NEW OPTIMAL CONDITIONS FOR UNIQUE SOLVABILITY OF THE CAUCHY PROBLEM FOR FIRST ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The nonimprovable sufficient conditions for the unique solvability of the problem

$$u'(t) = \ell(u)(t) + q(t), \qquad u(a) = c,$$

where $\ell\colon C(I;\mathbb{R}) \to L(I;\mathbb{R})$ is a linear bounded operator, $q \in L(I;\mathbb{R})$, $c \in \mathbb{R}$, are established which are different from the previous results. More precisely, they are interesting especially in the case where the operator ℓ is not of Volterra's type with respect to the point a.

Keywords: linear functional differential equations, differential equations with deviating arguments, initial value problems

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1. Statement of the problem and formulation of the main results

On the segment $I = [a, b]$ we will consider the functional differential equation

$$u'(t) = \ell(u)(t) + q(t) \tag{1.1}$$

and its particular case

$$u'(t) = p(t)u(\tau(t)) + q(t) \tag{1.1$'$}$$

with an initial condition

$$u(a) = c. \tag{1.2}$$

Here $\ell\colon C(I;\mathbb{R}) \to L(I;\mathbb{R})$ is a linear bounded operator, $c \in \mathbb{R}$, $p, q \in L(I;\mathbb{R})$, and $\tau\colon I \to I$ is a measurable function.

In this paper, nonimprovable sufficient conditions for the unique solvability of the problems (1.1), (1.2) and (1.1′), (1.2) are established which are different from the previous results (see [1–6] and references therein). More precisely, they are interesting especially in the case where the operator ℓ in the equation (1.1) is not of Volterra's type with respect to the point a (see notation below).

Throughout the paper the following notation is used.

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $[x]_+ = \frac{1}{2}(|x| + x)$, $[x]_- = \frac{1}{2}(|x| - x)$.

$C(I;\mathbb{R})$ is the Banach space of continuous functions $u\colon I \to \mathbb{R}$ with the norm

$$\|u\|_C = \max\{|u(t)|\colon\ t \in I\}.$$

$C(I;\mathbb{R}_+) = \{u \in C(I;\mathbb{R})\colon\ u(t) \geqslant 0 \text{ for } \ t \in I\}$.

$\widetilde{C}(I;D)$, where $D \subset \mathbb{R}$, is the set of absolutely continuous functions $u\colon I \to D$.

$L(I;\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p\colon I \to \mathbb{R}$ with the norm

$$\|p\|_L = \int_I |p(s)|\,\mathrm{d}s.$$

$L(I;\mathbb{R}_+) = \{p \in L(I;\mathbb{R})\colon\ p(t) \geqslant 0 \text{ for almost all } \ t \in I\}$.

$\mathcal{L}_I$ is the set of linear bounded operators $\ell\colon C(I;\mathbb{R}) \to L(I;\mathbb{R})$.

$\mathcal{P}_I$ is the set of operators $\ell \in \mathcal{L}_I$ mapping $C(I;\mathbb{R}_+)$ into $L(I;\mathbb{R}_+)$.

An operator $\ell \in \mathcal{L}_I$ is said to be a Volterra operator with respect to t_0, where $t_0 \in [a, b]$, if for arbitrary $a_1 \in [a, t_0]$, $b_1 \in [t_0, b]$, $a_1 \neq b_1$, and $v \in C(I;\mathbb{R})$ satisfying the condition

$$v(t) = 0 \qquad \text{for } \ a_1 \leqslant t \leqslant b_1$$

we have

$$\ell(v)(t) = 0 \qquad \text{for } \ a_1 < t < b_1.$$

A function $u \in \widetilde{C}(I;\mathbb{R})$ is said to be a solution of the equation (1.1) if it satisfies (1.1) almost everywhere on I.

