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MEAN VALUES AND ASSOCIATED MEASURES OF δ-SUBHARMONIC FUNCTIONS

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Abstract. Let u be a δ-subharmonic function with associated measure μ, and let v be a superharmonic function with associated measure ν, on an open set E. For any closed ball $B(x,r)$, of centre x and radius r, contained in E, let $\mathcal{M}(u,x,r)$ denote the mean value of u over the surface of the ball. We prove that the upper and lower limits as $s,t\to 0$ with $0<s<t$ of the quotient $(\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t))/(\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t))$, lie between the upper and lower limits as $r\to 0+$ of the quotient $\mu(B(x,r))/\nu(B(x,r))$. This enables us to use some well-known measure-theoretic results to prove new variants and generalizations of several theorems about δ-subharmonic functions.

Keywords: superharmonic, δ-subharmonic, Riesz measure, spherical mean values

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1. Introduction

Let E be an open subset of $\mathbb{R}^n$, let u be δ-subharmonic on E, and let v be superharmonic on E. Let μ and ν be the Borel measures associated with u and v by the Riesz Decomposition Theorem, so that μ is signed and ν is positive. Let $B(x,r)$ denote the closed ball with centre x and radius r contained in E, and let $\mathcal{M}(u,x,r)$ denote the spherical mean value of u over $\partial B(x,r)$. We shall prove that the upper and lower limits as $s,t\to 0$ with $0<s<t$ of

$$\frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)} \tag{1}$$

lie between the upper and lower limits as $r\to 0+$ of

$$\frac{\mu(B(x,r))}{\nu(B(x,r))}. \tag{2}$$

This enables us to use the measure-theoretic results of Besicovitch [3], [4] to study the behaviour of δ-subharmonic functions.

This work was inspired by a recent paper of Sodin [12]. However, the techniques we devised have much wider ramifications, so that Sodin's results appear only as fairly minor details. We generalize not only Sodin's results, but also some due to Armitage [2] and Watson [14]. We also present new analogues of some theorems about Poisson integrals which appeared in [1], [5], and [16], a new form of the Domination Principle, and variants of recent results of Fuglede [10].

Our starting point is the well-known formula

$$\mathcal{M}(u,x,s)=\mathcal{M}(u,x,t)+p_n\int_s^t r^{1-n}\mu(B(x,r))\,\mathrm{d}r,$$

in which $0<s<t$, $B(x,t)\subseteq E$, and $p_n=\max\{1,n-2\}$. See, for example, [2] Lemma 3. We shall put

$$I_\mu(x;s,t)=p_n\int_s^t r^{1-n}\mu(B(x,r))\,\mathrm{d}r.$$

Then the quotient (1) can be written as either

$$\frac{I_\mu(x;s,t)}{I_\nu(x;s,t)} \tag{3}$$

or

$$\frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{I_\nu(x;s,t)}. \tag{4}$$

From (3) it is easy to see the connection with (2).

If we take ν to be the Lebesgue measure λ, we have

$$I_\lambda(x;s,t)=p_nv_n(t^2-s^2)/2,$$

where $v_n=\lambda(B(0,1))$. Then, up to a multiplicative constant, (4) becomes

$$\frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{t^2-s^2}.$$

Theorem 3 below gives conditions on this quotient which ensure that μ can be written in the form

$$\mu=\omega-\sum_j c_j\delta_j$$

where ω is a positive measure, each c_j is a specific positive constant, each δ_j is a unit mass at a given point x_j, and there are countably many indices. This is analogous to decomposition formulas for the boundary measures of Poisson integrals given in [5], [1], and [16].

Theorem 4 shows how the quotient (4) can be used to determine which sets are positive for μ. Roughly, if

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{I_\nu(x;s,t)} \geqslant 0$$

for all $x \in S$, then S is positive for μ. The condition can be weakened on a ν-null subset of S. This result contains as special cases those due to Sodin [12], which include the one known as Grishin's lemma [11].

