ON A CANCELLATION LAW FOR MONOUNARY ALGEBRAS

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Abstract. In this paper we investigate the validity of a cancellation law for some classes of monounary algebras.

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1. Introduction

For monounary algebras we apply the standard notation (cf., e.g., [1]).

In this paper we deal with the implication

(1) $$AB \cong AC \Rightarrow B \cong C,$$

where A, B and C are monounary algebras.

If $\mathcal{K}$ is a class of monounary algebras such that for each $A, B, C \in \mathcal{K}$ the implication (1) is valid, then we say that the cancellation law (1) holds in $\mathcal{K}$.

For a given monounary algebra D we denote by $\mathcal{U}(D)$ the class of all monounary algebras A such that

(i) the number of connected components of A is finite;

(ii) if E is a connected component of A, then E can be expressed as the direct product of a finite number of subalgebras $A_1, A_2, \ldots, A_n$ of D such that no A_i $(i = 1, 2, \ldots, n)$ is a cycle.

We denote by $\mathbb{Z} = (\mathbb{Z}, f)$ the monounary algebra such that $f(x) = x + 1$ for each $x \in \mathbb{Z}$.

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Let $n \in \mathbb{N}$. Then D_n denotes a connected monounary algebra such that $D_n = \{a_0, a_1, \ldots, a_{n-1}\} \cup \mathbb{N}$, where $\{a_0, a_1, \ldots, a_{n-1}\}$ is an n-element cycle and for $1 \neq k \in \mathbb{N}$ we have $f(k) = k - 1, f(1) = a_0$.

We prove the following results:

(α) The class $\mathcal{U}(\mathbb{Z})$ does not satisfy the cancellation law (1).

(β) For each $n \in \mathbb{N}$, the cancellation law holds in the class $\mathcal{U}(D_n)$.

When proving (β), we apply different methods for the case $n = 1$ and for the case $n > 1$.

The validity of a cancellation law for finite unary algebras was investigated in [7]. In [6], a cancellation law for monounary algebras which are sums of cycles was dealt with.

The cancellation law (1) for finite algebras was studied in [3], [4]; cf. also the monograph [5], Section 5.7. In [2], the implication (1) for partially ordered sets was investigated.

2. Preliminaries

In this section we recall some definitions and prove some auxiliary results concerning the class $\mathcal{U}(D_1)$.

By a monounary algebra we understand a pair (A, f), where A is a non-empty set and f is a mapping of A into A. If no misunderstanding can occur, then we write A instead of (A, f).

A monounary algebra (A, f) is said to be connected if for each $x, y \in A$ there are $m, n \in \mathbb{N} \cup \{0\}$ such that $f^n(x) = f^m(y)$. A maximal connected subalgebra of a monounary algebra (A, f) is called a connected component of (A, f).

Let A be a monounary algebra. An element $a \in A$ is cyclic if $f^n(a) = a$ for some $n \in \mathbb{N}$. Let B be a connected subalgebra of A. If each element of B is cyclic, then B is called a cycle of A.

Let $n \in \mathbb{N}$. For $i \in \mathbb{Z}$ we denote $i_n = \{j \in \mathbb{Z} : j \equiv i \pmod{n}\}$. Next, let $\mathbb{Z}_n = \{0_n, 1_n, \ldots, (n-1)_n\}$ be the set of all integers modulo n. We define a monounary algebra $D_n = (D_n, f)$ putting

$$D_n = \mathbb{Z}_n \cup \mathbb{N},$$
$$f(a) = \begin{cases} a + 1_n & \text{if } a \in \mathbb{Z}_n, \\ a - 1 & \text{if } a \in \mathbb{N}, a \neq 1, \\ 0_n & \text{if } a = 1. \end{cases}$$

For $n = 1$ we write 0 instead of the symbol 0_n, i.e., $D_1 = \{0\} \cup \mathbb{N}$.

Let $n \in \mathbb{N}$ and let $X_1, \ldots, X_n$ be subalgebras of D_1 having more than one element. Further, let ξ be an isomorphism of $X_1 X_2 \ldots X_n$ onto a monounary algebra A. We will omit brackets and write just $\xi(x_1, \ldots, x_n)$ instead of $\xi((x_1, \ldots, x_n))$. We denote

$$\begin{aligned}
X_1^{(0)} &= \{\xi(x_1, 0, \ldots, 0)\colon\ x_1 \in X_1\},\\
X_2^{(0)} &= \{\xi(0, x_2, 0, \ldots, 0)\colon\ x_2 \in X_2\}, \ldots,\\
X_n^{(0)} &= \{\xi(0, 0, \ldots, x_n)\colon\ x_n \in X_n\}.
\end{aligned}$$

