INDUCED-PAIRED DOMATIC NUMBERS OF GRAPHS

Bohdan Zelinka, Liberec

(Received March 7, 2001)

Abstract. A subset D of the vertex set $V(G)$ of a graph G is called dominating in G, if each vertex of G either is in D, or is adjacent to a vertex of D. If moreover the subgraph $\langle D\rangle$ of G induced by D is regular of degree 1, then D is called an induced-paired dominating set in G. A partition of $V(G)$, each of whose classes is an induced-paired dominating set in G, is called an induced-paired domatic partition of G. The maximum number of classes of an induced-paired domatic partition of G is the induced-paired domatic number $d_{\rm ip}(G)$ of G. This paper studies its properties.

Keywords: dominating set, induced-paired dominating set, induced-paired domatic number

MSC 2000: 05C69, 05C35

A subset D of the vertex set $V(G)$ of a graph G is called dominating in G, if each vertex of G either is in D, or is adjacent to a vertex of D. The minimum number of vertices of a dominating set in G is the domination number $\gamma(G)$ of G. The maximum number of classes of a partition of $V(G)$, all of whose classes are dominating sets in G, is the domatic number $d(G)$ of G. This concept was introduced by F. J. Cockayne and S. T. Hedetniemi in [1].

A variant of $\gamma(G)$ was introduced in [3] by D. J. Studer, T. W. Haynes and L. M. Lawson. If a dominating set D in G has the property that the subgraph $\langle D\rangle$ of G induced by D is regular of degree 1, then D is called an induced-paired dominating set in G. The minimum number of vertices of an induced-paired dominating set in G is the induced-paired domination number $\gamma_{\rm ip}(G)$ of G.

Analogously as to $\gamma(G)$ the domatic number $d(G)$ was introduced, to $\gamma_{\rm ip}(G)$ we introduce the induced-paired domatic number $d_{\rm ip}(G)$. A partition of $V(G)$ is called induced-paired domatic, if all of its classes are induced-paired dominating sets (shortly IPDS) of G. The maximum number of classes of an induced-paired domatic partition of G is the induced-paired domatic number $d_{\rm ip}(G)$ of G. Let us recall yet another numerical invariant of a graph which will be useful for our considerations. A dominating set in G which is simultaneously independent (i.e. consisting of pairwise non-adjacent vertices) is an independent dominating set in G. The maximum number of classes of a partition of $V(G)$, all of whose classes are independent dominating sets in G, is called the independent domatic number (or shortly idomatic number) $d_{\rm i}(G)$ of G [4].

The numbers $d_{\rm i}(G)$ and $d_{\rm ip}(G)$ have the property that they are not well-defined for all graphs. Namely, there are graphs whose vertex sets cannot be partitioned into independent dominating sets or into induced-paired dominating sets.

Proposition 1. *Let G be a graph in which there is at least one independent domatic partition. Then $G \times K_2$ has at least one induced-paired domatic partition and*

$$d_{\rm ip}(G \times K_2) \geqslant d_{\rm i}(G).$$

P r o o f. The graph $G \times K_2$ consists of two vertex-disjoint copies of G and of edges joining the corresponding vertices in both the copies. Let D be an independent dominating set in one copy of G.

Let D' be the set consisting of all vertices of D and of all vertices of the other copy of G which are adjacent to vertices of D in $G \times K_2$. Then evidently D' is an induced-paired dominating set of $G \times K_2$. If some sets D form an independent domatic partition of the chosen copy of G, then the sets D' form an induced-paired domatic partition of $G \times K_2$ with the same number of classes. $\square$

Corollary 1. *Let G be a connected bipartite graph. Then $G \times K_2$ has at least one induced-paired domatic partition and $d_{\rm ip}(G) \geqslant 2$.*

A complete k-partite graph for an integer $k \geqslant 2$ is a graph whose vertex set is the disjoint union of k independent sets $V_1, \ldots, V_k$ and in which two-vertices are adjacent if and only if they belong to the sets V_i, V_j with $i \neq j$.

Corollary 2. *Let G be a complete bipartite graph. Then $d_{\rm ip}(G \times K_2) = k$.*

P r o o f. Any IPDS in $G \times K_2$ is a set consisting of vertices of V_i for some i in one copy of G and of vertices which are adjacent to them in the other copy. A proper subset of such a set is not dominating. Any two edges of G have either a common end vertex, or an edge which has common end vertices with both of them. $\square$

Proposition 2. *Let n be an even positive integer. For the complete graph K_n with n vertices we have $d_{\mathrm{ip}}(K_n) = \frac{1}{2}n$.*

P r o o f. We choose a linear factor in K_n. Each of its edges forms a one-element IPDS and this implies the result. $\square$

Proposition 3. *Let $K_{m,n}$ be a complete bipartite graph. An induced-paired domatic partition of $K_{m,n}$ exists if an only if $m = n$, and then $d_{\mathrm{ip}}(K_{n,n}) = n$.*

