A STEP TO KURZWEIL-HENSTOCK—AN OUTLINE

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Abstract. A short approach to the Kurzweil-Henstock integral is outlined, based on approximating a real function on a compact interval by suitable step-functions, and using filterbase convergence to define the integral. The properties of the integral are then easy to establish.

Keywords: integral, Kurzweil-Henstock integral, step-function, filterbase

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1. Introduction

The Kurzweil-Henstock integral is a generalization of the Riemann integral, which includes Lebesgue and Denjoy-Perron integrals. The outline in this paper may make this integral accessible to a wider audience. Most textbook treatments of the Riemann integral are burdened with many summations (and associated notation), most of which can be avoided (Craven 1982) by using integrals of *step-functions* as approximations to Riemann integrals. Since a Kurzweil-Henstock integral is the limit of integrals of a suitably constructed sequence of step-functions, a simpler presentation (omitting proofs of intuitive properties of step-functions) can be used here also. Details of δ*-fine* divisions are needed for certain critical theorems, but not all proofs.

The convergence of approximating sums to the Henstock integral is naturally described by filterbase convergence (as likewise for the Riemann integral though not often expressed so). While most proofs here are based on Lee (1989), the present approach allows a simpler presentation of the basic theory, and also offers a simpler approach to measure, and Lebesgue (and Lebesgue-Stieltjes) integral, than the traditional constructions. (Some proof details are omitted in this short account.) For other presentations, see Kurzweil (1980), Henstock (1991), Lee (1989), Lee and Výborný (2000), and Schwabik (1999).

The traditional distinctions between bounded and unbounded integrands, or domains of integration, are unimportant here. Various integrals, e.g. $\int_0^\infty x^{-1}\sin x\,\mathrm{d}x$, are definable directly as Kurzweil-Henstock integrals, rather than only as limits of integrals. The critical distinction is between *absolute*, when $\int f$ is defined and also $\int |f|$ is finite, and *non-absolute*, or *conditional*, when only $\int f$ is defined. In this presentation, $\mathrm{KH}(I)$ denotes the class of functions possessing a Kurzweil-Henstock integral on I, and $\mathrm{AKH}(I)$ denotes the strictly smaller class for which the integral is absolute. For comparison, the Lebesgue integral is absolute, and the class $L(I)$ of Lebesgue integrable functions on I corresponds to $\mathrm{AKH}(I)$.

2. Basic constructions

Consider first integrating a real function f on a compact interval $I = [a, b]$. A *gauge* $\delta(\cdot)$ is a positive function, $\delta(\cdot) > 0$ on I. A subdivision: $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ is called δ-*fine* if $(\forall i)\xi_i \in \overline{I}_i$, with $I_i := (x_{i-1}, x_i] \subset N(\xi_i)$, where $N(\xi_i) = (\xi_i - \delta(\xi_i),\, \xi_i + \delta(\xi_i))$ denotes a δ-neighbourhood of ξ_i. From this, define a δ-step-function approximation (δsfa) of f as $f^\delta(\cdot) := \sum\limits_i f(\xi_i)\chi_i(\cdot)$, where χ_i is the indicator function of $I_i := (x_{i-1}, x_i]$. Denote by A_δ the set of all $\int_I f^\delta$, in which f^δ is a δsfa, and the integral of a step-function has its elementary meaning.

Cousin's Lemma states that a δ-*fine* subdivision exists for each gauge $\delta(\cdot) > 0$ on a compact interval I. (If the subdivision does not exist, then successive binary subdivision of I gives a nested sequence of intervals $\overline{I}_j$ (the closure of I_j), with limit point ξ, for which a δ-*fine* subdivision does not exist. But then some $I_j \subset N(\xi)$, a contradiction.)