2.1. Lemma. *A is a connected monounary algebra with a one-element cycle $\{\xi(0, 0, \ldots, 0)\}$. Further,*

$$|f^{-1}(\xi(0, 0, \ldots, 0))| = 2^n.$$

Proof. Let $x = \xi(x_1, \ldots, x_n) \in A$, $k = \max\{x_1, \ldots, x_n\} + 1$. Then

$$f^k(x) = \xi(f^k(x_1), \ldots, f^k(x_n)) = \xi(0, \ldots, 0) = f(\xi(0, \ldots, 0)),$$

which implies that the element $\xi(0, \ldots, 0)$ forms a one-element cycle of A and that A is connected. Next,

$$\begin{aligned}
&f^{-1}(\xi(0, \ldots, 0)) = \{\xi(y_1, \ldots, y_n)\colon\ y_i \in \{0, 1\} \text{ for each } i \in \{1, \ldots, n\}\},\\
&|f^{-1}(\xi(0, \ldots, 0))| = 2^n.
\end{aligned}$$

□

2.2. Lemma. *Let $x \in A$ be such that $f^{-1}(x) \neq \emptyset$. Then $x \in \bigcup\limits_{i=1}^{n} X_i^{(0)}$ if and only if $|f^{-1}(x)| \in \{2^n, 2^{n-1}\}$.*

Proof. Suppose that $x \in \bigcup\limits_{i=1}^{n} X_i^{(0)}$. There are $i \in I$ and $x_i \in X_i$ with $x = \xi(0, 0, \ldots, x_i, \ldots, 0)$. We have supposed that $f^{-1}(x) \neq \emptyset$, thus $f^{-1}(x_i) \neq \emptyset$; if $x_i = 0$, then $f^{-1}(x_i) = \{0, 1\}$ and if $x_i \neq 0$, then $f^{-1}(x_i) = x_i + 1$. Let $y \in f^{-1}(x)$. If $j \neq i$, then the j-th projection of $\xi^{-1}(y)$ belongs to the set $\{0, 1\}$. Hence

(a) if $x_i = 0$, then $|f^{-1}(x)| = 2^n$ by 2.1,

(b) if $x_i \neq 0$, then

$$\begin{aligned}
&f^{-1}(x) = \{\xi(y_1, y_2, \ldots, x_i + 1, \ldots, y_n)\colon\ y_j \in \{0, 1\} \text{ for } j \neq i\},\\
&|f^{-1}(x)| = 2^{n-1}.
\end{aligned}$$

Therefore

$$|f^{-1}(x)| \in \{2^n, 2^{n-1}\}.$$

Conversely, assume that $x \in A - \bigcup\limits_{i=1}^{n} X_i^{(0)}$. Then the number of projections of $\xi^{-1}(x)$ which are equal to 0 is less than $n-1$; without loss of generality, $x = \xi(x_1, \ldots, x_k, 0, \ldots, 0)$, $\{x_1, \ldots, x_k\} \cap \{0\} = \emptyset, k > 1$. We obtain

$$f^{-1}(x) = \{\xi(x_1 + 1, \ldots, x_k + 1, y_{k+1}, \ldots, y_n) \colon y_{k+1}, \ldots, y_n \in \{0, 1\}\},$$

which implies that $|f^{-1}(x)| \leqslant 2^{n-2}$. $\square$

Now let $n, m \in \mathbb{N}$ and let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m$ be subalgebras of D_1 having more than one element. Further, let A be a monounary algebra such that ξ is an isomorphism of $X_1 X_2 \ldots X_n$ onto A and let η be an isomorphism of $Y_1 Y_2 \ldots Y_m$ onto A. We suppose that $X_1^{(0)}, \ldots, X_n^{(0)}, Y_1^{(0)}, \ldots, Y_m^{(0)}$ have an analogous meaning as above.

2.3. Lemma.

(1) $n = m$;

(2) $\left\{x \in \bigcup\limits_{i=1}^{n} X_i^{(0)} \colon f^{-1}(x) \neq \emptyset\right\} = \left\{y \in \bigcup\limits_{i=1}^{n} Y_i^{(0)} \colon f^{-1}(y) \neq \emptyset\right\}$;

(3) *there is a permutation* φ *of the set* $\{1, 2, \ldots, n\}$ *such that* $X_k \cong X_k^{(0)} \cong Y_{\varphi(k)}^{(0)} \cong Y_{\varphi(k)}$ *for each* $k \in \{1, \ldots, n\}$.

P r o o f. In wiew of 2.1 we obtain that $\{\xi(0, \ldots, 0)\} = \{\eta(0, \ldots, 0)\}$ is a cycle of A and

$$2^n = |f^{-1}(\xi(0, \ldots, 0))| = |f^{-1}(\eta(0, \ldots, 0))| = 2^m,$$

therefore $n = m$.

The assertion (2) follows from 2.2.

The set $\bigcup\limits_{i=1}^{n} X_i^{(0)}$ is a subalgebra of A. Further, for $i \in \{1, \ldots, n\}$, $X_i^{(0)} \cong X_i$ is a subalgebra of D_1 and $X_i^{(0)} \cap X_j^{(0)}$ is a one-element cycle of A whenever $i, j \in \{1, \ldots, n\}$, $i \neq j$. (Analogously for $Y_1, \ldots, Y_n$.) Notice that if $x \in X_j^{(0)}$, $j \in \{1, \ldots, n\}$ and $f^{-1}(x) = \emptyset$, then the element $f(x) = z$ has the property that $f^{-1}(z) \neq \emptyset$, $z \in \left(\bigcup\limits_{i=1}^{n} X_i^{(0)}\right) \cap \left(\bigcup\limits_{i=1}^{n} Y_i^{(0)}\right)$.