P r o o f. The first part is evident, the second part may be proved in the same way as Proposition 2. $\square$

Proposition 1. *Let C_n be the circuit of length n. An induced paired domatic partition of C_n exists if and only if n is divisible by 4, and then $d_{\mathrm{ip}}(C_n) = 2$.*

P r o o f. Let the edges going around C_n be $e_1, e_2, \ldots, e_n$. Evidently, if n is not divisible by 4, an induced-paired domatic partition does not exist. If n is divisible by 4, then let $E_1 = \{e_i;\ \ i \equiv 0 \pmod 4\}$, $E_1 = \{e_i;\ \ i \equiv 2 \pmod 4\}$. Let D_1 (or D_2) be the set of all end vertices of edges of E_1 (or of E_2 respectively). Then $\{D_1, D_2\}$ is an induced-paired domatic partition of C and $d_{\mathrm{ip}}(C_n) = 2$. $\square$

Now we state two general assertions.

Proposition 5. *If there exists an induced-paired domatic partition of G, then G has an even number of vertices.*

Proof is straightforward.

Proposition 6. *Let there exist $d_{\mathrm{ip}}(G)$ for a graph G, let $\delta(G)$ be the minimum degree of a vertex of G. Then $d_{\mathrm{ip}}(G) \leqslant \delta(G)$.*

P r o o f. Each vertex v of G must be adjacent to at least one vertex of each class of an induced-paired domatic partition to which v does not belong. Moreover, it is incident with an edge of the subgraph of G induced by the class to which v belongs. Hence the degree of v is at least $d_{\mathrm{ip}}(G)$. $\square$

Now we will study graphs G with $d_{\mathrm{ip}}(G) = 2$.

Theorem 1. *Let G be a graph with n vertices such that $d_{\mathrm{ip}}(G) = 2$. Then*

$$n \leqslant |E(G)| \leqslant \frac{1}{4}n^2 - 1.$$

Both the bounds are attained.

P r o o f. Let $\{D_1, D_2\}$ be an induced-paired domatic partition of G. Let $|D_1| = a$, $|D_2| = b$, $a \leqslant b$. Therefore $a + b = n$, $b \geqslant \frac{1}{2}n$. The subgraph of G induced by D_1 (or D_2) has $\frac{1}{2}a$ (or $\frac{1}{2}b$ respectively) edges. As $a \leqslant b$, there exists at least b edges joining vertices of D_1 with vertices of D_2. The number of edges of G is at least $b + \frac{1}{2}a + \frac{1}{2}b = b + \frac{1}{2}n$. This expression has its minimum value for $b = \frac{1}{2}n$. Then G has n edges, in the other cases they are at least n.

Now suppose that G contains all edges which join vertices of D_1 with vertices of D_2. We must exclude the possibility $a = b$; otherwise G would contain a factor isomorphic to $K_{a,a}$ and the inequality $d_{\rm ip}(G) \geqslant a$ would hold. Again the subgraph of G induced by D_1 (or D_2) has $\frac{1}{2}a$ (or $\frac{1}{2}b$ respectively) edges. The number of other edges is ab. The number of edges of G is $ab + \frac{1}{2}a + \frac{1}{2}b = ab + \frac{1}{2}n$. The maximum value of this expression is for $a = b$; but we have excluded this case. The maximum in the other cases occurs for $a = \frac{1}{2}n - 1$, $b = \frac{1}{2}n + 1$ and it is $\frac{1}{4}n^2 - 1$. □

Return to Theorem 1. Evidently a graph G having the minimum number of edges at $d_{\rm ip}(G) = 2$ is a regular graph of degree 2, i.e. a graph all of whose connected components are circuits.

According to Proposition 4 these circuits have lengths divisible by 4. In a graph G with $d_{\rm ip}(G) > 2$ this holds for the subgraph of G induced by the union of two classes of the induced-paired domatic partition. This implies the following proposition.

Proposition 7. *Let G be a graph with the minimum number of edges at a given $d_{\rm ip}(G) \geqslant 3$. Then G is the union of circuits of lengths divisible by 4. The edges of each circuit may be coloured alternatingly in red and blue in such a way that each red edge is contained in $d_{\rm ip}(G) - 1$ circuits, while each blue edge is contained in only one of them. Each vertex is incident with one red edge and $d_{\rm ip}(G) - 1$ blue edges.*

It is evident that red edges are exactly the edges of the subgraph of G induced by classes of the induced-paired domatic partition, while blue edges are the others.

Theorem 2. *Let G be a graph with n vertices such that $d_{\rm ip}(G) = \frac{1}{2}n$. Then*

$$\frac{1}{4}n^2 \leqslant |E(G)| \leqslant \frac{1}{2}n(n-1).$$

Both the bounds are attained.

