BANACH-VALUED HENSTOCK-KURZWEIL INTEGRABLE FUNCTIONS ARE MCSHANE INTEGRABLE ON A PORTION

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Abstract. It is shown that a Banach-valued Henstock-Kurzweil integrable function on an m-dimensional compact interval is McShane integrable on a portion of the interval. As a consequence, there exist a non-Perron integrable function $f\colon [0,1]^2 \longrightarrow \mathbb{R}$ and a continuous function $F\colon [0,1]^2 \longrightarrow \mathbb{R}$ such that

$$(\mathrm{P})\int_0^x \left\{(\mathrm{P})\int_0^y f(u,v)\,\mathrm{d}v\right\}\mathrm{d}u = (\mathrm{P})\int_0^y \left\{(\mathrm{P})\int_0^x f(u,v)\,\mathrm{d}u\right\}\mathrm{d}v = F(x,y)$$

for all $(x,y)\in[0,1]^2$.

Keywords: Henstock-Kurzweil integral, McShane integral

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1. Introduction

It is well known that if f is Denjoy-Perron integrable on an interval $[a,b]\subset\mathbb{R}$, then f must be Lebesgue integrable on a portion of $[a,b]$. K. Karták [6] asked whether an analogous result holds for the multiple Perron integral. In a fairly recent paper [1] Buczolich gave an affirmative answer to this problem using the Henstock-Kurzweil integral. Nevertheless, his proof depends on the measurability of the integrand. Since a Banach-valued Henstock-Kurzweil integrable function need not be strongly measurable, see for instance [4, p. 567], it is natural to ask whether Buczolich's result holds for Banach-valued Henstock-Kurzweil integrable functions. In this paper we give an affirmative answer to this problem. As an application, we answer another question of K. Karták [6, Problem 9.3] concerning the Perron integral; namely, there exist a non-Perron integrable function $f\colon [0,1]^2 \longrightarrow \mathbb{R}$ and a continuous function

$F\colon [0,1]^2 \longrightarrow \mathbb{R}$ such that

$$(\mathrm{P})\int_0^x \left\{(\mathrm{P})\int_0^y f(u,v)\,\mathrm{d}v\right\}\mathrm{d}u = (\mathrm{P})\int_0^y \left\{(\mathrm{P})\int_0^x f(u,v)\,\mathrm{d}u\right\}\mathrm{d}v = F(x,y)$$

for all $(x,y) \in [0,1]^2$.

2. Preliminaries

Unless stated otherwise, the following conventions and notation will be used. The set of all real numbers is denoted by $\mathbb{R}$, and the ambient space of this paper is $\mathbb{R}^m$, where m is a fixed positive integer. The norm in $\mathbb{R}^m$ is the maximum norm $|\!|\!|\cdot|\!|\!|$, where $|\!|\!|(x_1,x_2,\ldots,x_m)|\!|\!| = \max\limits_{i=1,\ldots,m} |x_i|$. For $x \in \mathbb{R}^m$ and $r>0$, set $B(x,r) := \{y \in \mathbb{R}^m \colon |\!|\!|y-x|\!|\!| < r\}$. Let $E := \prod\limits_{i=1}^m [a_i,b_i]$ be a fixed non-degenerate interval in $\mathbb{R}^m$. Let X be a Banach space equipped with a norm $\|\cdot\|$. A function is always X-valued. When no confusion is possible, we do not distinguish between a function defined on a set Z and its restriction to a set $W \subset Z$.

An *interval* in $\mathbb{R}^m$ is the cartesian product of m non-degenerate compact intervals in $\mathbb{R}$, $\mathcal{I}$ denotes the family of all non-degenerate subintervals of E. For each $I \in \mathcal{I}$, $|I|$ denotes the volume of I.

A *partition* P is a collection $\{(I_i,\xi_i)\}_{i=1}^p$, where $I_1, I_2, \ldots, I_p$ are non-overlapping non-degenerate subintervals of E. Given $Z \subseteq E$, a positive function δ on Z is called a *gauge* on Z. We say that a partition $\{(I_i,\xi_i)\}_{i=1}^p$ is

(i) a partition *in* Z if $\bigcup\limits_{i=1}^p I_i \subseteq Z$,
(ii) a partition *of* Z if $\bigcup\limits_{i=1}^p I_i = Z$,
(iii) *anchored* in Z if $\{\xi_1,\xi_2,\ldots,\xi_p\} \subset Z$,
(iv) δ*-fine* if $I_i \subset B(\xi_i,\delta(\xi_i))$ for each $i=1,2,\ldots,p$,
(v) *Perron* if $\xi_i \in I_i$ for each $i=1,2,\ldots,p$,
(vi) *McShane* if ξ_i need not belong to I_i for all $i=1,2,\ldots,p$.

