overline{D}_2 = D_2$ which for every $k \in \mathbb{N}$ is an A^k-domain of holomorphy but not a sequentially A^k-convex domain.

We remark that as a consequence of Proposition 3.2 and Proposition 3.3 a domain Ω for which every boundary point is a peak point for $A^k(\Omega)$ is an A^k-convex A^k-domain of holomorphy.

4. A^k-domains of holomorphy

We now study domains of existence for the class of A^k-functions on domains in $\mathbb{C}^n$. We first prove the following lemma.

Lemma 4.1. *Let $k \in \mathbb{N}$ and let Ω be a bounded domain in $\mathbb{C}^n$. For every element m in the spectrum $\mathfrak{M}^{A^k(\Omega)}$ of $A^k(\Omega)$ the following inequality holds:*

$$|m(f)| \leqslant \sup_{z \in \Omega} |f(z)|, \quad f \in A^k(\Omega).$$

Proof. Suppose there is an element $m \in \mathfrak{M}^{A^k(\Omega)}$ and a function $f \in A^k(\Omega)$ such that

$$m(f) = \lambda, \text{ where } |\lambda| > \sup_{z \in \Omega} |f(z)|.$$

Then the function

$$g(z) = \frac{1}{f(z) - \lambda}$$

belongs to $A^k(\Omega)$ and $m(g(f-\lambda)) = 1$. On the other hand

$$\begin{aligned} m\left(g(f-\lambda)\right) &= m(g) \cdot m(f-\lambda) \\ &= m(g) \cdot (m(f) - \lambda) = 0. \end{aligned}$$

This contradiction completes the proof of the lemma. $\square$

Let Ω be a bounded domain in $\mathbb{C}^n$. Denote by $\mathscr{A}^k_D(\Omega)$ the set of functions $f\colon \Omega \to [0, \infty)$ with the property that there exists a sequence of functions $f_j \in A^k(\Omega)$ such that $\{|f_j|\}$ is a nonincreasing sequence and such that

$$f(z) = \lim_{j \to \infty} |f_j(z)|.$$

Furthermore, denote by $\mathscr{A}_I^k(\Omega)$ the set of functions $f\colon \Omega \to (0,\infty)$ with the property that there exists a sequence of nonvanishing functions $f_j \in A^k(\Omega)$ such that $\{|f_j|\}$ is an nondecreasing sequence and such that

$$f(z) = \lim_{j\to\infty} |f_j(z)|.$$

It follows from the definitions that the functions in $\mathscr{A}_D^k$ are nonnegative and plurisubharmonic and that the functions in $\mathscr{A}_I^k$ are positive and plurisuperharmonic.

We now introduce a class of domains defined by functions in $\mathscr{A}_D^k$ and $\mathscr{A}_I^k$ and we prove the following theorem:

Theorem 4.2. *Let $k \in \mathbb{N}$ and let Ω_1 and Ω_2 be bounded A^k-domains of holomorphy in $\mathbb{C}^{m_1}$ and $\mathbb{C}^{m_2}$ respectively. Let $g_1 \in \mathscr{A}_D^k(\Omega_1)$, $g_2 \in \mathscr{A}_I^k(\Omega_1)$ and $h \in \mathscr{A}_D^k(\Omega_2) \cap \mathscr{A}_I^k(\Omega_2)$. If the domain Ω defined by*

$$\Omega = \{(z,w) \in \Omega_1 \times \Omega_2\colon\ g_1(z) < h(w) < g_2(z)\}$$

fulfills $\operatorname{int}\overline{\Omega} = \Omega$*, then Ω is an A^k-domain of holomorphy.*

P r o o f. Suppose that $g_1(z) = \lim\limits_{j\to\infty} |g_{1,j}(z)|$ and that $g_2(z) = \lim\limits_{j\to\infty} |g_{2,j}(z)|$ where $\{|g_{1,j}|\}$ is a nonincreasing sequence and $\left\{|g_{2,j}|\right\}$ a nondecreasing sequence of nonvanishing functions where $g_{1,j}$ and $g_{2,j}$ belong to $A^k(\Omega_1)$. Suppose also that $h(z) = \lim\limits_{j\to\infty} |h_{1,j}(z)| = \lim\limits_{j\to\infty} |h_{2,j}(z)|$ where $\{|h_{1,j}|\}$ is an nonincreasing sequence and $\{|h_{2,j}|\}$ a nondecreasing sequence of nonvanishing functions where $h_{1,j}$ and $h_{2,j}$ belong to $A^k(\Omega_2)$.