Since then each A_δ is nonempty, the sets A_δ form a *filterbase*, whose defining properties are:

$$(\forall\delta(\cdot) > 0)\ A_\delta \neq \emptyset;\ (\forall\alpha(\cdot), \beta(\cdot) > 0)\ (\exists\gamma(\cdot) > 0)\ A_\gamma \subset A_\alpha \cap A_\beta.$$

For a sequence $\{x_1, x_2, \ldots, x_n, \ldots\}$, the *tails* $A_{1/n} := \{x_n, x_{n+1}, \ldots\}$ form a filterbase, and the usual sequential convergence agrees with the convergence of $\{A_{1/n}\}$ as defined below. For the filterbase of sets A_δ, $\gamma(\cdot) = \min\{\alpha(\cdot) \cdot \beta(\cdot)\}$; and γ-*fine* implies α-*fine.* A finer subdivision keeps all subdivision points, and adds more.

The filterbase $\{A_\delta\colon\ \delta(\cdot) > 0\}$ *converges* to y if (denoting neighbourhoods of y by $U(y)$):

$$(\forall U(y))\ (\exists\alpha(\cdot) > 0)\ \ A_\alpha \subset U(y).$$

The limit y, if it exists, is the Kurzweil-Henstock (briefly, KH) integral $\int_a^b f$; denote by $\mathrm{KH}(I)$ the set of functions for which this integral exists. This convergence implies the Cauchy condition:

$$(\forall \varepsilon > 0)\ (\exists \alpha(\cdot) > 0) \quad \operatorname{diam}(A_\alpha) := \sup\{d(u,v) \colon\ u, v \in A_\alpha < \varepsilon\},$$

where the metric $d(u,v) := |u - v|$ on $\mathbb{R}$ is complete. By a theorem of Cantor (see Dugundji 1966), the Cauchy condition implies that a limit exists. (For $n = 1, 2, \ldots$, find $A_{\alpha(n)}$ with $\operatorname{diam}(A_{\alpha(n)}) < 1/n$. Then $\{A_\alpha\}$ converges if the filterbase $\{B_n := \bigcap_1^n A_{\alpha(n)} : n = 1, 2, \ldots\}$ converges, and choosing $b_n \in B_n$, $\{b_n\}$ is a Cauchy sequence, convergent since the metric space is complete.)

What functions possess Kurzweil-Henstock integrals? Each constant function (with value c) is in $\mathrm{KH}(I)$, since each element of each A_δ equals $c(b-a)$. Each step-function is in $\mathrm{KH}(I)$ (any contribution from a jump points c is $O(\delta(c))$. Each continuous function is in $\mathrm{KH}(I)$ (as a uniform limit of step functions). Each Riemann-integrable function is in $\mathrm{KH}(I)$ (take only constant functions $\delta(\cdot)$), to reproduce the definition of Riemann integral).

Extensions to Stieltjes. So far, integration has related to the usual length on $\mathbb{R}$. However, there is an immediate generalization from $\int_a^b f(x)\,\mathrm{d}x$ to $\int_a^b f(x)\,\mathrm{d}g(x)$, where $g(\cdot)$ is an increasing right-continuous function, defining a *pre-measure* $\mathbf{g}$ by $\mathbf{g}((\alpha,\beta]) := g(\beta) - g(\alpha)$, and $\int_I c\,\mathrm{d}g := c\mathbf{g}(I)$ for constant c and interval I. Thus a *Kurzweil-Henstock-Stieltjes integral* is defined immediately.

Unbounded extension. Define $\mathbb{R}^* := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, assigning neighbourhoods $[-\infty, -c)$ to $-\infty$ and $(c, \infty]$ to ∞, to make $\mathbb{R}^*$ compact. Then $f \colon\ \mathbb{R}^* \to \mathbb{R}^*$ may be considered, instead of $f \colon\ \mathbb{R} \to \mathbb{R}$. Then the definition of convergence allows "convergence" to $+\infty$.

Denote then $f \in \mathrm{KH}(I)$ if the Kurzweil-Henstock integral exists with finite value. Define also $f \in \mathrm{AKH}(I)$ (absolute Henstock) if both $f \in \mathrm{KH}(I)$ and $|f| \in \mathrm{KH}(I)$ (Unlike the Lebesgue integral, $\mathrm{KH}(I) \neq \mathrm{AKH}(I)$. So the term *integrable* needs qualification.)