Theorems 5 and 6 generalize results of Armitage [2] by extending them to points where his conditions that an infinity occur no longer hold. Theorems 10 and 11 similarly extend results of Watson [14].

By analogy with results on half-space Poisson integrals given in [1] and [16], Theorem 7 gives conditions on the quotient (1) which ensure that $u-Av$ is superharmonic for some real number A. For example, the condition

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)} \geqslant A$$

for all $x \in E$, is sufficient. A minor modification of the proof, in the special case where u is a positive superharmonic function, E is Greenian, and $v = G_E\nu$ is a Green potential, yields the following domination principle as Theorem 8: If

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)}$$

is never $-\infty$, and is greater than or equal to 1 for ν-almost all x, then $u \geqslant v$.

In Theorem 9, we use (1) to determine the μ-null subsets of E. One of its corollaries is an extension to δ-subharmonic functions of the fact that polar sets are null for the restriction of ν to the set where v is finite. A different such extension was established by Fuglede [10].

Given a Borel subset B of E, we denote by μ_B the restriction of μ to B.

Theorem 1 contains the necessary measure theory. It is implicit in [15], but may not have been stated explicitly before. References are given to the original papers of Besicovitch; an alternative source is [9].

Theorem 1. *Let μ be a signed measure and ν a positive measure on E. Let*

$$f(x) = \lim_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))}$$

whenever the limit exists, let $Z^+ = \{x \in E\colon f(x) = \infty\}$, and let $Z^- = \{x \in E\colon f(x) = -\infty\}$. Then f is defined and finite ν-a.e. on E, and there are positive ν-singular measures σ^+ and σ^-, concentrated on Z^+ and Z^- respectively, such that

$$\mathrm{d}\mu = f\,\mathrm{d}\nu + \mathrm{d}\sigma^+ - \mathrm{d}\sigma^-. \tag{5}$$

P r o o f. By [3] Theorem 2, f is defined and finite ν-a.e. By [4] Theorem 6, f is the Radon-Nikodým derivative of μ with respect to ν, so that (5) holds with σ^+ and σ^- the positive and negative variations of the ν-singular part of μ.

To show that σ^+ is concentrated on Z^+, we put $\mathrm{d}\omega = f\,\mathrm{d}\nu - \mathrm{d}\sigma^-$. Then both ν and ω are σ^+-singular, so that by [3] Theorem 3,

$$\lim_{r\to 0} \frac{\nu(B(x,r))}{\sigma^+(B(x,r))} = 0$$

and

$$\lim_{r\to 0} \frac{\mu(B(x,r))}{\sigma^+(B(x,r))} = \lim_{r\to 0} \frac{\omega(B(x,r))}{\sigma^+(B(x,r))} + 1 = 1,$$

for σ^+-almost all x. Hence $f(x) = \infty$ for σ^+-almost all x, so that σ^+ is concentrated on Z^+. Similarly, σ^- is concentrated on Z^-.

Corollary 1. *Let μ be a signed measure and ν a positive measure on E, and let S be a Borel subset of E. If*

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} > -\infty \tag{6}$$

for all $x \in S$ at which the upper limit is defined, and

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \geqslant A \tag{7}$$

for ν-almost all $x \in S$, then $(\mu - A\nu)_S \geqslant 0$.

P r o o f. By Theorem 1, $\mathrm{d}\mu_S = f\,\mathrm{d}\nu_S + \mathrm{d}\sigma_S^+ - \mathrm{d}\sigma_S^-$ with $f(x)$ equal to the upper limit in (7) and σ_S^- concentrated on $\{x \in S\colon f(x) = -\infty\}$. By (6) this set is empty, and by (7) $f \geqslant A$. Hence $\mathrm{d}\mu_S - A\,\mathrm{d}\nu_S \geqslant \mathrm{d}\sigma_S^+ \geqslant 0$. □

Corollary 2. *Let μ be a signed measure and ν a positive measure on E. Let S be a Borel subset of E such that, for each $x \in S$, either*

$$\lim_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} = 0 \tag{8}$$

or the limit does not exist. Then $\mu_S = 0$.