Let $\ell \in \mathcal{L}_I$. Define

$$T(v)(t) \stackrel{\text{def}}{=} \int_a^t \ell(v)(s)\,\mathrm{d}s, \quad \varphi(v)(t) \stackrel{\text{def}}{=} v(t) \exp\bigl[T(1)(t)\bigr],$$
$$\hat{\ell}_0(v)(t) \stackrel{\text{def}}{=} \ell(v)(t), \quad \hat{\ell}_{k+1}(v)(t) \stackrel{\text{def}}{=} \hat{\ell}_k(T(v))(t), \quad k \in \mathbb{N}_0,$$
$$G(v)(t) \stackrel{\text{def}}{=} \bigl[\hat{\ell}_1(\varphi(v))(t) - \ell(1)(t) T(\varphi(v))(t)\bigr] \exp\bigl[-T(1)(t)\bigr].$$

Theorem 1.1. *Let $-\ell \in \mathcal{P}_I$ and let for some $k \in \mathbb{N}_0$ the inequality*

$$\int_a^b \hat{\ell}_k(1)(s)\,\mathrm{d}s < 2 + \frac{1}{2}\big(1 + (-1)^k\big) \tag{1.3}$$

hold. Then the problem (1.1), (1.2) *has a unique solution.*

Corollary 1.1. *Let $p(t) \leqslant 0$ for $t \in I$ and*

$$\operatorname{ess\,sup}\left\{\int_a^{\tau(t)} |p(s)| \int_a^{\tau(s)} |p(\xi)|\,\mathrm{d}\xi\,\mathrm{d}s \colon\ t \in I\right\} < 2. \tag{1.4}$$

Then the problem (1.1′), (1.2) *has a unique solution.*

R e m a r k 1.1. It follows from Theorem 1.2 in [2] that if $p(t) \leqslant 0$ for $t \in I$ and either

$$\operatorname{ess\,sup}\left\{\int_a^{\tau(t)} |p(s)|\,\mathrm{d}s \colon\ t \in I\right\} < 3 \tag{1.5}$$

or

$$\operatorname{ess\,sup}\left\{\int_{\tau(t)}^b |p(s)|\,\mathrm{d}s \colon\ t \in I\right\} \leqslant 1, \tag{1.6}$$

then the problem (1.1′), (1.2) is uniquely solvable. In Section 3, we will give an example showing that the condition (1.4) does not follow from the conditions (1.5) and (1.6). In this sense Theorem 1.1 makes the above mentioned theorem from [2] more complete.

Theorem 1.2. *Let $\ell \in \mathcal{P}_I$, and let there exist $k \in \mathbb{N}_0$, $c \in]a,b[$ and $\gamma \in \widetilde{C}(I;\mathbb{R})$ such that*

$$\gamma(t)\operatorname{sgn}(t-c) > 0 \qquad \textit{for}\ \ t \in I \setminus \{c\}, \quad \gamma(a) < 0, \quad \gamma(b) > 0 \tag{1.7}$$

and the inequalities

$$\gamma'(t) \leqslant \hat{\ell}_k(\gamma)(t) \qquad \textit{for}\ \ t \in I, \tag{1.8}$$

$$\int_a^b \hat{\ell}_k(1)(s)\,\mathrm{d}s \leqslant 2 \tag{1.9}$$

are fulfilled. Then the problem (1.1), (1.2) *has a unique solution.*

Corollary 1.2. *Let $p(t) \geqslant 0$ for $t \in I$ and let the inequalities*

$$\operatorname{ess\,inf}\left\{ \int_a^{\tau(t)} p(s)\,\mathrm{d}s \colon t \in I \right\} > 1, \tag{1.10}$$

$$\int_a^b p(s)\,\mathrm{d}s \leqslant 2 \tag{1.11}$$

be fulfilled. Then the problem (1.1′), (1.2) *has a unique solution.*

R e m a r k 1.2. The conditions (1.10) and (1.11) in Corollary 1.2 are optimal and cannot be weakened. More precisely, the strict inequality in (1.10) cannot be replaced by the nonstrict one, and in the inequality (1.11) we cannot write $2+\varepsilon$ instead of 2 no matter how small $\varepsilon > 0$ would be.

Theorem 1.3. *Let $G \in \mathcal{P}_I$ and let there exist $c \in]a, b[$ and $\gamma \in \widetilde{C}(I;\mathbb{R})$ such that the conditions* (1.7) *hold and the inequalities*

$$\gamma'(t) \leqslant G(\gamma)(t) \qquad \text{for } \ t \in I, \tag{1.12}$$

$$\int_a^b G(1)(s)\,\mathrm{d}s \leqslant 2 \tag{1.13}$$

are fulfilled. Then the problem (1.1), (1.2) *has a unique solution.*

Corollary 1.3. *Let $p(t) \geqslant 0$, $\tau(t) \geqslant t$ for $t \in I$ and let the inequalities*