Suppose that Ω is not an A^k-domain of holomorphy. Since $\operatorname{int}\overline{\Omega} = \Omega$ the A^k-envelope of holomorphy, $(\tilde{\Omega}, \Pi, \mathbb{C}^{m_1+m_2})$, of Ω contains a point $\tilde{z}$ such that $\Pi(\tilde{z}) = (z^0, w^0) \notin \overline{\Omega}$. We will see that this leads to a contradiction.

There exists a complex homomorphism m^0 in $\mathfrak{M}^{A^k(\Omega)}$ such that

$$m^0(f) = \tilde{f}(z^0, w^0) \text{ for every } f \in A^k(\Omega) \tag{4.1}$$

where $\tilde{f}$ denotes the holomorphic continuation of f to $(\tilde{\Omega}, \Pi, \mathbb{C}^{m_1+m_2})$. Since Ω_1 and Ω_2 are A^k-domains of holomorphy, it follows from Proposition 2.3 that $z^0 \in \Omega_1$ and $w^0 \in \Omega_2$.

Define for every $n \in \mathbb{N}$ the functions

$$\gamma_{n,1,i}(z,w) = \begin{cases} 0 & \text{if } g_{1,i} \equiv 0, \\ \dfrac{g_{1,i}^{k+n}(z)}{\|g_{1,i}\|_{L^\infty(\Omega_1)}^k h_{1,i}^n(w)} & \text{otherwise} \end{cases}$$

and

$$\gamma_{n,2,i}(z,w) = \frac{h_{2,i}(w)^{k+n}}{\|g_{2,i}\|^k_{L^\infty(\Omega_1)} g^n_{2,i}(z)}.$$

For i large enough we have $|g_{1,i}(z)| < |h_{1,i}(w)|$ and $|h_{2,i}(w)| < |g_{2,i}(z)|$ when $(z,w) \in \Omega$ and therefore $\gamma_{n,1,i}$ and $\gamma_{n,2,i}$ belong to $A^k(\Omega)$. Furthermore we have $\|\gamma_{n,1,i}\|_{L^\infty(\Omega)} \leqslant 1$ and $\|\gamma_{n,2,i}\|_{L^\infty(\Omega)} \leqslant 1$. For i large enough we get, using Lemma 4.1, that

$$|m^0(\gamma_{n,1,i})| = \frac{|m^0(g_{1,i})|^{k+n}}{\|g_{1,i}\|^k_{L^\infty(\Omega_1)}|m^0(h_{1,i})|^n} = \frac{|g_{1,i}(z^0)|^{k+n}}{\|g_{1,i}\|^k_{L^\infty(\Omega_1)}|h_{1,i}(w^0)|^n} \leqslant 1$$

and

$$|m^0(\gamma_{n,2,i})| = \frac{|m^0(h_{2,i})|^{k+n}}{\|g_{2,i}\|^k_{L^\infty(\Omega_1)}|m^0(g_{2,i})|^n} = \frac{|h_{2,i}(w^0)|^n}{\|g_{2,i}\|^k_{L^\infty(\Omega_1)}|g_{2,i}(z^0)|^n} \leqslant 1$$

which implies that

$$|g_{1,i}(z^0)|^{k+n} \leqslant \|g_{1,i}\|^k_{L^\infty(\Omega_1)}|h_{1,i}(w^0)|^n$$

and

$$|h_{2,i}(w^0)|^{k+n} \leqslant \|g_{2,i}\|^k_{L^\infty(\Omega_1)}|g_{2,i}(z^0)|^n$$

for all $n \in \mathbb{N}$. Hence

$$|g_{1,i}(z^0)| \leqslant |h_{1,i}(w^0)| \text{ and } |h_{2,i}(w^0)| \leqslant |g_{2,i}(z^0)|.$$

This holds for every i large enough, so we conclude that

$$g_1(z^0) \leqslant h(w^0) \leqslant g_2(z^0)$$

which means that (z^0, w^0) belongs to $\overline{\Omega}$. This contradiction concludes the proof of the theorem. $\square$

A comparison with the class $A^\infty(\Omega) = C^\infty(\overline{\Omega}) \cap H(\Omega)$ gives that the domains considered in the statement of Theorem 4.2 do not have to be A^∞-domains of holomorphy. (See Remark 1 on page 60.)