Proof. By Theorem 1, $\{x \in E\colon (8) \text{ holds}\}$ is μ-null, and the set of points where f is undefined is also μ-null. □

We include for completeness the definition of

$$\limsup_{0<s<t\to 0} f(s,t),$$

although it is the natural one. Those of the corresponding lim inf and lim are then obvious.

Definition. Suppose that $f(s,t)$ is defined as an extended-real number whenever $0 < s < t < a$, and that $\ell \in \mathbb{R}$. We write

$$\limsup_{0<s<t\to 0} f(s,t) = \ell$$

if to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that $f(s,t) < \ell + \varepsilon$ whenever $0 < s < t < \delta$, and there is a sequence $\{(s_k,t_k)\}$ such that $0 < s_k < t_k \to 0$ and $f(s_k,t_k) \to \ell$ as $k \to \infty$. We also write

$$\limsup_{0<s<t\to 0} f(s,t) = \infty$$

if there is a sequence $\{(s_k,t_k)\}$ such that $0 < s_k < t_k \to 0$ and $f(s_k,t_k) \to \infty$. Finally, we write

$$\limsup_{0<s<t\to 0} f(s,t) = -\infty$$

if to each $A \in \mathbb{R}$ there corresponds $\delta > 0$ such that $f(s,t) < A$ whenever $0 < s < t < \delta$.

We can now establish the connection on which all our results are based.

Theorem 2. *If u is δ-subharmonic on E with associated measure μ, and ν is a positive measure on E, then*

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{I_\nu(x;s,t)} \leqslant \limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \tag{9}$$

whenever the latter exists. The reverse inequality holds for lower limits, and

$$\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{I_\nu(x;s,t)} = \lim_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \tag{10}$$

for ν-almost all $x \in E$.

Proof. Given x for which the upper limit on the right-hand side of (9) exists, denote that upper limit by ℓ. If $\ell = \infty$ there is nothing to prove. Otherwise, given a real number $A > \ell$ we can find $\delta > 0$ such that

$$\frac{\mu(B(x,r))}{\nu(B(x,r))} < A \qquad \text{whenever } 0 < r < \delta.$$

If $\nu(B(x,r)) = 0$ for all $r < \eta$ ($\leqslant \delta$), then the above inequality can hold only if $\mu(B(x,r)) < 0$ for all such r. Then $I_\nu(x;s,t) = 0$ whenever $t < \eta$, and

$$\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t) = I_\mu(x;s,t) < 0,$$

so that (9) holds with both sides $-\infty$. On the other hand, if $\nu(B(x,r)) > 0$ for all r, then

$$\frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{I_\nu(x;s,t)} = \frac{p_n}{I_\nu(x;s,t)} \int_s^t r^{1-n}\nu(B(x,r))\, \frac{\mu(B(x,r))}{\nu(B(x,r))}\,\mathrm{d}r < A$$

whenever $0 < s < t < \delta$, and again (9) holds.

Obviously (9) implies the reverse inequality for lower limits. Now (10) follows from [3] Theorem 2. □

The particular cases of Theorem 2, in which ν is the Lebesgue measure λ or the unit mass δ_x at x, are of special importance.

Corollary 1. *If u is δ-subharmonic with associated measure μ on E, then*

$$\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{t^2-s^2} = \frac{p_n}{2} \lim_{r\to 0} \frac{\mu(B(x,r))}{r^n}$$

whenever the latter exists.