$$\operatorname{ess\,inf}\left\{ \int_a^t p(s)\,\mathrm{d}s + \int_t^{\tau(t)} p(s) \int_a^{\tau(s)} p(\xi)\,\mathrm{d}\xi\,\mathrm{d}s \colon t \in I \right\} > 1, \tag{1.14}$$

$$\int_a^b p(t)\exp\left[-\int_a^t p(\xi)\,\mathrm{d}\xi\right] \int_t^{\tau(t)} p(s)\exp\left[\int_a^{\tau(s)} p(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s\,\mathrm{d}t \leqslant 2 \tag{1.15}$$

be fulfilled. Then the problem (1.1′), (1.2) *has a unique solution.*

R e m a r k 1.3. The conditions (1.14) and (1.15) in Corollary 1.3 are optimal and cannot be weakened. More precisely, the strict inequality in (1.14) cannot be replaced by the nonstrict one, and in the inequality (1.15) we cannot write $2+\varepsilon$ instead of 2 no matter how small $\varepsilon > 0$ would be.

R e m a r k 1.4. It is easy to verify that the condition (1.14) is satisfied, e.g., if the condition (1.10) is satisfied.

Finally, we give a theorem on the existence and uniqueness of a solution of the equation (1.1) satisfying the initial condition

$$u(t_0) = 0, \tag{1.16}$$

where $t_0 \in I$.

Theorem 1.4. *Let $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \mathcal{P}_I$ and let ℓ_0, ℓ_1 be the Volterra operators with respect to b. Let, moreover, there exist $\gamma \in \widetilde{C}([t_0, b];]0, +\infty[)$ satisfying the inequality*

$$\gamma'(t) \geqslant \ell_0(\gamma)(t) \qquad \text{for} \quad t \in]t_0, b[. \tag{1.17}$$

Then the problem (1.1), (1.16) *has a unique solution.*

Corollary 1.4. *Let either*

$$\tau(t) \geqslant t \qquad \text{for} \quad t \in I \quad \text{and} \qquad \int_{t_0}^{b} [p(s)]_+ \, \mathrm{d}s < 1, \tag{1.18}$$

or

$$\tau(t) \leqslant t \qquad \text{for} \quad t \in I \quad \text{and} \qquad \int_{a}^{t_0} [p(s)]_- \, \mathrm{d}s < 1. \tag{1.19}$$

Then the problem (1.1′), (1.16) *has a unique solution.*

R e m a r k 1.5. Corollary 1.4 is optimal in the sense that neither the strict inequality $\int_{t_0}^{b} [p(s)]_+ \, \mathrm{d}s < 1$ nor $\int_{a}^{t_0} [p(s)]_- \, \mathrm{d}s < 1$ can be replaced by the nonstrict one.

2. Auxiliary propositions

Lemma 2.1. *Let $\ell \in \mathcal{P}_I$, and let u be a nontrivial solution of the problem*

$$u'(t) = \ell(u)(t), \tag{2.1}$$

$$u(a) = 0. \tag{2.2}$$

Let, moreover,

$$\int_{a}^{b} \ell(1)(s) \, \mathrm{d}s \leqslant 2. \tag{2.3}$$

Then u is of constant sign.

P r o o f. Suppose on the contrary that u assumes both positive and negative values. Put

$$m = \max\{-u(t)\colon\ t \in I\}, \qquad M = \max\{u(t)\colon\ t \in I\}. \tag{2.4}$$

It is clear that $m > 0$ and $M > 0$. Choose $t_*, t^* \in]a, b]$ such that

$$u(t_*) = -m, \qquad u(t^*) = M. \tag{2.5}$$

Without loss of generality we can assume that $t^* < t_*$. Integrating (2.1) from a to t^* and from t^* to t_* and taking into account (2.4), (2.5), we obtain respectively

$$M = \int_a^{t^*} \ell(u)(s)\,\mathrm{d}s \leqslant M \int_a^{t^*} \ell(1)(s)\,\mathrm{d}s,$$

and

$$M + m = -\int_{t^*}^{t_*} \ell(u)(s)\,\mathrm{d}s \leqslant m \int_{t^*}^{t_*} \ell(1)(s)\,\mathrm{d}s.$$

Consequently,

$$\int_a^b \ell(1)(s)\,\mathrm{d}s \geqslant \int_a^{t^*} \ell(1)(s)\,\mathrm{d}s + \int_{t^*}^{t_*} \ell(1)(s)\,\mathrm{d}s \geqslant 1 + 1 + \frac{M}{m} > 2,$$

which contradicts (2.3). $\square$

Lemma 2.2. *Let the equation* (2.1) *have a positive solution, let $c \in I$ and let the problem*

$$u'(t) = \ell(u)(t), \qquad u(c) = 0 \tag{2.6}$$

have only the trivial solution. Then for every $t_0 \in I$ the problem

$$u'(t) = \ell(u)(t), \qquad u(t_0) = 0 \tag{2.7}$$

possesses only the trivial solution.