Let $k \in \{1, \ldots, n\}$.

(a) Suppose that $X_k \cong D_1$. Then $X_k^{(0)} \cong D_1$ and for each $x \in X_k^{(0)}$ we have $f^{-1}(x) \neq \emptyset$. According to (2),

$$X_k^{(0)} \subseteq \left\{y \in \bigcup_{i=1}^{n} Y_i^{(0)} \colon f^{-1}(y) \neq \emptyset\right\}. \tag{4}$$

Take $t \in X_k^{(0)}$ such that t does not belong to a cycle but $f(t)$ does; t is uniquely determined. Further, (4) implies that there is $j \in \{1, \dots, n\}$ with $t \in Y_j$. By the above consideration, the set $Y_i^{(0)} \cap Y_j^{(0)}$ is a one-element cycle of A for each $i \neq j$, therefore we obtain in view of (4) that $Y_j^{(0)} = X_k^{(0)}$; let us denote $j = \varphi(k)$.

(b) Now let $X_k \neq D_1$. There is exactly one $x \in X_k^{(0)}$ such that $f^{-1}(x) = \emptyset$. Let $z = f(x)$. According to (2) the subalgebra $\{f^l(z)\colon\ l \in \mathbb{N} \cup \{0\}\}$ of $X_k^{(0)}$ is a subalgebra of $\Big\{y \in \bigcup\limits_{i=1}^{n} Y_i^{(0)}\colon\ f^{-1}(y) \neq \emptyset\Big\}$ and analogously as above, there is exactly one $j \in \{1, \dots, n\}$ such that

$$\{f^l(z)\colon\ l \in \mathbb{N} \cup \{0\}\} = \{y \in Y_j^{(0)}\colon\ f^{-1}(y) \neq \emptyset\}. \tag{5}$$

We have $X_k^{(0)} = \{x\} \cup \{f^l(z)\colon\ l \in \mathbb{N} \cup \{0\}\}$. Similarly, $Y_j^{(0)}$ consists of the elements of the right set in (5) and of one element q with the property $f^{-1}(q) = \emptyset$. Hence $X_k^{(0)} \cong Y_j^{(0)}$. We denote $j = \varphi(k)$.

The mapping φ is a permutation and $X_k^{(0)} \cong Y_{\varphi(k)}^{(0)}$ for each $k \in \{1, \dots, n\}$, i.e., (3) is valid. $\square$

2.4. Proposition. *If A is isomorphic to a direct product of subalgebras of D_1 such that these subalgebras are not cycles, then the decomposition of A into such a direct product is unique up to isomorphism.*

P r o o f. This is a corollary of 2.3. $\square$

3. Cancellation law in $\mathcal{U}(D_1)$

For investigating the properties of $\mathcal{U}(D_1)$, in this section we deal with the system $\mathbb{Z}[y_1, \dots, y_n]$ of polynomials with unknowns $y_1, \dots, y_n$ over the integrity domain of integers $\mathbb{Z}$; it is known that $\mathbb{Z}[y_1, \dots, y_n]$ is an integrity domain as well.

A similar consideration has been used in [2] for investigating the cancellation law for partially ordered sets, where generalized polynomials over $\mathbb{Z}$ have been taken into account.

Let A, B, C be monounary algebras belonging to the class $\mathcal{U}(D_1)$. Next, let $\{Y_1, \dots, Y_n\}$ be a system of monounary algebras such that

(a) if $i, j \in \{1, \dots, n\}, i \neq j$, then $Y_i \not\cong Y_j$,

(b) if $E \in \{A, B, C\}, F$ is a connected component of E and $F = X_1 X_2 \dots X_k$, where $X_1, \dots, X_k$ are subalgebras of D_1 which are not cycles, then for each $j \in \{1, \dots, k\}$ there is $l \in \{1, \dots, n\}$ such that $X_i \cong Y_l$,

(c) if $l \in \{1, \dots, n\}$, then $|Y_l| > 1$.

Let us remark that it follows from the definition of $\mathcal{U}(D_1)$ that a system $\{Y_1, \dots, Y_n\}$ with the required properties exists.

3.1. Notation. If X is a monounary algebra, $k \in \mathbb{N}$, then we denote by X^0 a one-element monounary algebra and

$$X^k = XX \dots X \ (k\text{-times}).$$

If $k \in \mathbb{N}$ and $X_1, \dots, X_k$ are mutually disjoint monounary algebras, then $\sum_{i=1}^{k} X_i$ is a disjoint union of the given algebras. Further, if $X_1 \cong X_2 \cong \dots \cong X_k$, then we write also kX_1 instead of $\sum_{i=1}^{k} X_i$; thus kX_2 is an algebra consisting of k copies of X_1.