It is not difficult to see that the proof of Theorem 4.2 can be modified to give the following proposition.

Proposition 4.3. *Let $k \in \mathbb{N}$ and let Ω_1 be a bounded A^k-domain of holomorphy in $\mathbb{C}^n$. Let $g \in \mathscr{A}_I^k(\Omega_1)$. Then the Hartogs domain Ω defined by*

$$\Omega = \{(z, w) \in \Omega_1 \times \mathbb{C}\colon\ |w| < g(z)\}$$

is an A^k-domain of holomorphy if $\operatorname{int} \overline{\Omega} = \Omega$.

5. Spectrum properties

Recall that $\overline{\Omega}_e$ denotes the embedding of the point evaluations on $\overline{\Omega}$ in the spectrum $\mathscr{M}^{A^k(\Omega)}$ (see Section 2). In this section we show that if Ω is a pseudoconvex domain with C^1-boundary in $\mathbb{C}^n$ which has the property that the projection of the spectrum of $A^k(\Omega)$ on $\mathbb{C}^n$ equals $\overline{\Omega}$, then the spectrum in fact equals $\overline{\Omega}_e$. We then show that if a domain Ω has a Stein neigbourhood basis, then the projection of the spectrum of $A^k(\Omega)$ equals the closure of Ω. We also show that the domains studied in Section 4 have, under certain conditions, a Stein neighbourhood basis. We conclude that if Ω is such a domain with C^1-boundary, then the A^k-corona problem can be solved.

For a domain $\Omega \subset \mathbb{C}^n$ we will denote by π the projection of the spectrum of $A^k(\Omega)$ on $\mathbb{C}^n$ defined by

$$\pi(m) = (m(z_1), \dots, m(z_n)), \quad m \in \mathscr{M}^{A^k(\Omega)}.$$

Observe that the closure of Ω is always a subset of $\pi(\mathscr{M}^{A^k(\Omega)})$. The following proposition gives a sufficient condition for the equality $\overline{\Omega}_e = \mathscr{M}^{A^k(\Omega)}$ to hold:

Theorem 5.1. *Let $k \in \mathbb{N}$ and let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^1-boundary. If the projection $\pi(\mathscr{M}^{A^k(\Omega)})$ of the spectrum of $A^k(\Omega)$ equals $\overline{\Omega}$, then $\mathscr{M}^{A^k(\Omega)} = \overline{\Omega}_e$.*

Proof. Let f be an arbitrary function in $A^k(\Omega)$ and define the continuous function $F\colon \mathscr{M}^{A^k(\Omega)} \to \mathbb{C}$ as $F(m) = f \circ \pi(m)$. Since $\pi(\mathscr{M}^{A^k(\Omega)}) = \overline{\Omega}$, the function is well-defined. By B we denote the uniform algebra generated by F and $A^k(\Omega)$. Clearly the Shilov boundary $\delta A^k(\Omega)$ of $A^k(\Omega)$ is a subset of the Shilov boundary δB of B.

Since Ω is a bounded pseudoconvex domain with C^1-boundary, it follows from a result by M. Hakim and N. Sibony ([5], Lemma 3), that for every $m \in \mathscr{M}^{A^k(\Omega)}$ there exists a neighbourhood U of m such that F can be uniformly approximated on U by functions in $A^k(\Omega)$. From this it follows (Lemma 9.1, p. 93, [4]) that $\delta B \subset \delta A^k(\Omega)$

and hence $\delta B = \delta A^k(\Omega)$. Furthermore the Shilov boundary of $A^k(\Omega)$ is contained in the topological boundary $\partial\Omega$ of Ω.

We have that $\hat{f} = F$ on $\partial\Omega$ and hence on δB. Thus $\hat{f} = f \circ \pi$ on $\mathscr{M}^{A^k(\Omega)}$. This proves that π is injective and the result follows. □

We remind the reader of the definition of a Stein neighbourhood basis.