Proof. Whenever $B(x,t) \subseteq E$, we have

$$I_\lambda(x;s,t) = p_n \int_s^t r^{1-n}(v_n r^n)\,\mathrm{d}r = p_n v_n (t^2-s^2)/2,$$

so that the result follows from Theorem 2. □

Corollary 2. *If u is δ-subharmonic with associated measure μ on E, then for each $x \in E$ we have*

$$\mu(\{x\}) = \lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{\log(t/s)} \qquad \text{if } n = 2,$$

and

$$\mu(\{x\}) = \lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{s^{2-n} - t^{2-n}} \qquad \text{if } n \geqslant 3.$$

P r o o f. Writing $\delta = \delta_x$, we have

$$I_\delta(x; s, t) = p_n \int_s^t r^{1-n}\,\mathrm{d}r = \left\{\begin{array}{ll} \log(t/s) & \text{if } n = 2, \\ s^{2-n} - t^{2-n} & \text{if } n \geqslant 3, \end{array}\right\}$$

so that Theorem 2 gives the result. □

3. A representation theorem

Theorem 2 and its corollaries enable us to prove a new representation theorem for δ-subharmonic functions, which is analogous to known results about Poisson integrals on a ball due to Bruckner, Lohwater and Ryan [5], and on a half-space due to Armitage [1] and Watson [16].

Theorem 3. *Let u be δ-subharmonic with associated measure μ on E. If*

$$\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{t^2 - s^2} \geqslant 0 \tag{11}$$

for λ-almost all $x \in E$, and

$$\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x_j,s) - \mathcal{M}(u,x_j,t)}{t^2 - s^2} = -\infty \tag{12}$$

for only the points x_j in a countable set C, then μ can be written in the form

$$\mu = \omega + \sum_j \Big(\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x_j,s) - \mathcal{M}(u,x_j,t)}{\log(t/s)}\Big)\delta_j \tag{13}$$

if $n = 2$,

$$\mu = \omega + \sum_j \Big(\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x_j,s) - \mathcal{M}(u,x_j,t)}{s^{2-n} - t^{2-n}}\Big)\delta_j \tag{14}$$

if $n \geqslant 3$, where ω is a positive measure such that $\omega(C) = 0$, and δ_j is the unit mass at x_j.

P r o o f. In view of Theorem 2 Corollary 1, condition (11) implies that

$$\lim_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))} \geqslant 0$$

for λ-almost all $x \in E$, and condition (12) implies that

$$\lim_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))} = -\infty$$

only if $x \in C$. Therefore, by Theorem 1,

$$\mathrm{d}\mu = f\,\mathrm{d}\lambda + \mathrm{d}\sigma^+ - \mathrm{d}\sigma^-$$

with $f \geqslant 0$ and σ^- concentrated on C. Furthermore, for each j, Theorem 2 Corollary 2 shows that the limits in (13) and (14) are equal to $\mu(\{x_j\})$. Thus

$$\mathrm{d}\mu = (f\,\mathrm{d}\lambda + \mathrm{d}\sigma^+) + \sum_j \mu(\{x_j\})\delta_j$$

yields the required representation. □

In particular, Theorem 3 allows the following characterization of a point mass.

Corollary. *Let u be δ-subharmonic with associated measure μ on E. If*

$$\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{t^2 - s^2}$$

is 0 for λ-almost all $x \in E$, is finite except at x_0, and is ∞ at x_0, then μ is a positive constant multiple of the unit mass at x_0.

P r o o f. Applying Theorem 3 to $-u$, we obtain

$$-\mu = \omega - \Big(\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x_0,s) - \mathcal{M}(u,x_0,t)}{\log(t/s)}\Big)\delta_0$$

if $n = 2$,

$$-\mu = \omega - \Big(\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x_0,s) - \mathcal{M}(u,x_0,t)}{s^{2-n} - t^{2-n}}\Big)\delta_0$$

if $n \geqslant 3$, where ω is a positive measure such that $\omega(\{x_0\}) = 0$. By Theorem 2 Corollary 2, $-\mu = \omega - \mu(\{x_0\})\delta_0$ in either case. Applying Theorem 3 to u itself, we find that μ is positive, so that ω is null. □

The proof of its corollary illustrates how Theorem 3 can sometimes be used to show that the measure associated with a δ-subharmonic function is positive. Theorem 4 below is a refinement that allows us to determine which are the positive sets for the measure. It is similar in essence to the case $Y = \emptyset$ of [15] Theorem 6.