We denote $0X = \emptyset$ for each monounary algebra X.

3.2. Lemma. *Let $t_1, \dots, t_n, s_1, \dots, s_n \in \mathbb{N} \cup \{0\}$. Then $Y_1^{t_1} Y_2^{t_2} \dots Y_n^{t_n} \cong Y_1^{s_1} Y_2^{s_2} \dots Y_n^{s_n}$ if and only if $t_1 = s_1, \dots, t_n = s_n$.*

Proof. Since the condition (a) is satisfied, we obtain the assertion by virtue of 2.4. $\square$

3.3. Corollary. *Let $E \in \{A, B, C\}$, let F be a connected component of E. Then F can be expressed in the form $F \cong Y_1^{t_1} \dots Y_n^{k_n}$, where $t_1, \dots, t_n \in \mathbb{N} \cup \{0\}$; further, $t_1, \dots, t_n$ are uniquely determined.*

3.4. Notation. Let $f(y_1, \dots, y_n) \in \mathbb{Z}[y_1, \dots, y_n]$ be a polynomial with non-negative coefficients. Then we can write it in the form

$$f(y_1, \dots, y_n) = \sum_{i=1}^{m} p_i y_1^{t_{i1}} y_2^{t_{i2}} \dots y_n^{t_{in}}$$

such that

(i) $p_i \geqslant 0$ for each $i \in \{1, \dots, m\}$,

(ii) if j, k are distinct elements of the set $\{1, \dots, m\}$, then $y_1^{t_{j1}} y_2^{t_{j2}} \dots y_n^{t_{jn}} \neq y_1^{t_{l1}} y_2^{t_{l2}} \dots y_n^{t_{ln}}$; we will say that $f(y_1, \dots, y_n)$ is written in a normal form. By $f(Y_1, \dots, Y_n)$ we denote the monounary algebra

$$\sum_{i=1}^{m} p_i Y_1^{t_{i1}} Y_2^{t_{i2}} \dots Y_n^{t_{in}}.$$

3.5. Lemma. *Let $f(y_1,\dots,y_n)$, $g(y_1,\dots,y_n) \in \mathbb{Z}[y_1,\dots,y_n]$ be polynomials with non-negative coefficients which are written in a normal form. Then $f(Y_1,\dots,Y_n) = g(Y_1,\dots,Y_n)$ if and only if $f(y_1,\dots,y_n) = g(y_1,\dots,y_n)$.*

Proof. If $f(y_1,\dots,y_n) = g(y_1,\dots,y_n)$, then 3.4 implies that $f(Y_1,\dots,Y_n) = g(Y_1,\dots,Y_n)$. The converse implication follows from 3.2 in view of the fact that the polynomials are written in a normal form. $\square$

3.6. Corollary. *There are uniquely determined polynomials $f_A(y_1,\dots,y_n)$, $f_B(y_1,\dots,y_n)$, $f_C(y_1,\dots,y_n)$ with non-negative coefficients such that*

$$\begin{aligned} A &\cong f_A(Y_1,\dots,Y_n),\\ B &\cong f_B(Y_1,\dots,Y_n),\\ C &\cong f_C(Y_1,\dots,Y_n). \end{aligned}$$

Proof. This is a consequence of the definition of the system $\{Y_1,\dots,Y_n\}$ and of 3.5. $\square$

3.7. Corollary. *Let $(f_A \cdot f_B)(y_1,\dots,y_n) = f_A(y_1,\dots,y_n) \cdot f_B(y_1,\dots,y_n)$. Then*

$$AB \cong (f_A \cdot f_B)(Y_1,\dots,Y_n).$$

Proof. By 3.6, $AB \cong f_A(Y_1,\dots,Y_n) f_B(Y_1,\dots,Y_n)$, thus we get the assertion in view of 3.4. $\square$

3.8. Theorem. *Let $A, B, C \in \mathcal{U}(D_1)$, $AB \cong AC$. Then $B \cong C$.*

Proof. According to 3.7 we obtain $AB \cong (f_A \cdot f_B)(Y_1,\dots,Y_n)$, and similarly, $AC \cong (f_A \cdot f_C)(Y_1,\dots,Y_n)$. Thus

$$(f_A \cdot f_B)(Y_1,\dots,Y_n) \cong (f_A \cdot f_C)(Y_1,\dots,Y_n).$$

According to 3.5,

$$\begin{aligned} (f_A \cdot f_B)(y_1,\dots,y_n) &= (f_A \cdot f_C)(y_1,\dots,y_n),\\ f_A(y_1,\dots,y_n) \cdot f_B(y_1,\dots,y_n) &= f_A(y_1,\dots,y_n) \cdot f_C(y_1,\dots,y_n). \end{aligned}$$

The polynomial $f_A(y_1,\dots,y_n)$ is a non-zero polynomial, thus we can apply the cancellation law in the integrity domain $\mathbb{Z}[y_1,\dots,y_n]$, which implies

$$f_B(y_1,\dots,y_n) = f_C(y_1,\dots,y_n).$$

Again by 3.5, $f_B(Y_1,\dots,Y_n) = f_C(Y_1,\dots,Y_n)$, thus $B \cong C$. $\square$

Now we will give two examples showing that if some of the conditions (i), (ii) in the definition of the class $\mathcal{U}(D)$ fails to hold, then the cancellation law need not be valid in general.