3. Some properties of Kurzweil-Henstock integrals

Linearity and additivity of integral

$$\int_I (\alpha f + \beta g) = \alpha \int_I f + \beta \int_I g; \quad \int_I f + \int_J f = \int_{I \cup J} f$$

if the KH integrals on each side exist finitely, and I, J are intervals. Similarly $\int_I f \leqslant \int_I g$ when $f, g \in H(I)$ and $f \leqslant g$.

P r o o f. These results hold trivially for step-functions. They extend to KH integrals by taking limits of suitable step-functions.

I n t e g r a t i o n o v e r a s u b i n t e r v a l. Let $f \in \mathrm{KH}(I)$ where I is an interval. Then $f \in \mathrm{KH}(J)$ for each subinterval $J \subset I$. □

P r o o f. A filterbase of elements $\int_I f^\delta$, convergent to $\int_I f$, is Cauchy. Then replacement of each $\int_I f^\delta$ by $\int_J f^\delta$ gives also a Cauchy filterbase.

H e n s t o c k ' s L e m m a. Let $f \in \mathrm{KH}(I)$. For each $\varepsilon > 0$, there exists a δsfa f^δ, with typical subinterval J, for which $\sum\limits_J |\int_J f - \int_J f^\delta| < \varepsilon$. □

P r o o f. By definition of KH integral, there is a gauge β such that $|\int_I (f - f^{\delta'})| < \varepsilon/4$ for each βsfa $f^{\delta'}$, with $\delta' \leqslant \beta$, having typical interval J. Let E be the union of a finite subset of the J; let $V := I \setminus E$. The combination δ of δ' on E with some $\delta'' \leqslant \beta$ on $\overline{V}$ has $\delta \leqslant \beta$. Since $f \in \mathrm{KH}(V)$ by the previous result, $|\int_V f - \int_V f^\delta| \leqslant \varepsilon/4$. Substituting $\delta' = \delta$ and subtracting gives $\int_E f - \int_E f^\delta| \leqslant \varepsilon/4$. Now choose E as the union of those J for which $\int_J f^\delta - \int_J f \geqslant 0$. Then

$$\sum_{J\subset I} \left| \int_J f^\delta - \int_J f \right| = \sum_{J\subset E} \left[\int_J f^\delta - \int_J f \right] + \sum_{J\subset \overline{V}} \left| \int_J f^\delta - \int_J f \right| \leqslant \varepsilon/2 + \varepsilon/2.$$

M o n o t o n e C o n v e r g e n c e T h e o r e m. Let $I := [a,b]; f_n \in \mathrm{KH}(I)$ $(n = 1,2,\ldots)$; $(f_n(\cdot)\} \to f(\cdot)$ pointwise; $f_{n+1}(\cdot) \geqslant f_n(\cdot)$; $\{\int_I f_n\} \leqslant k < \infty$. Then $\{\int_I f_n\} \to \int_I f$, for Kurzweil-Henstock integrals. □

P r o o f. Let $L = \sup \int_I f_n$; then $L \leqslant k < \infty$. Choose $\varepsilon > 0$; then $(\exists \nu)$ $(\forall n \geqslant \nu)$ $0 \leqslant L - \int_I f_n < \varepsilon$. Now

$$(\forall \xi \in I)\ (\exists m(\xi) \geqslant \nu)\ |f_{m(\xi)}(\xi) - f(\xi)| < \varepsilon.$$