P r o o f. Let $\mathcal{D}$ be an induced-paired domatic partition of G. Each vertex of G must be adjacent to vertices of all classes of $\mathcal{D}$ and have degree at least $\frac{1}{2}n$. Hence $|E(G)| \geqslant \frac{1}{4}n^2$. The equality $|E(G)| = \frac{1}{4}n^2$ is attained in the case when $G \cong K_{n/2} \times K_2$. The upper bound follows from the fact that $\frac{1}{2}n\ (n-1)$ is the number of edges of K_n. And, as we have seen in Proposition 2, $d_{\rm ip}(K_n) = \frac{1}{2}n$. □

Obviously $d_{\mathrm{ip}}(G) \leqslant \frac{1}{2}n$, whenever $d_{\mathrm{ip}}(G)$ exists; this follows from the definition. Also the following proposition is evident.

Proposition 8. *Let G be a graph. The equality $d_{\mathrm{ip}}(G) = 1$ holds if and only if G is regular of degree 1.*

Now we shall treat interconnections among graphs and interconnections among IPDS.

Note that in some cases there is non analogy between $d_{\mathrm{ip}}(G)$ and $d(G)$. If $D \subseteq S \subseteq V(G)$, where D is an IPDS, then S need not be an IPDS.

Proposition 9. *Let G be the disjoint union of two graphs G_1, G_2. A subset $S \subseteq V(G)$ is an IPDS in G if and only if $S = S_1 \cup S_2$, where S_1 is an IPDS in G_1 and S_2 is an IPDS in G_2.*

Proof is easy.

Corollary 3. *Let G be the disjoint union of two graphs G_1, G_2. If $d_{\mathrm{ip}}(G_1) = d_{\mathrm{ip}}(G_2)$, then also $d_{\mathrm{ip}}(G) = d_{\mathrm{ip}}(G_1) = d_{\mathrm{ip}}(G_2)$.*

If G_1, G_2 are two vertex-disjoint graphs, then the Zykov sum $G_1 \oplus G_2$ of G_1 and G_2 is the graph obtained from G_1 and G_2 by G adding all edges which join a vertex of G_1 with a vertex of G_2.

Theorem 4. *Let G be the Zykov sum $G_1 \oplus G_2$. A subset $S \subseteq V(G)$ is an IPDS of G if and only if it is an IPDS in G_1 or an IPDS in G_2.*

Proof is again easy.

Corollary 4. *Let there exist numbers $d_{\mathrm{ip}}(G_1)$ and $d_{\mathrm{ip}}(G_2)$ for graphs G_1, G_2. Then $d_{\mathrm{ip}}(G_1 \oplus G_2) = d_{\mathrm{ip}}(G_1) + d_{\mathrm{ip}}(G_2)$.*

It is easy to compare $d_{\mathrm{ip}}(G)$ with the domatic number $d(G)$ and with the total domatic number $d_t(G)$ (see e.g. [2]). Every IPDS of G is also a total dominating set in G and every total dominating set in G is a dominating set in G.

Proposition 10. *Let G be a graph for which $d_{\mathrm{ip}}(G)$ is defined. Then $d(G) \leqslant d_t(G) \leqslant d_{\mathrm{ip}}(G)$.*

Now we compare $d_{\mathrm{ip}}(G)$ with the chromatic number $\chi(G)$ of G.

Proposition 11. *Let G be a graph for which $d_{\mathrm{ip}}(G)$ is defined. Then $\chi(G) \leqslant 2d_{\mathrm{ip}}(G)$.*

P r o o f. Let $s = d_{\mathrm{ip}}(G)$ and let $\{D_1, \ldots, D_s\}$ be an induced-paired domatic partition of G. Let us have $2s$ colours $c_1^1, c_1^2, \ldots, c_s^1, c_s^2$. We colour vertices of G by these colours. The vertices of each D_i for $i = 1, \ldots, s$ will be coloured by c_i^1 and c_i^2; obviously adjacent vertices are coloured by different colours. The colouring thus obtained is an admissible colouring of G and this yields the assertion. $\square$

References

[1] *E. J. Cockayne, S. T. Hedetniemi*: Towards a theory of domination in graphs. Networks *7* (1977), 247–261.

[2] *T. W. Haynes, S. T. Hedetniemi, P. J. Slater*: Fundamentals of Domination in Graphs. Marcel Dekker, New York, 1998.

[3] *D. S. Studer, T. W. Haynes, L. M. Lawson*: Induced-paired domination in graphs. Ars Combinatoria *57* (2000), 111–128.

[4] *B. Zelinka*: Adomatic and idomatic numbers of graphs. Math. Slovaca *33* (1983), 99–103.

Author's address: *Bohdan Zelinka*, Department of Applied Mathematics, Technical University of Liberec, Voroněžská 13, 460 01 Liberec, Czech Republic, e-mail: `bohdan.zelinka@vslib.cz`.