P r o o f. Assume on the contrary that there exists $t_0 \in I \setminus \{c\}$ such that the problem (2.7) has a nontrivial solution u_0. Evidently,

$$u_0(c) \neq 0. \tag{2.8}$$

Denote by u_c a positive solution of the equation (2.1), i.e.,

$$u_c(t) > 0 \qquad \text{for} \ \ t \in I. \tag{2.9}$$

Put

$$u(t) = u_c(t)u_0(c) - u_c(c)u_0(t) \qquad \text{for } \ t \in I.$$

It is clear that u is a solution of the problem (2.6). Therefore, according to the assumptions of the lemma, $u \equiv 0$. In particular, $u(t_0) = 0$ and in view of (2.8) we have $u_c(t_0) = 0$, which contradicts (2.9). □

Finally, we formulate the following proposition.

Proposition 2.1. *Let $\ell \in \mathcal{P}_I$ be a Volterra operator with respect to b. Then for any $t \in I$ and $v \in C(I;\mathbb{R})$ satisfying the inequality*

$$v(s) \geqslant 0 \qquad \textit{for } \ t \leqslant s \leqslant b$$

we have the inequality

$$\ell(v)(s) \geqslant 0 \qquad \textit{for } \ t < s < b.$$

3. Proofs

It is known (see, e.g., [3, 5, 6]) that for the unique solvability of the problem (1.1), (1.2) or (1.1), (1.16), it is necessary and sufficient that the corresponding homogeneous problem have only the trivial solution. Thus to prove Theorems 1.1–1.3 or Theorem 1.4, it is sufficient to show that the homogeneous problem (2.1), (2.2) or (2.7), respectively, has only the trivial solution.

Proof of Theorem 1.1. Let u be a solution of the problem (2.1), (2.2). It is easy to verify that u satisfies also the equation

$$u'(t) = \hat{\ell}_k(u)(t). \tag{3.1}$$

Since $-\ell \in \mathcal{P}_I$, it is obvious that

$$-(-1)^k \hat{\ell}_k \in \mathcal{P}_I. \tag{3.2}$$

In the case where k is an even number, on account of (1.3), (3.2) and Theorem 1.3 in [2], the problem (3.1), (2.2) has only the trivial solution, and consequently, $u \equiv 0$.

Assume now that k is an odd number. Then, according to (1.3), (3.1), (3.2) and Lemma 2.1, without loss of generality we can assume that

$$u(t) \geqslant 0 \qquad \text{for } \ t \in I. \tag{3.3}$$

Taking into account the inequality (3.3) and the fact that $-\ell \in \mathcal{P}_I$, from (2.1) we obtain

$$u'(t) \leqslant 0 \qquad \text{for } \ t \in I, \tag{3.4}$$

which together with (2.2) and (3.3) results in $u \equiv 0$. $\square$

Proof of Corollary 1.1. Let $\tau^* = \operatorname{ess\,sup}\{\tau(t)\colon t \in I\}$ and ℓ be an operator defined by

$$\ell(v)(t) = -|p(t)|v(\tau(t)) \qquad \text{for } \ a < t < \tau^*.$$

Then it is clear that

$$\hat{\ell}_1(v)(t) = |p(t)| \int_a^{\tau(t)} |p(s)|v(\tau(s))\,\mathrm{d}s \qquad \text{for } \ a < t < \tau^*,$$

and consequently, in view of (1.4) and Theorem 1.1 there exists $u_0 \in \widetilde{C}([a,\tau^*];\mathbb{R})$ satisfying (1.1$'$) almost everywhere on $[a,\tau^*]$ and the condition (1.2). It can be easily verified that the function

$$u(t) = \begin{cases} u_0(t) & \text{for } \ a \leqslant t < \tau^* \\ u_0(\tau^*) + \int_{\tau^*}^t [p(s)u_0(\tau(s)) + q(s)]\,\mathrm{d}s & \text{for } \ \tau^* \leqslant t \leqslant b \end{cases}$$

is the unique solution of the problem (1.1$'$), (1.2). $\square$

On Remark 1.1. Let $a = 0$, $b = 1$, n be a natural number such that

$$\Big(\frac{2^{n+2}(n+1)}{2^{n+1}+n}\Big)^{1/2} > 3$$

and $3 < c < (\frac{2^{n+2}(n+1)}{2^{n+1}+n})^{1/2}$.