Definition 5.2. A domain $\Omega \subset \mathbb{C}^n$ is said to have a Stein neighbourhood basis if for every open neighbourhood U of $\overline{\Omega}$ there exists a domain of holomorphy Ω' such that $\overline{\Omega} \subset \Omega' \subset U$.

Theorem 5.3. *Let $k \in \mathbb{N}$ and let $\Omega \subset \mathbb{C}^n$ be a bounded domain that has a Stein neighbourhood basis. Then the projection on $\mathbb{C}^n$ of the spectrum $\mathscr{M}^{A^k(\Omega)}$ of $A^k(\Omega)$ equals $\overline{\Omega}$.*

Proof. Suppose there is an element m_0 in the spectrum $\mathscr{M}^{A^k(\Omega)}$ such that $\pi(m_0) = (m_0(z_1), \ldots, m_0(z_n)) \notin \overline{\Omega}$. Let U be a bounded open neighbourhood of $\overline{\Omega}$ such that $\pi(m_0) \notin U$ and denote by $\widetilde{\Omega}$ a pseudoconvex domain with C^∞-boundary such that $\overline{\Omega} \subset \widetilde{\Omega} \subset U$. It follows from [5] that the spectrum $\mathscr{M}^{A^k(\widetilde{\Omega})}$ equals $\overline{\widetilde{\Omega}}_e$. We have that the restrictions to $\overline{\Omega}$ of the functions in $A^k(\widetilde{\Omega})$ is a subset of $A^k(\Omega)$. It follows that there exists an element $\widetilde{m}_0$ in $\mathscr{M}^{A^k(\widetilde{\Omega})}$ defined by $\widetilde{m}_0(f) = m_0(f|_{\overline{\Omega}})$. Hence $\pi(\widetilde{m}_0) = (\widetilde{m}_0(z_1), \ldots, \widetilde{m}_0(z_n)) = (m_0(z_1), \ldots, m_0(z_n)) = \pi(m_0) \notin U$. This however contradicts the fact that $\mathscr{M}^{A^k(\widetilde{\Omega})} = \overline{\widetilde{\Omega}}_e$. Thus we obtain that $\pi(\mathscr{M}^{A^k(\Omega)}) = \overline{\Omega}$. □

Theorem 5.4. *Let $k \in \mathbb{N}$ and let Ω_1 and Ω_2 be bounded A^k-domains in $\mathbb{C}^{m_1}$ and $\mathbb{C}^{m_2}$ respectively. Let $g_1 \in \mathscr{A}_D^k(\Omega_1)$, $g_2 \in \mathscr{A}_I^k(\Omega_1)$ and $h \in \mathscr{A}_D^k(\Omega_2) \cap \mathscr{A}_I^k(\Omega_2)$ and suppose that g_1 does not vanish on Ω_1. If the domain Ω defined by*

$$\Omega = \{(z, w) \in \Omega_1 \times \Omega_2 \colon\ g_1(z) < h(w) < g_2(z)\}$$

fulfills $\operatorname{int}\overline{\Omega} = \Omega$ and is a relatively compact subset of $\Omega_1 \times \Omega_2$, then Ω has a Stein neighbourhood basis.

Proof. Define the domains

$$G_{1,\varepsilon} = \Big\{(z, w) \in \Omega_1 \times \Omega_2 \colon\ h(w) > 0, \frac{g_1(z)}{h(w)} < 1 + \varepsilon\Big\}$$

and

$$G_{2,\varepsilon} = \left\{(z, w) \in \Omega_1 \times \Omega_2 \colon\ g_2(w) > 0, \frac{h(w)}{g_2(z)} < 1 + \varepsilon\right\}.$$

From the plurisubharmonicity and the plurisuperharmonicity of the functions that define $G_{1,\varepsilon}$ and $G_{2,\varepsilon}$ it follows that these domains are pseudoconvex. For $\varepsilon > 0$ small enough the intersection $G_\varepsilon = G_{1,\varepsilon} \cap G_{2,\varepsilon} \subset \Omega_1 \times \Omega_2$ obviously contains Ω and is pseudoconvex. Furthermore, for every open neighbourhood U of $\overline{\Omega}$ we can find an ε such that $\overline{\Omega} \subset G_\varepsilon \subset U$. This completes the proof of the theorem. □