3.9. E x a m p l e. Let E be an arbitrary subalgebra of D_1, $|E| > 1$. Put $A = \aleph_0 E$, $B = E$, $C = 2E$. Then

$$\begin{aligned} AB &= (\aleph_0 E)E = \aleph_0(EE), \\ AC &= (\aleph_0 E)(2E) \cong \aleph_0(EE). \end{aligned}$$

Hence $AB \cong AC$, but $B \ncong C$. Notice that here each connected component F of A, B, C is a subalgebra of D_1 and $|F| > 1$, i.e., A, B and C fulfil the condition (ii).

3.10. E x a m p l e. Let E be as in 3.9. Take $A = E^{\aleph_0}$, $B = E$, $C = E^2$. Then

$$AB \cong E^{\aleph_0} \cong AC, \ B \ncong C.$$

Here A, B, C have finitely many connected components, each connected component is a direct product of subalgebras of D_1, but there are infinitely many factors in the product in A.

4. The class $\mathcal{U}(\mathbb{Z})$

Let $\mathbb{N} = (\mathbb{N}, f)$ be a monounary algebra such that $f(x) = x + 1$ for each $x \in \mathbb{N}$. We will show that the cancellation law (1) in $\mathcal{U}(\mathbb{N})$ is not valid in general.

4.1. Lemma. $\mathbb{N}\mathbb{N} \cong \aleph_0\mathbb{N}$.

P r o o f. Let E be a connected component of $\mathbb{N}\mathbb{N}$, $u = (u_1, u_2) \in E$. Without loss of generality, suppose that $u_1 \leqslant u_2$. Let $a = (1, u_2 - u_1 + 1)$. Then

$$f^{u_1-1}(a) = (1 + (u_1 - 1), u_2 - u_1 + 1 + (u_1 - 1)) = (u_1, u_2) = u,$$

thus $a \in E$. Further, $f^{-1}(a) = \emptyset$. If $i, j \in \mathbb{N} \cup \{0\}, f^i(a) = f^j(a)$, then

$$(1 + i, u_2 - u_2 - u_1 + 1 + i) = (1 + j, u_2 - u_1 + 1 + j),$$

which implies that $i = j$. We will show that

$$E = \{f^i(a)\colon \ i \in \mathbb{N} \cup \{0\}\}.$$

Let $(w_1, w_2) = w \in E$. Then there are $m, n \in \mathbb{N} \cup \{0\}$ such that $f^m(a) = f^n(w)$. Denote $a_2 = u_2 - u_1 + 1$. We obtain

$$\begin{aligned}(1 + m, a_2 + m) &= (w_1 + n, w_2 + n),\\ 1 + m = w_1 + n, a_2 + m &= w_2 + n,\\ w_1 = 1 + m - n, w_2 &= a_2 + m - n.\end{aligned}$$

Since $w_1 \geqslant 1$, we have $m - n \geqslant 0$ and $f^{m-n}(a) = w$. Therefore $E \subseteq \{f^i(a)\colon i \in \mathbb{N} \cup \{0\}\}$. The converse inclusion is obvious. Hence each connected component of $\mathbb{N}\mathbb{N}$ is isomorphic to $\mathbb{N}$.

Further, if $i, j \in \mathbb{N}$, $i \neq j$, then $(1, i)$ and $(1, j)$ do not belong to the same connected component. We have $|\mathbb{N}\mathbb{N}| = \aleph_0$, thus $\mathbb{N}\mathbb{N}$ consists of $\aleph_0$ connected components which are all isomorphic to $\mathbb{N}$. □

4.2. Lemma. *A monounary algebra E belongs to $\mathcal{U}(\mathbb{N})$ if and only if $E \cong k\mathbb{N}$, $k \in \mathbb{N}$.*

Proof. Let $E \in \mathcal{U}(\mathbb{N})$. If F is a connected component of E, then F is a direct product of finitely many subalgebras of $\mathbb{N}$. Since each subalgebra of $\mathbb{N}$ is isomorphic to $\mathbb{N}$ and a product of at least two algebras isomorphic to $\mathbb{N}$ is non-connected by 4.1, we obtain that each connected component F of E is isomorphic to $\mathbb{N}$. Next, E consists of finitely many connected components, which implies that $E \cong k\mathbb{N}$, $k \in \mathbb{N}$.