Put $p(t) = -c$ for $0 < t < 1$ and

$$\tau(t) = \begin{cases} \frac{1}{2^n} & \text{for } \ 0 < t < \frac{1}{2} \\ t^n & \text{for } \ \frac{1}{2} < t < 1 \end{cases}.$$

It is easy to see that the condition (1.4) is satisfied, however the inequalities (1.5) and (1.6) are violated.

Proof of Theorem 1.2. Assume on the contrary that there exists a nontrivial solution u of the problem (2.1), (2.2). It is evident that u satisfies also the equation

(3.1). Therefore, according to (1.9) and Lemma 2.1, without loss of generality we can assume that the inequality (3.3) is fulfilled.

Put

$$t^* = \sup\{t \in I \colon\ u(s) = 0 \text{ for } a \leqslant s \leqslant t\}.$$

In view of (1.8) and the facts that $\gamma(a) < 0$ and $\ell \in \mathcal{P}_I$, for an arbitrary $\varepsilon \in]0, -\gamma(a)[$ the function $\overline{\gamma}(t) = \varepsilon + \gamma(t)$ for $t \in I$ satisfies the inequalities

$$\overline{\gamma}'(t) \leqslant \hat{\ell}_k(\overline{\gamma})(t) \qquad \text{for } \ t \in I, \quad \overline{\gamma}(a) < 0, \quad \overline{\gamma}(b) > 0.$$

Therefore without loss of generality it can be assumed that $c \neq t^*$ and

$$\gamma(t) \operatorname{sgn}(t - c) > 0 \qquad \text{for } \ t \in I \setminus \{c\}, \quad \gamma(c) = 0.$$

First suppose $t^* < c$ and put

$$\begin{aligned} \lambda &= \max\Big\{\frac{\gamma(t)}{u(t)} \colon\ t \in [c, b]\Big\}, \\ v(t) &= \lambda u(t) - \gamma(t) \qquad \text{for } \ t \in I. \end{aligned}$$

Obviously,

$$v'(t) \geqslant \hat{\ell}_k(v)(t) \qquad \text{for } \ t \in I, \tag{3.5}$$

$$v(t) \geqslant 0 \qquad \text{for } \ t \in I, \quad v(a) > 0, \tag{3.6}$$

and there exists $t_0 \in]c, b]$ such that

$$v(t_0) = 0. \tag{3.7}$$

From (3.5), due to (3.6) and $\ell \in \mathcal{P}_I$, it follows that $v'(t) \geqslant 0$ for $t \in I$, which together with (3.6) contradicts (3.7).

Now suppose $t^* > c$. Put

$$v_n(t) = \gamma(t) - nu(t) \qquad \text{for } \ t \in I, \quad n = 1, 2, \dots$$

Evidently,

$$v_n(t) < 0 \qquad \text{for } \ a < t < c, \quad v_n(c) = 0, \qquad v_n(t^*) = \gamma(t^*) > 0,$$

$$v_n'(t) \leqslant \hat{\ell}_k(v_n)(t) \qquad \text{for } \ t \in I. \tag{3.8}$$

Put

$$t_n = \inf\{t \in [t^*, b] \colon\ v_n(t) = 0\}.$$

It can be easily verified that

$$\lim_{n\to+\infty} t_n = t^* \tag{3.9}$$

and we can choose n_0 such that for any $n > n_0$ there is $\xi_n \in]c, t_n[$ such that

$$v_n(\xi_n) = M, \qquad \text{where} \qquad M = \max\{v_n(t)\colon\ t \in I\}. \tag{3.10}$$

On account of (3.10), integration of (3.8) from c to ξ_n results in

$$M = v_n(\xi_n) \leqslant \int_c^{\xi_n} \hat{\ell}_k(v_n)(s)\,\mathrm{d}s \leqslant M \int_c^{\xi_n} \hat{\ell}_k(1)(s)\,\mathrm{d}s.$$