According to Cousin's Lemma [8, Lemma 6.2.6], for any given gauge δ on E, δ-fine Perron partitions of E exist. Hence the following definition is meaningful.

Definition 2.1. A function $f\colon E \longrightarrow X$ is said to be Henstock-Kurzweil integrable (McShane integrable, respectively) on E if there exists $A \in X$ with the following property: given $\varepsilon > 0$ there exists a gauge δ on E such that

$$\left\|\sum_{i=1}^p f(\xi_i)|I_i| - A\right\| < \varepsilon$$

for each δ-fine Perron partition (δ-fine McShane partition, respectively) $\{(I_i,\xi_i)\}_{i=1}^p$ of E. We write A as (HK) $\int_E f$ $\big($(M) $\int_E f$, respectively$\big)$.

It is well known that if f is Henstock-Kurzweil integrable on E, then f is Henstock-Kurzweil integrable on each subinterval J of E. Moreover, the interval function $J \mapsto$ (HK) $\int_J f$ is additive on $\mathcal{I}$. This interval function is known as the *indefinite Henstock-Kurzweil integral*, or in short the indefinite $\mathcal{HK}$-integral, of f.

Theorem 2.2 (Saks-Henstock Lemma). *Let $f\colon E \longrightarrow X$ be Henstock-Kurzweil integrable on E and let F be the indefinite $\mathcal{HK}$-integral of f. Then given $\varepsilon > 0$ there exists a gauge δ on E such that*

$$\Big\| \sum_{(I,x)\in P} \{f(x)|I| - F(I)\} \Big\| < \varepsilon$$

for each δ-fine Perron partition P in E.

3. Banach-valued Henstock-Kurzweil integrable functions are McShane integrable on a portion

Theorem 3.1. *Let $f\colon E \longrightarrow X$ be Henstock-Kurzweil integrable on E and let F denote the indefinite Henstock-Kurzweil integral of f. Then the following conditions are equivalent:*

(i) *f is McShane integrable on E;*

(ii) $\sup \| \sum\limits_{i=1}^{q} F(J_i) \|$ *is finite, where the supremum is taken over all finite partitions $\{J_1,\ldots,J_q\}$ of pairwise non-overlapping subintervals of E.*

Proof. Since E is compact, the implication (i) $\Longrightarrow$ (ii) follows from [11, Lemma 28].

(ii) $\Longrightarrow$ (i). Assume (ii). If $x \in X^*$, then $x(f)$ is Henstock-Kurzweil integrable on E and the indefinite Henstock-Kurzweil integral of $x(f)$ is of bounded variation on E. The rest of the proof is similar to that of the implication (iii) $\Longrightarrow$ (i) of [2, Corollary 9]. The proof is complete. $\square$

In view of [3, Proposition 2B], the next theorem is a mild improvement of [2, Theorem 8].

Theorem 3.2. *Let $f\colon E \longrightarrow X$ be Henstock-Kurzweil integrable on E and let F denote the indefinite Henstock-Kurzweil integral of f. Then the following conditions are equivalent:*

(i) *f is McShane integrable on E;*

(ii) *F is absolutely continuous on $\mathcal{I}$, that is, given any $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality $\|\sum\limits_{i=1}^{p} F(I_i)\| < \varepsilon$ holds whenever $\{I_1, \ldots, I_p\}$ is a finite collection of pairwise non-overlapping subintervals of E with $\sum\limits_{i=1}^{p} |I_i| < \delta$.*

Proof. (i) $\Longrightarrow$ (ii). This follows from [11, Lemma 28].

(ii) $\Longrightarrow$ (i). Since E is compact, this follows from Theorem 3.1. $\square$

It is well known that the real-valued McShane integral is equivalent to the Lebesgue integral. For a proof of this result, see, for example, [10]. Hence the following theorem is a generalization of [1, Theorem]. Recall that a portion of E is a set of the form $E \cap I$, where I is an open interval in $\mathbb{R}^m$.

Theorem 3.3. *If $f\colon E \longrightarrow X$ is Henstock-Kurzweil integrable on E, then f is McShane integrable on a portion of E.*

Proof. Since f is assumed to be Henstock-Kurzweil integrable on E, the Saks-Henstock Lemma (Theorem 2.2) holds. Therefore there exists a gauge δ on E such that

$$\left\| \sum_{(I,x)\in P} \{f(x)|I| - F(I)\} \right\| < 1$$

for each δ-fine Perron partition P in E. For each $n \in \mathbb{N}$, we set

$$X_n = \left\{x \in E\colon \ \|f(x)\| < n \text{ and } \delta(x) > \frac{1}{n}\right\}.$$

Clearly $\bigcup\limits_{n\in\mathbb{N}} X_n = E$ and hence by Baire's Category Theorem [5, Theorem 5.2] there exists $N \in \mathbb{N}$ such that X_N is dense on some J belonging to $\mathcal{I}$. Without loss of generality we may assume that $\operatorname{diam}(J) < 1/N$, where $\operatorname{diam}(J)$ denotes the diameter of J.