The relation $\{k\mathbb{N}\colon\ k \in \mathbb{N}\} \subseteq \mathcal{U}(\mathbb{N})$ is obvious. □

4.3. Lemma. *Let $A, B \in \mathcal{U}(\mathbb{N})$. Then $AB \cong \aleph_0\mathbb{N}$.*

Proof. By 4.2 there are $k, m \in \mathbb{N}$ with $A \cong k\mathbb{N}$, $B \cong m\mathbb{N}$. According to 4.1 we obtain

$$AB \cong (k\mathbb{N})(m\mathbb{N}) = (km)(\mathbb{N}\mathbb{N}) \cong \aleph_0\mathbb{N}.$$

□

From 4.3 we infer that the cancellation law (1) does not hold in $\mathcal{U}(\mathbb{N})$ in general. Further, as a corollary we obtain

4.4. Theorem.

(a) *For each $A \in \mathcal{U}(\mathbb{N})$ there are $B, C \in \mathcal{U}(\mathbb{N})$ with $B \ncong C$, $AB \cong AC$.*

(b) *For each $B, C \in \mathcal{U}(\mathbb{N})$ there is $A \in \mathcal{U}(\mathbb{N})$ such that $AB \cong AC$.*

4.5. Corollary. *The cancellation law (1) in $\mathcal{U}(\mathbb{Z})$ does not hold in general.*

Proof. This is a consequence of the fact that the class $\mathcal{U}(\mathbb{N})$ is a subclass of $\mathcal{U}(\mathbb{Z})$ and that the cancellation law (1) does not hold in $\mathcal{U}(\mathbb{N})$ in general. □

Let $n \in \mathbb{N}, n > 1$. According to the notation of Section 2 we have $D_n = \mathbb{Z}_n \cup \mathbb{N}$, where $i_n \in \mathbb{Z}_n$ is the set of all integers k with $k \equiv i \pmod{n}$.

5.1. Lemma. *If X, Y are subalgebras of D_n, then XY is non-connected.*

Proof. Let $a = (0_n, 1_n)$, $b \in (1_n, 0_n) \in XY$. By way of contradiction, suppose that XY is connected. Then there are $k, m \in \mathbb{N} \cup \{0\}$ such that $f^k(a) = f^m(b)$. We obtain

$$((0+k)_n, (1+k)_n) = ((1+m)_n, (0+m)_n),$$

i.e., $k \equiv m+1, 1+k \equiv m \pmod{n}$. This implies that $n/2$ and since $n > 1$, we have $n = 2$. Take $c = (0_2, 0_2)$. There exist $p, q \in \mathbb{N} \cup \{0\}$ such that $f^p(a) = f^q(c)$, thus

$$(p_2, (1+p)_2) = (q_2, q_2),$$

i.e., $p \equiv q, 1+p \equiv q \pmod{2}$, which is a contradiction. □

5.2. Lemma. *A monounary algebra E belongs to $\mathcal{U}(D_n)$ if and only if E consists of finitely many connected components and each connected component F of E is a subalgebra of $D_n, |F| > n$.*

Proof. Let $E \in \mathcal{U}(D_n)$. Then it has finitely many connected components. By the definition of $\mathcal{U}(D_n)$, no connected component F of E is a cycle, thus $|F| > n$. The remaining part of the proof is analogous to 4.2 provided we apply 5.1. □

In 5.3.1–5.5.3 let X, Y be subalgebras of D_n such that $|X| > n$, $|Y| > n$. There are $k, m \in \mathbb{N} \cup \{\aleph_0\}$ with

$$X = \mathbb{Z}_n \cup \{i \in \mathbb{N}\colon\ i \leqslant k\},$$
$$Y = \mathbb{Z}_n \cup \{i \in \mathbb{N}\colon\ i \leqslant m\}.$$

Let $E = XY$. The following two lemmas are easy to verify by a routine calculation.

5.3.1. Lemma.

(a) *Let $v = (v_1, v_2) \in E$. Then v belongs to a cycle of E if and only if $v_1, v_2 \in \mathbb{Z}_n$.*
(b) *Each connected component of E contains a cycle with n elements.*

5.3.2. Lemma.

(a) $f^{-1}((0_n, 0_n)) = \{((n-1)_n, (n-1)_n), (1, (n-1)_n), ((n-1)_n, 1),;$
$((n-1)_n, 1), (1, 1)\}$

(b) *if* $0_n \neq j_n \in \mathbb{Z}_n$*, then* $f^{-1}((j_n, 0_n)) = \{((j-1)_n, (n-1)_n), ((j-1)_n, 1)\}$, $f^{-1}((0_n, j_n)) = \{((n-1)_n, (j-1)_n), (1, (j-1)_n)\}$;

(c) *if* $0_n \neq j_n \in \mathbb{Z}_n$, $0_n \neq l_n \in \mathbb{Z}_n$*, then* $f^{-1}((j_n, l_n)) = \{((j-1)_n, (l-1)_n)\}$.