Hence, in view of (3.9), we get

$$\int_c^{t^*} \hat{\ell}_k(1)(s)\,\mathrm{d}s \geqslant 1. \tag{3.11}$$

On the other hand, integration of (3.1) from t^* to t yields

$$u(t) = \int_{t^*}^{t} \hat{\ell}_k(u)(s)\,\mathrm{d}s \qquad \text{for } \ t^* \leqslant t \leqslant b.$$

From the last inequality we easily obtain

$$\|u\|_C \leqslant \|u\|_C \int_{t^*}^{b} \hat{\ell}_k(1)(s)\,\mathrm{d}s,$$

and consequently,

$$\int_{t^*}^{b} \hat{\ell}_k(1)(s)\,\mathrm{d}s \geqslant 1. \tag{3.12}$$

Moreover, integration of (1.8) from a to c results in

$$\|\gamma\|_C \int_a^c \hat{\ell}_k(1)(s)\,\mathrm{d}s \geqslant -\gamma(a) > 0,$$

and consequently,

$$\int_a^c \hat{\ell}_k(1)(s)\,\mathrm{d}s > 0.$$

The last inequality together with (3.11) and (3.12) contradicts (1.9). $\square$

Proof of Corollary 1.2. According to (1.10), there exists $0 < \varepsilon < \int_a^b p(s)\,\mathrm{d}s$ such that

$$\int_a^{\tau(t)} p(s)\,\mathrm{d}s \geqslant 1+\varepsilon \qquad \text{for } \ t \in I.$$

Put

$$\gamma(t) = \int_a^t p(s)\,\mathrm{d}s - \varepsilon \qquad \text{for } \ t \in I.$$

It is clear that $\gamma(a) < 0$, $\gamma(b) > 0$ and the inequality

$$\gamma'(t) \leqslant \ell(\gamma)(t) \qquad \text{for } \ t \in I$$

holds, where ℓ is an operator defined by

$$\ell(v)(t) \overset{\text{def}}{=} p(t)v(\tau(t)).$$

Hence, in view of (1.11), it follows that the assumptions of Theorem 1.2 are satisfied for $k = 0$. □

On Remark 1.2. Choose $c \in]a, b[$ and a function $p \in L(I; \mathbb{R}_+)$ such that

$$\int_a^c p(s)\,\mathrm{d}s = 1, \qquad \int_a^b p(s)\,\mathrm{d}s < 2,$$

and put

$$\tau(t) = c \qquad \text{for } \ t \in I.$$

Therefore the condition (1.11) is satisfied,

$$\operatorname{ess\,inf}\left\{ \int_a^{\tau(t)} p(s)\,\mathrm{d}s \colon \ t \in I \right\} = 1$$

and the problem

$$u'(t) = p(t)u(\tau(t)), \qquad u(a) = 0 \tag{3.13}$$

has a nontrivial solution

$$u(t) = \int_a^t p(s)\,\mathrm{d}s \qquad \text{for } \ t \in I.$$

Thus, according to Theorem 1 in [3], there exists $q \in L(I; \mathbb{R})$ such that the problem (1.1′), (2.2) either has no solution or has an infinite set of solutions.

Let $\varepsilon > 0$ be an arbitrarily small number. Choose $c \in]a, b[$ and $p \in L(I; \mathbb{R}_+)$ such that

$$\int_a^c p(s)\,\mathrm{d}s = 1 + \frac{\varepsilon}{2}, \qquad \int_c^b p(s)\,\mathrm{d}s = 1,$$

and put

$$\tau(t) = \begin{cases} c & \text{for } \ a < t < c \\ b & \text{for } \ c < t < b \end{cases}.$$

It is clear that $\int_a^b p(s)\,\mathrm{d}s < 2 + \varepsilon$ and the condition (1.10) is satisfied. However, the problem (3.13) has a nontrivial solution

$$u(t) = \begin{cases} 0 & \text{for } \ a \leqslant t \leqslant c \\ \int_c^t p(s)\,\mathrm{d}s & \text{for } \ c < t \leqslant b \end{cases}.$$

Proof of Theorem 1.3. Let u be a solution of the problem (2.1), (2.2). It can be directly verified that the function

$$v(t) = u(t)\exp[-T(1)(t)] \qquad \text{for } \ t \in I$$

is a solution of the problem

$$v'(t) = G(v)(t), \qquad v(a) = 0$$

(operators T and G are defined in Section 1). According to (1.12), (1.13) and Theorem 1.2, this problem has only the trivial solution. Consequently, $u \equiv 0$. $\square$

Corollary 1.3 can be proved analogously to Corollary 1.2.