For each n, a gauge $\delta_n > 0$ exists by Henstock's Lemma such that when δ is a δ-fine gauge with typical subinterval J, then $\Sigma_J |\int_J (f_n^{\delta n} - f_n)| < 2^{-n}\varepsilon$ for any $\delta_n$sfa $f_n^{\delta n}$ of f_n. Choose the gauge $\delta(\xi) := \delta_{m(\xi)}(\xi)$. For such a δ-fine subdivision, with $\xi \in J$, denote $\hat{f}(x) := f_n(x)$ with $n = m(\xi)$ when $x \in J$. Then:

$$\begin{aligned} 0 \leqslant L - \int_I f^\delta &\leqslant \sum_J \left| \int_J [f^\delta - \hat{f}^\delta] \right| + \sum_J \left| \int_I [\hat{f}^\delta - \hat{f}] \right| + \left| \int_I \hat{f} - L \right| \\ &\leqslant \varepsilon |I| + \sum_n 2^{-n}\varepsilon + \varepsilon; \\ L - \varepsilon \leqslant \int_I f_\nu &\leqslant \int_I \hat{f} \leqslant \int_I f_p \leqslant L, \end{aligned}$$

where $p = \max m(\xi)$ over the finite set of ξ.

M o r e f u n c t i o n s h a v e KH i n t e g r a l s. The limit f of a monotone increasing sequence of functions $f_n \in \mathrm{KH}(I)$, is also in $\mathrm{KH}(I)$.

L a t t i c e p r o p e r t y. If $p \leqslant f_i \leqslant q$ $(i = 1, 2)$, where $p, f_i,\ q \in \mathrm{KH}(I)$, then $f := \max\{f_1, f_2\} \in \mathrm{KH}(I)$. □

P r o o f. Choose $\varepsilon > 0$. By Henstock's Lemma, a δ-*fine* subdivision of the interval I exists, with typical subinterval J, and δsfa f_i^δ, for which:

$$\sum_J \left| \int_J (f_i^\delta - f) \right| < \varepsilon\ (i = 1, 2), \quad \left| \sum_J \max\left\{ \int_J f_1, \int_J f_2 \right\} - L \right| < \varepsilon$$

where this $\sum \max$ increases with further subdivision, and has a finite sup L over subdivisions, because of the bounds p and q. Then

$$\sum_J \max\left\{ \int_J f_1, \int_J f_2 \right\} - 2\varepsilon \leqslant \sum_J \max\left\{ \int_J f_1^\delta, \int_J f_2^\delta \right\} = \sum_J \int_J f\delta$$
$$\leqslant \sum_J \max\left\{ \int_J f_1, \int_J f_2 \right\} + 2\varepsilon.$$

As $\varepsilon \to 0$ and $\delta(\cdot) \to 0$, the terms at left and right $\to L$, and the middle terms $\to \int_I f$. □

R e m a r k s. Let $f_+ := \max\{f, 0\}$ and $f_- := \max\{-f, 0\}$; then $f = f_+ - f_-$. In order to deduce from $f \in \mathrm{KH}(I)$ that $f_+ \in \mathrm{KH}(I)$, the least restrictive bounds for the *lattice property* are $-f_- \leqslant f \leqslant f_+$ and $-f_- \leqslant 0 \leqslant f_+$, which only apply if $f_+ \in \mathrm{KH}(I)$ and $f_- \in \mathrm{KH}(I)$, hence when $|f| = f_- + f_+ \in \mathrm{KH}(I)$.

Thus $f \in \mathrm{AKH}(I) \Rightarrow |f| \in \mathrm{KH}(I)$. Note that I is here restricted to intervals.

D o m i n a t e d c o n v e r g e n c e t h e o r e m. Let $p \leqslant f_n \leqslant q$ $(n = 1, 2, \ldots)$ with $p,\ q,\ f_n \in \mathrm{KH}(I)$, and $f_n(x) \to f(x)$ for each $x \in I$. Then $f \in \mathrm{KH}(I)$, and $\int_I f_n \to \int_I f$.

P r o o f. Since $\{h_n(x)\}$, where $h_n(x) := \inf\limits_{n \geqslant j} f_j(x)$, is an increasing sequence, with limit $f(x)$, and bounded by p and q, the result follows from monotone convergence, provided that $h_n \in \mathrm{KH}(I)$. This follows from monotone convergence, since $\{\inf\limits_{i \leqslant n \leqslant j} f_n(x)\}_j$ increases to $h_n(x)$.

(There exist also convergence theorems for non-absolute integrals, see e.g. [8]).