Corollary 5.5. *Let $k \in \mathbb{N}$ and let Ω_1 and Ω_2 be bounded A^k-domains in $\mathbb{C}^{m_1}$ and $\mathbb{C}^{m_2}$ respectively. Let $g_1 \in \mathscr{A}^k_D(\Omega_1)$, $g_2 \in \mathscr{A}^k_I(\Omega_1)$ and $h \in \mathscr{A}^k_D(\Omega_2) \cap \mathscr{A}^k_I(\Omega_2)$ and suppose that g_1 does not vanish on Ω_1. Let Ω be a domain defined by*

$$\Omega = \{(z,w) \in \Omega_1 \times \Omega_2 \colon\ g_1(z) < h(w) < g_2(z)\}\,.$$

Assume that Ω has C^1-boundary and is a relatively compact subset of $\Omega_1 \times \Omega_2$. Let $f_1, \dots, f_m$ be functions in $A^k(\Omega)$ such that $|f_1(z)| + |f_2(z)| + \ldots + |f_m(z)| > 0$ for every $z \in \overline{\Omega}$. Then there exist functions $g_1, \dots, g_m$ in $A^k(\Omega)$ such that

$$\sum_{i=1}^{m} f_i(z) g_i(z) = 1 \text{ for every } z \in \overline{\Omega}.$$

P r o o f. It follows from Theorem 5.3 and Theorem 5.4 that the projection of $\mathscr{M}^{A^k(\Omega)}$ on $\mathbb{C}^n$ equals $\overline{\Omega}$. Theorem 5.1 now gives that $\mathscr{M}^{A^k(\Omega)} = \overline{\Omega}_e$. The conclusion in the theorem is then a standard result in the theory of uniform algebras. □

6. A^k-CONVEXITY

For a domain Ω in $\mathbb{C}^n$ consider the property of being convex with respect to $H(\Omega)$. This is both a necessary and a sufficient condition for Ω to be a domain of existence for $H(\Omega)$ ([3]). The convexity property remains a necessary condition if the class of holomorphic functions $H(\Omega)$ is replaced by an arbitrary subclass S of $H(\Omega)$ such that if f is a function in S, then all derivatives of f also belong to S. For any $k \in \mathbb{N}$ the corresponding convexity property of Ω when $H(\Omega)$ is replaced by $A^k(\Omega)$ is neither necessary nor sufficient as remarked in Section 3. In this section we study convexity with respect to the class of A^k-functions for domains of the type studied in the previous sections.

We start with a lemma that will be used to show that the domains considered in the statement of Theorem 5.4 are convex with respect to the class of A^k-functions.

Lemma 6.1. *Let Ω be a domain in $\mathbb{C}^n$ and let $S(\Omega)$ be a subclass of $H(\Omega)$ such that if f is a function in S, then all derivatives of f also belong to S. Let K be a compact subset of Ω and denote by $\varrho = \varrho(K, \partial\Omega)$:*

$$\varrho(K, \partial\Omega) = \inf_{z \in K} \{\sup\{R \in \mathbb{R}\colon\ \Delta(z, R) \subset \Omega\}\}$$

where $\Delta(z, R)$ is the polydisc with centre at z and all radii equal R. If p is a point in the S-convex hull $\widehat{K}_S$ of K, then every function $f \in S(\Omega)$ extends holomorphically to the polydisc with centre at p and all radii equal ϱ.

For the reader's convenience we prove the proposition:

P r o o f. (See e.g. [9].) Every function $f \in S(\Omega)$ can in a neighbourhood of a be expanded in a Taylor series

$$f(z) = \sum_{|k|=0}^{\infty} c_k (z-p)^k \tag{6.1}$$

since $p \in \Omega$. Here

$$c_k = \frac{1}{k!} \frac{\partial^{|k|} f}{\partial z^k}(p).$$

Since $p \in \widehat{K}_S$ it follows that

$$\Big| \frac{\partial^{|k|} f}{\partial z^k}(p) \Big| \leqslant \Big\| \frac{\partial^{|k|} f}{\partial z^k} \Big\|_K.$$

Choose a number $r < \varrho$ and denote by K^r an r-neighbourhood of K. The function f is bounded on K^r since K^r is relatively compact in Ω and we let

$$M_f(r) = \|f\|_K^r.$$

If $z \in K$, then $\Delta(z, r) \subset K^r$ and we get

$$|c_k| \leqslant \frac{1}{k!} \Big\| \frac{\partial^{|k|} f}{\partial z^k} \Big\|_K \leqslant \frac{M_f(r)}{r^{|k|}}.$$