On Remark 1.3. Choose $c \in]a, b[$ and $p \in L(I; \mathbb{R}_+)$ such that $\int_a^c p(s)\,\mathrm{d}s = 1$, and put

$$\tau(t) = \begin{cases} c & \text{for } \ a < t < c \\ t & \text{for } \ c < t < b \end{cases}.$$

Then it is obvious that

$$\operatorname{ess\,inf}\left\{ \int_a^t p(s)\,\mathrm{d}s + \int_t^{\tau(t)} p(s) \int_a^{\tau(s)} p(\xi)\,\mathrm{d}\xi\,\mathrm{d}s \colon \ t \in I \right\} = 1,$$

and

$$\int_a^b p(t)\exp\left(- \int_a^t p(\xi)\,\mathrm{d}\xi \right) \int_t^{\tau(t)} p(s)\exp\left(\int_a^{\tau(s)} p(\xi)\,\mathrm{d}\xi \right)\mathrm{d}s\,\mathrm{d}t = 1.$$

However, the problem (3.13) has a nontrivial solution

$$u(t)=\begin{cases}\int_a^t p(s)\,\mathrm{d}s & \text{for } a\leqslant t\leqslant c\\ \exp\left(\int_c^t p(s)\,\mathrm{d}s\right) & \text{for } c<t\leqslant b\end{cases}.$$

Consequently, according to Theorem 1 in [3], there exists $q\in L(I;\mathbb{R})$ such that the problem (1.1′), (2.2) either has no solution or has an infinite set of solutions.

Let now $\varepsilon>0$ be an arbitrarily small number. Choose $\delta>0$, $c_1\in]a,b[$, $c_2\in]c_1,b[$, and $p\in L(I;\mathbb{R}_+)$ such that

$$\delta e^{1+\delta}<\varepsilon,\qquad \int_a^{c_1}p(s)\,\mathrm{d}s=1+\delta,\qquad \int_{c_1}^{c_2}p(s)\,\mathrm{d}s=1,$$

and put

$$\tau(t)=\begin{cases}c_1 & \text{for } a<t<c_1\\ c_2 & \text{for } c_1<t<c_2\\ t & \text{for } c_2<t<b\end{cases}.$$

It can be easily verified that the condition (1.14) is satisfied, and

$$\int_a^b p(t)\exp\left(-\int_a^t p(\xi)\,\mathrm{d}\xi\right)\int_t^{\tau(t)}p(s)\exp\left(\int_a^{\tau(s)}p(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s\,\mathrm{d}t=2+\delta e^{1+\delta}<2+\varepsilon.$$

However, the problem (3.13) has a nontrivial solution

$$u(t)=\begin{cases}0 & \text{for } a\leqslant t\leqslant c_1\\ \int_{c_1}^t p(s)\,\mathrm{d}s & \text{for } c_1<t\leqslant c_2\\ \exp\left(\int_{c_2}^t p(s)\,\mathrm{d}s\right) & \text{for } c_2<t\leqslant b\end{cases}.$$

Proof of Theorem 1.4. Denote by $\tilde{\ell}_0$ and $\tilde{\ell}$ respectively the restrictions of the operators ℓ_0 and ℓ to the space $C([t_0,b];\mathbb{R})$ and show that the problem

$$u'(t)=\tilde{\ell}(u)(t),\qquad u(t_0)=0 \tag{3.14}$$

has only the trivial solution.

In view of Volterra's property of the operator $\tilde{\ell}$ with respect to b, the problem $u'(t)=\tilde{\ell}(u)(t)$, $u(b)=0$ has only the trivial solution. Therefore, according to Lemma 2.2, it is sufficient to show that the equation $u'(t)=\tilde{\ell}(u)(t)$ has a positive solution.