I n t e g r a t i o n o v e r a s u b s e t. Let $f \in \mathrm{AKH}(I)$ and $E \subset I$, with indicator $\chi_E \in \mathrm{KH}(I)$. Then $f\chi_E \in \mathrm{KH}(I)$. □

P r o o f. The indicator χ_E is the limit of a sequence $\{q_n\}$ of step-functions, with each q_n taking values 1 and 0. Since $-|f| \leqslant fq_n \leqslant |f|$, and $fq_n \in \mathrm{KH}(I)$, dominated convergence shows that $f\chi_E \in \mathrm{KH}(I)$. (This need not hold if $|f| \notin \mathrm{KH}(I)$.)

N e w t o n i n t e g r a l. If F has a derivative F' at all points of an interval $I = [a,b]$, then $F' \in \mathrm{KH}(I)$, and $F(b) - F(a) = \int_I F'$. $\square$

P r o o f. Use a gauge δ given by $|(t-s)^{-1}(F(t) - F(s)) - F'(s)| < \varepsilon$ when $0 < |t-s| < \delta(\varepsilon)$. $\square$

4. Measure and lebesgue integral

N u l l s e t s. For each $\varepsilon > 0$, suppose that the set $N \subset \bigcup_{n=1}^{\infty} I_n^\varepsilon$ for open intervals I_n^ε satisfying $\sum |I_n^\varepsilon| < \varepsilon$. (Then N is called a *null set.*) Then $(\forall \varepsilon > 0) \int \chi_N < \varepsilon$, so $\int \chi_N = 0$. A choice of gauge $\delta(\cdot)$ with $(\forall \xi \in N)$ $(\exists i)(\xi - \delta(\xi),\ \xi + \delta(\xi)) \subset I_i^\varepsilon$ shows again that $\int_I \chi_N = 0$. A property holding for all points except those of a null set is said to hold *almost everywhere* (*a.e.*). The value of the integral is unchanged by altering the integrand at the points of a null set. However, the null sets are different if the length $|I_n|$ is replaced by $\mathbf{g}(I_n)$ for some $g(x) \neq x$.

M e a s u r a b l e. A function f may be defined as *measurable* if it is the limit a.e. of a sequence of step-functions; $f \in \mathrm{KH}(I)$ can be shown to have this property (this proof is omitted), hence is measurable. A set X is *measurable* if its indicator function χ_X is measurable; its *measure* is $mE := \int \chi_E$.

M e a s u r e p r o p e r t i e s. Let $\{E_1, E_2, \ldots\}$ be a sequence of disjoint measurable sets, with union E; let $A_n := E_1 \cup E_2 \cup \ldots \cup E_{n|}$, and f_n the indicator of A_n, f the indicator of E. Now $0 \leqslant f_n \leqslant f$, $f_{n+1} \geqslant f_n$, and $f \in \mathrm{KH}(I)$ if E is contained in a compact interval I. So from the monotone convergence theorem,

$$\sum_{j=1}^{n} mE_j = \int_I f_n \to \int_I f = mE.$$

Thus the measure m has the *countably-additive* property.

Let $f \in H(I)$; let $E \subset I$ and $E \subset A$ be measurable sets. Since

$$\int_I \chi_E - \int_I \chi_A = \int_I (\chi_E - \chi_A) = \int_I \chi_B,$$

holds, where $B = E \setminus A$, from $\chi_E \in \mathrm{KH}(I)$ and $\chi_A \in \mathrm{KH}(I)$, there follows that $\chi_B \in \mathrm{KH}(I)$. Thus the difference $E \setminus A$ of measurable sets is also measurable. A similar argument shows that $\int_E f = \int_A f + \int_B f$, provided that $f \in \mathrm{AKH}(E)$.