5.3.3. Corollary. *Let* v *be a cyclic element of* E*. Then* $v = (0_n, 0_n)$ *if and only if* $|f^{-1}(v)| = 4$.

5.4. Lemma. *Let* F *be the connected component of* E *containing the element* $(0_n, 0_n)$.

(a) $|\{v \in F\colon\ v \text{ is cyclic}, |f^{-1}(v)| > 1\}| = 1$;

(b) *if* F_1 *is a connected component of* E *such that* $F_1 \neq F$*, then* $|\{v \in F_1\colon\ v \text{ is cyclic}, |f^{-1}(v)| > 1\}| > 1$.

Proof. Let v be a cyclic element of F. Then $v = (i_n, i_n)$, $i_n \in \mathbb{Z}_n$. If $i_n = 0_n$, then 5.3.3 implies that $|f^{-1}(v)| = 4$. If $i_n \neq 0_n$, then 5.3.2(c) yields that $|(f^{-1}(v)| = 1$. Hence (a) is valid.

Now let F_1 be a connected component of E such that $F_1 \neq F$. Then there is $j \in \{1, 2, \ldots, n-1\}$ such that $(0_n, j_n) \in F_1$. Denote $v = (0_n, j_n)$, $w = f^{n-j}(v)$. Thus w is a cyclic element of F_1,

$$w = ((n-j)_n, (j+n-j)_n) = ((-j)_n, 0_n).$$

According to 5.3.2(b), $|f^{-1}(v)| = 2 = |f^{-1}(w)|$, which implies that (b) holds. □

Denote $u = (0_n, 0_n), u^{(1)} = (1, 1), u^{(2)} = (1, (n-1)_n), u^{(3)} = ((n-1)_n, 1)$.

5.5.1. Lemma.

(a) *If* $k = m = \aleph_0$*, then* $f^{-1}(u^{(\alpha)}) \neq \emptyset$ *for each* $\alpha \in \{1, 2, 3\}$, $i \in \mathbb{N}$.

(b) *If* $k < m = \aleph_0$*, then* $f^{-1}(u^{(3)}) \neq \emptyset$ *for each* $i \in \mathbb{N}$ *and* $f^{-k}(u^{(1)}) = \emptyset = f^{-(k-1)}(u^{(1)}), f^{-k}(u^{(2)}) = \emptyset \neq f^{-(k-1)}(u^{(2)})$.

(c) *If* $k \leqslant m < \aleph_0$*, then* $f^{-m}(u^{(3)}) = \emptyset \neq f^{-(m-1)}(u^{(3)}), f^{-k}(u^{(1)}) = \emptyset \neq f^{-(k-1)}(u^{(1)}), f^{-k}(u^{(2)}) = \emptyset \neq f^{-(k-1)}(u^{(2)})$.

Proof.

(a) Let $k = m = \aleph_0$, $i \in \mathbb{N}$. Then

$$\begin{aligned} f^i((i+1, i+1)) &= (i+1-i, i+1-i) = (1,1) = u^{(1)}, \\ f^i((i+1, (n-1-i)_n) &= (i+1-i, (n-1-i+i)_n) = (1, (n-1)_n) = u^{(2)}, \\ f^i(((n-1-i)_n, i+1)) &= ((n-1-i+i)_n, i+1-i) = u^{(3)}. \end{aligned}$$

(b) Let $k < m = \aleph_0$, $i \in \mathbb{N}$. Similarly as above, $f^i((n-1-i)_n, i+1) = u^{(3)}$. Further,

$$f^{k-1}((k,(-k)_n)) = (k-(k-1), (-k+k-1)_n) = (1,(n-1)_n) = u^{(2)},$$
$$f^{k-1}((k,k)) = (k-(k-1), k-(k-1)) = (1,1) = u^{(1)}.$$

Suppose that $f^{-k}(u^{(1)}) \neq \emptyset$ (the case for $u^{(2)}$ is analogous). Then there is $(t_1,t_2) \in E$ with

$$u^{(1)} = (1,1) = f^k((t_1,t_2)) = (f^k(t_1), f^k(t_2)).$$

This implies that in X the set $f^{-k}(1)$ is non-empty, which is a contradiction. Therefore (b) is valid.

(c) The proof of this assertion is similar to that of (b).

□

In view of 5.3.2(a), the set $f^{-1}(u) = f^{-1}((0_n, 0_n))$ consists of a cyclic element $((n-1)_n, (n-1)_n)$ and of three non-cyclic elements; let $w^{(1)}$, $w^{(2)}$, $w^{(3)}$ be these elements.