Let u_0 be a solution of the problem

$$u'(t)=\tilde{\ell}(u)(t),\qquad u(b)=1.$$

Show that

$$u_0(t) > 0 \qquad \text{for } \; t_0 \leqslant t \leqslant b. \tag{3.15}$$

Assume on the contrary that (3.15) is violated. Then there exists $a_1 \in [t_0, b[$ such that

$$u_0(t) > 0 \qquad \text{for } \; a_1 < t < b, \qquad u_0(a_1) = 0. \tag{3.16}$$

Put

$$\lambda = \max\left\{ \frac{u_0(t)}{\gamma(t)} : \; a_1 \leqslant t \leqslant b \right\}$$

and

$$v(t) = \lambda\gamma(t) - u_0(t) \qquad \text{for } \; t_0 \leqslant t \leqslant b.$$

Evidently,

$$v(t) \geqslant 0 \qquad \text{for } \; a_1 \leqslant t \leqslant b, \tag{3.17}$$

$$v(a_1) > 0 \tag{3.18}$$

and there exists $c \in]a_1, b]$ such that

$$v(c) = 0. \tag{3.19}$$

Due to (1.17), (3.17) and Proposition 2.1, it is clear that

$$v'(t) \geqslant \tilde{\ell}_0(v)(t) \qquad \text{for } \; a_1 < t < b.$$

Hence, in view of the inclusion $\ell_0 \in \mathcal{P}_I$, the inequality (3.17) and Proposition 2.1, we get

$$v'(t) \geqslant 0 \qquad \text{for } \; a_1 < t < b,$$

which contradicts the inequalities (3.17)–(3.19).

Let now u be a solution of the problem

$$u'(t) = \ell(u)(t), \quad u(t_0) = 0.$$

It is clear that u is also a solution of the problem (3.14) and consequently,

$$u(t) = 0 \qquad \text{for } \; t_0 \leqslant t \leqslant b.$$

Therefore u is also a solution of the problem

$$u'(t) = \ell(u)(t), \quad u(b) = 0$$

and thus, in view of Volterra's property of the operator ℓ with respect to b, we have $u \equiv 0$. $\square$

Proof of Corollary 1.4. Let the condition (1.18) hold. Put

$$\ell_0(v)(t) \stackrel{\text{def}}{=} [p(t)]_+ v(\tau(t)), \qquad \ell_1(v)(t) \stackrel{\text{def}}{=} [p(t)]_- v(\tau(t)),$$
$$\gamma(t) = \int_{t_0}^{t} [p(s)]_+ \, \mathrm{d}s + \varepsilon \qquad \text{for } t \in [t_0, b],$$

where $0 < \varepsilon < 1 - \int_{t_0}^{b} [p(s)]_+ \, \mathrm{d}s$. According to (1.18), it can be easily verified that the assumptions of Theorem 1.4 are fulfilled.

Let now the condition (1.19) hold. It is evident that if v is a solution of the problem

$$v'(t) = \overline{p}(t) v(\mu(t)) + \overline{q}(t), \qquad v(\overline{t_0}) = c, \tag{3.20}$$

where $\overline{t_0} = a + b - t_0$ and

$$\overline{p}(t) = -p(a+b-t), \quad \overline{q}(t) = -q(a+b-t), \quad \mu(t) = a+b-\tau(a+b-t) \qquad \text{for } t \in I,$$

then the function $u(t) = v(a+b-t)$ for $t \in I$ is a solution of the problem (1.1′), (1.16). According to the above proved part of the theorem, it is clear that the condition (1.19) guarantees the unique solvability of the problem (3.20). □

On Remark 1.5. Let $t_0 \in [a, b[$ and $p_0 \in L([t_0, b]; \mathbb{R}_+)$ be such that

$$\int_{t_0}^{b} p_0(s) \, \mathrm{d}s = 1.$$

Put

$$\tau(t) = \begin{cases} t_0 & \text{for } a < t < t_0 \\ b & \text{for } t_0 < t < b \end{cases}$$

and

$$p(t) = \begin{cases} g(t) & \text{for } a < t < t_0 \\ p_0(t) & \text{for } t_0 < t < b \end{cases},$$

where $g \in L(I; \mathbb{R})$. It is clear that

$$\int_{t_0}^{b} [p(s)]_+ \, \mathrm{d}s = \int_{t_0}^{b} p_0(s) \, \mathrm{d}s = 1,$$

and the problem (1.1′), (1.16) has a nontrivial solution

$$u(t) = \begin{cases} 0 & \text{for } a \leqslant t \leqslant t_0 \\ \int_{t_0}^{t} p_0(s) \, \mathrm{d}s & \text{for } t_0 < t \leqslant b \end{cases}.$$

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