W h e n n u l l s e t s c a n b e n e g l e c t e d. Suppose now that $f(x) = g(x)$ for all $x \in I \setminus N$, with $g \in \mathrm{KH}(I)$. Then $N = \bigcup_j N_j$ with $N_j := (x \in N\colon\ j-1 \leqslant |f(x)| \leqslant j\}$, and N is a null set. So $N_j \subset \bigcup_k I_{jk}$, with open intervals satisfying

$\sum_k m(I_{jk}) < 2^{-j-k} i^{-1} \varepsilon$. Define a gauge $\delta(\cdot)$ with $(\forall \xi \in N_j)\ (\xi - \delta(\xi), \xi + \delta(\xi)) \subset \bigcup_k I_{jk}$. Then the contribution to $\int_I f^\delta$ is less than ε. Hence $f \in \mathrm{KH}(I)$, and $\int_I f = \int_I g$. Consequently, in dominated convergence, $\{f_n(x)\}$ may fail to converge to $f(x)$ for x in a set N of zero measure.

5. Functions of two (or more) variables

Intervals $(\alpha, \beta] \subset \mathbb{R}$ (or $\mathbb{R}^*$) are replaced by rectangles $(\alpha_1, \beta_1] \times (\alpha_2, \beta_2]$; they may still be called *intervals*. The length of an *interval* I becomes $\mathbf{g}(I) := g(\alpha_2, \beta_2) - g(\alpha_1, \beta_2) - g(\alpha_2, \beta_1) + g(\alpha_1, \beta_1)$, where g is such that $\mathbf{g}(I) \geqslant 0$ for each I. (The construction of a *premeasure* $\mathbf{g}$ for n-dimensions is detailed in Craven (1982), where it is applied to Riemann integration.) Define $\int_I 1\,\mathrm{d}\mathbf{g}$ as $\mathbf{g}(I)$, and the integral of a step-function $f = \sum c_i \chi_I$ (where χ_i is the indicator of interval I_i) as $\int f\,\mathrm{d}\mathbf{g} := \sum_i c_i \mathbf{g}(I_i)$. Then a Kurzweil-Henstock integral may be constructed from step-functions in the same manner as in section 2; and the basic properties are the same. In particular, if $g(\alpha, \beta) = \alpha\beta$, then $\mathbf{g}(I) = (\beta_2 - \alpha_2)(\beta_1 - \alpha_1)$, and the integral may be written in conventional notation as $\int \int f(x, y)\,\mathrm{d}x\,\mathrm{d}y$.

Fubini's theorem. Let $I = A \times B \subset \mathbb{R}^{*2}$ be an interval (bounded or unbounded); let $0 \leqslant f \in \mathrm{KH}(I)$. Then the Kurzweil-Henstock integrals

$$\iint_{A \times B} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_A \left(\int_B f(x, y)\,\mathrm{d}y \right) \mathrm{d}x.$$

Proof. The result is immediate for step-functions. Monotone convergence extends it to limits of increasing sequences of step-functions. Suppose now that N is a null set; then there are intervals I_{ij} with $\sum_{j=1}^{\infty} |I_{ij}| < 1/i$ and $(\forall i)\ N \subset E_i := \bigcup_{j=1}^{\infty} I_{ij}$. Thus $N \subset E := \bigcup_{i=1}^{\infty} E_i$. From monotone convergence, Fubini holds for each $f_i := \chi_{E_i}$, then for χ_E (considering the sequence $\{1 - f_i\}$ for I bounded), then also for I unbounded. Hence $\{y\colon (x, y) \in S\}$ is null a.e. in x. Thus a limit f of an increasing sequence of step-functions may be altered on a null set without changing the integral. Fubini follows for f. $\square$

Remarks. This is the same proof as for Lebesgue integrals. The integrals may have value $+\infty$. There is an extension to $f \in \mathrm{AKH}(I)$, taking both signs. Thus, if $f \in \mathrm{AKH}(I)$, then $f_+ := \max\{f, 0\} \in \mathrm{AKH}(I)$, and $f_- := \max\{-f, 0) \in \mathrm{AKH}(I)$. Hence Fubini holds for this f, by subtracting the finite integrals for f_+ and f_-. There exist Fubini results also for non-absolute integrals (see e.g. [8]).

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