5.5.2. Lemma.

(a) *If $k = m = \aleph_0$, then $f^{-i}(w^{(\alpha)}) \neq \emptyset$ for each $\alpha \in \{1,2,3\}$, $i \in \mathbb{N}$.*
(b) *If $k < m = \aleph_0$, then there is $\alpha \in \{1,2,3\}$ such that $f^{-i}(w^{(\alpha)}) \neq \emptyset$ for each $i \in \mathbb{N}$ and if $\alpha \neq \beta \in \{1,2,3\}$, then $f^{-k}(w^{(\beta)}) = \emptyset \neq f^{-(k-1)}(w^{(\beta)})$.*
(c) *If $k \leqslant m < \aleph_0$, then there is $\alpha \in \{1,2,3\}$ such that $f^{-m}(w^{(\alpha)}) = \emptyset \neq f^{-(m-1)}(w^{(\alpha)})$ and if $\alpha \neq \beta \in \{1,2,3\}$, then $f^{-k}(w^{(\beta)}) = \emptyset \neq f^{-(k-1)}(w^{(\beta)})$.*

5.5.3. Corollary.

(a) *Let $f^{-i}(w^{(\alpha)}) \neq \emptyset$ for each $\alpha \in \{1,2,3\}$, $i \in \mathbb{N}$. Then $k = m = \aleph_0$.*
(b) *Let the assumption of* (a) *be not valid and suppose that there is $\alpha \in \{1,2,3\}$ such that $f^{-i}(w^{(\alpha)}) \neq \emptyset$ for each $i \in \mathbb{N}$. Then there is $j \in \mathbb{N}$ such that if $\alpha \neq \beta \in \{1,2,3\}$, then $f^{-j}(w^{(\beta)}) = \emptyset \neq f^{-(j-1)}(w^{(\beta)})$. Further, this yields that $\{k,m\} = \{j, \aleph_0\}$.*
(c) *Let neither the assumption of* (a) *nor the assumption of* (b) *be valid. There are $j, l \in \mathbb{N}$ and $\alpha \in \{1,2,3\}$ such that $f^{-j}(w^{(\alpha)}) = \emptyset \neq f^{-(j-1)}(w^{(\alpha)})$ and if $\alpha \neq \beta \in \{1,2,3\}$, then $f^{-l}(w^{(\beta)}) = \emptyset \neq f^{-(l-1)}(w^{(\beta)})$. Then $\{k,m\} = \{j,l\}$.*

5.6. Lemma. *Suppose that X, Y, X', Y' are subalgebras of D_n which are not cycles. If $XY \cong X'Y'$, then either $X \cong X'$, $Y \cong Y'$ or $X \cong Y'$, $Y \cong X'$.*

Proof. Let k, m, E, u, F be as above and assume that k', m', E', u', F' have an analogous meaning in the product $X'Y'$. There is an isomorphism $\xi\colon XY \to X'Y'$. By 5.3.1, ξ maps cyclic elements into cyclic elements and by 5.3.3, $\xi(u) = u'$. Next, 5.4 implies that $\xi(F) = F'$. It follows from 5.5.3 that

$$\{k, m\} = \{k', m'\}.$$

If $k = k'$, $m = m'$, then $X \cong X'$, $Y \cong Y'$. If $k = m'$, $m = k'$, then $X \cong Y'$, $Y \cong X'$. □

5.7. Theorem. *Let $A, B, C \in \mathcal{U}(D_n)$, $n \in \mathbb{N}$, $n > 1$. Then $AB \cong AC$ implies $B \cong C$.*

Proof. It follows from 5.2 that A, B, C are sums of finitely many subalgebras of D_n. Let $\{Y_1, \ldots, Y_n\}$ be a system of monounary algebras such that

(1) $Y_i \ncong Y_j$ for $i, j \in \{1, \ldots, n\}$, $i \neq j$,

(2) if F is a connected component of A, B or C, then $F \cong Y_i$ for some $i \in \{1, \ldots, n\}$.

Then there are non-negative integers α_i, β_i, γ_i ($i \in \{1, \ldots, n\}$) such that

$$A \cong \sum_{i=1}^{n} \alpha_i Y_i,\ B \cong \sum_{i=1}^{n} \beta_i Y_i,\ C \cong \sum_{i=1}^{n} \gamma_i Y_i.$$

Suppose that $AB \cong AC$, i.e.,

$$\sum_{i,j} (\alpha_i \beta_j)(Y_i Y_j) \cong \sum_{i,j} (\alpha_i \gamma_j)(Y_i Y_j).$$

Since (1) is valid, we obtain by virtue of 5.6 that $\beta_j = \gamma_j$ for each $j \in \{1, \ldots, n\}$. Therefore $B \cong C$. □

References

[1] *B. Jónsson*: Topics in Universal Algebra. Springer, Berlin, 1972.

[2] *J. Jakubík, L. Lihová*: On the cancellation law for disconnected partially ordered sets. Math. Bohem. Submitted.

[3] *L. Lovász*: Operations with structures. Acta Math. Acad. Sci. Hungar. *18* (1967), 321–328.

[4] *L. Lovász*: On the cancellation law among finite relational structures. Period. Math. Hungar. *1* (1971), 145–156.

[5] *R. McKenzie, G. McNulty, W. Taylor*: Algebras, Lattices, Varieties. Vol. I, Wadsworth, Belmont, 1987.

[6] *J. Novotný*: On the characterization of a certain class of monounary algebras. Math. Slovaca *40* (1990), 123–126.

[7] *M. Ploščica, M. Zelina*: Cancellation among finite unary algebras. Discrete Mathematics *159* (1996), 191–198.

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