Consider any finite collection $\{J_1, \ldots, J_q\}$ of pairwise non-overlapping subintervals of J. For each $i \in \{1, \ldots, q\}$ we invoke the density of $X_N \cap J$ in J to pick $x_i \in X_N \cap J$. Since $\operatorname{diam}(J) < 1/N$, we see that $\{(J_1, x_1), \ldots, (J_q, x_q)\}$ is a $(1/N)$-fine, and hence δ-fine, Perron partition anchored in $X_N \cap J$. Hence, by our choice of δ,

$$\left\| \sum_{i=1}^{q} \{f(x_i)|J_i| - F(J_i)\} \right\| < 1$$

and so

$$\left\|\sum_{i=1}^{q} F(J_i)\right\| < 1 + \sum_{i=1}^{q} \|f(x_i)\||J_i| < 1 + N|J|.$$

As $\{J_1, \ldots, J_q\}$ is an arbitrary finite collection of pairwise non-overlapping subintervals of J, an appeal to Theorem 3.1 completes the proof of the theorem. $\square$

In [7], Kurzweil and Jarník proved that if f is a real-valued Henstock-Kurzweil integrable function on E, then there exists an increasing sequence $\{X_n\}_{n=1}^{\infty}$ of closed sets whose union is E, and for each $n \in \mathbb{N}$, f is Lebesgue integrable on X_n. Hence it is natural to pose the following problem.

Problem 3.4. Let $f\colon E \longrightarrow X$ be Henstock-Kurzweil integrable on E. Can we find an increasing sequence $\{X_n\}_{n=1}^{\infty}$ of closed sets whose union is E, and for each $n \in \mathbb{N}$, f is McShane integrable on X_n?

4. On a question of K. Karták concerning the Perron integral

K. Karták posed the following problem for the Perron integral:

Problem 4.1 [6, Problem 9.3]. Is there a function $f\colon [0,1]^2 \longrightarrow \mathbb{R}$ such that

$$(\mathrm{P})\int_0^x \left\{(\mathrm{P})\int_0^y f(u,v)\,\mathrm{d}v\right\}\mathrm{d}u = (\mathrm{P})\int_0^y \left\{(\mathrm{P})\int_0^x f(u,v)\,\mathrm{d}u\right\}\mathrm{d}v = F(x,y)$$

for all $(x,y) \in [0,1]^2$ and that the function F is continuous on $[0,1]^2$ while f is not Perron integrable on $[0,1]^2$?

Recall that the real-valued Henstock-Kurzweil integral is equivalent to the Perron integral. Hence we may use the Henstock-Kurzweil integral to answer the above question of K. Karták.

Theorem 4.2. *There exist* $f\colon [0,1]^2 \longrightarrow \mathbb{R}$ *and a continuous function* $F\colon [0,1]^2 \longrightarrow \mathbb{R}$ *such that*

$$(1)\quad (\mathrm{HK})\int_0^x \left\{(\mathrm{HK})\int_0^y f(u,v)\,\mathrm{d}v\right\}\mathrm{d}u = (\mathrm{HK})\int_0^y \left\{(\mathrm{HK})\int_0^x f(u,v)\,\mathrm{d}u\right\}\mathrm{d}v = F(x,y)$$

for all $(x,y) \in [0,1]^2$ *but* f *is not Henstock-Kurzweil integrable on* $[0,1]^2$*.*

Proof. Let f be given as in [12, Chapter VI]. Then there exist a continuous function $F\colon [0,1]^2 \longrightarrow \mathbb{R}$ and $f\colon [0,1]^2 \longrightarrow \mathbb{R}$ such that

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial^2 F(x,y)}{\partial y \partial x} = f(x,y)$$

for all $(x,y) \in (0,1)^2$. Moreover, f is not Lebesgue integrable, and hence not McShane integrable, on any non-degenerate subinterval of $[0,1]^2$. It is clear that (1) holds for all $(x,y) \in [0,1]^2$. Using Theorem 3.3 with $E = [0,1]^2$ and $X = \mathbb{R}$, we conclude that f cannot be Henstock-Kurzweil integrable on $[0,1]^2$. The proof is complete.

In view of [9, Theorem 4.3] and [9, Theorem 4.1], we see that every real-valued indefinite Henstock-Kurzweil integral generates a σ-finite Henstock variational measure. Thus it is natural to pose the following problem.

Problem 4.3. Let F be given as in Theorem 4.2, and let $\widetilde{F}$ be the additive interval function induced by F. Must the Henstock variational measure $V_{\mathcal{HK}}\widetilde{F}$ be σ-finite on $[0,1]^2$?

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