For any positive $r_1 < r$ and $z \in \Delta(p, r_1)$ we have

$$\big| c_k (z-p)^k \big| \leqslant M_f(r) \Big(\frac{r_1}{r} \Big)^{|k|}$$

and from this we see that the series (6.1) converges in $\Delta(p, r_1)$. Since we can choose r and r_1 arbitrary close to ϱ it follows that the series (6.1) converges in $\Delta(p, \varrho)$. The holomorphic continuation is given by this series and the proof is completed. □

Since the coordinate functions belong to $A^\infty(\Omega) = H(\Omega) \cap C^\infty(\overline{\Omega})$ we get from Lemma 6.1 the following corollary:

Corollary 6.2. *An A^∞-domain of holomorphy $\Omega \subset \mathbb{C}^n$ is A^∞-convex.*

However it is not true that every A^∞-convex domain is an A^∞-domain of holomorphy as is seen from the example by Sibony [10] mentioned in Section 3. That is an example of a domain which is not an H^∞-domain of holomorphy and hence not an A^∞-domain of holomorphy. However it is A^∞-convex since it is pseudoconvex and Runge.

Theorem 6.3. *Let $k \in \mathbb{N}$ and let Ω_1 and Ω_2 be bounded A^k-domains in $\mathbb{C}^{m_1}$ and $\mathbb{C}^{m_2}$ respectively. Let $g_1 \in \mathscr{A}_D^k(\Omega_1)$, $g_2 \in \mathscr{A}_I^k(\Omega_1)$ and $h \in \mathscr{A}_D^k(\Omega_2) \cap \mathscr{A}_I^k(\Omega_2)$ and suppose that g_1 does not vanish on Ω_1. If the domain Ω defined by*

$$\Omega = \{(z,w) \in \Omega_1 \times \Omega_2 \colon\ g_1(z) < h(w) < g_2(z)\}$$

fulfills $\operatorname{int} \overline{\Omega} = \Omega$ and is a relatively compact subset of $\Omega_1 \times \Omega_2$, then Ω is A^k-convex.

P r o o f. Recall that any pseudoconvex domain can be exhausted by bounded pseudoconvex domains with C^∞-boundary and that bounded pseudoconvex domains with C^∞-boundary are A^∞-domains of holomorphy ([2], [5]). It follows from Theorem 5.4 that Ω has a Stein neighbourhood basis and therefore Ω is the interior of the intersection of A^∞-domains of holomorphy. Hence Ω is an A^∞-domain of holomorphy. Corollary 6.2 implies that Ω is convex with respect to A^∞ and hence also with respect to A^k, $0 \leqslant k < \infty$. $\square$

R e m a r k 1. If the assumption that g_1 is strictly positive on Ω_1 in the statement of Theorem 6.3 is removed, then it can be shown that the conclusion of the theorem is not true in general. Suppose there is a point $(z_0, w_0) \in \Omega \subset \mathbb{C}^{m_1} \times \mathbb{C}$ such that $g_1(z_0) = 0$. If $h(w) = |w|$, then Ω contains the punctured disk $\{(z_0, w)\colon\ 0 < |w| < g_2(z_0)\}$ which implies that the A^k-convex hull of $K = \{(z_0, w)\colon\ |w| = 2^{-1} g_2(z_0)\}$ is not a compact subset of Ω. This also means that Ω is not an A^∞-domain of holomorhy since, by Corollary 6.2, every A^∞-domain of holomorphy is convex with respect to the class of A^∞-functions.

Also if the condition that Ω is a relatively compact subset of $\Omega_1 \times \Omega_2$ is not fulfilled, then the conclusion of the theorem may not be true. This can be seen by letting Ω_1 and Ω_2 be A^k-domains of holomorphy such that $\Omega_1 \times \Omega_2$ is not A^k-convex. Then it is trivial that one can find functions g_1, g_2 and h so that $\{(z,w) \in \Omega_1 \times \Omega_2 \colon\ g_1(z) < h(w) < g_2(z)\} = \Omega_1 \times \Omega_2$.

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