Recall that μ_S denotes the restriction of μ to the set S.

Theorem 4. *Let u be δ-subharmonic with associated measure μ on E, let S be a Borel subset of E, and let ν be a positive measure on E. If*

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{I_\nu(x;s,t)}$$

is not $-\infty$ for any $x \in S$, and is nonnegative for ν-almost all $x \in S$, then $\mu_S \geqslant 0$.

Proof. By Theorem 2,

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))}$$

is not $-\infty$ for any $x \in S$, and is nonnegative for ν-almost all $x \in S$. Therefore $\mu_S \geqslant 0$, by Theorem 1 Corollary 1. □

Theorem 4 contains the results of Sodin [12] which, in turn, are extensions of Grishin's lemma [11]. Other extensions of Grishin's lemma were obtained by Fuglede [10]. Our next corollary extends Sodin's theorem to n-dimensions.

Corollary 1. *Let u be δ-subharmonic with associated measure μ on E, and let S be the set of points in E with the following property: There are sequences $\{s_k\}$ and $\{t_k\}$, which depend on the point x, such that $0 < s_k < t_k \to 0$ and*

$$\mathcal{M}(u,x,s_k) \leqslant \mathcal{M}(u,x,t_k) \tag{15}$$

for all k. Then $\mu_S \leqslant 0$.

Proof. Sodin proved that S is a Borel set. If $x \in S$, then

$$\liminf_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{I_\lambda(x;s,t)} \leqslant 0.$$

Therefore $\mu_S \leqslant 0$, by Theorem 4. □

Theorem 4 is much stronger than its first corollary. To see this, consider the case where $\mathrm{d}\mu(y) = f(y)\,\mathrm{d}y$ with f continuous and nonnegative, and with the zero set Z of f nonempty but with empty interior. Then, whenever $B(x,t) \subseteq E$ and $0 < s < t$, we have $\mathcal{M}(u,x,s) > \mathcal{M}(u,x,t)$, so that the corollary can only be applied to $-u$ and not to u, and it yields only the inequality $\mu \geqslant 0$. However, for any $x \in Z$ we have

$$\lim_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))} = 0,$$

so that

$$\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{I_\lambda(x;s,t)} = 0$$

by Theorem 2. Now Theorem 4 can be applied to both u and $-u$ (with $S = Z$), and confirms that μ_Z is null.

The next corollary generalizes both of Sodin's "remarks" to n-dimensions, with weaker hypotheses.

Corollary 2. *Let u be δ-subharmonic with associated measure μ on E, and let S be a Borel subset of E on which there is defined a positive measure ν such that for some constant $\beta \geqslant 0$*

$$\nu(B(x,r)) \geqslant \kappa r^\beta$$

whenever $x \in S$ and $0 < r < r_x$, where $\kappa = \kappa_x > 0$. Let $\alpha > 0$, and let h be an absolutely continuous function on $[0,\alpha]$ such that $h'(r) = o(r^{\beta-n+1})$ as $r \to 0$. If, to each $x \in S$, there correspond sequences $\{s_k\}$ and $\{t_k\}$ such that $0 < s_k < t_k \to 0$ and

$$\mathcal{M}(u,x,s_k) - \mathcal{M}(u,x,t_k) \geqslant h(s_k) - h(t_k) \qquad \forall k,$$

then $\mu_S \geqslant 0$.

Proof. Given $x \in S$ and $\varepsilon > 0$, for all sufficiently large k we have

$$\begin{aligned}\mathcal{M}(u,x,s_k) - \mathcal{M}(u,x,t_k) &\geqslant -\int_{s_k}^{t_k} h'(r)\,\mathrm{d}r \geqslant -\varepsilon\kappa p_n \int_{s_k}^{t_k} r^{\beta-n+1}\,\mathrm{d}r \\ &\geqslant -\varepsilon p_n \int_{s_k}^{t_k} r^{1-n}\nu(B(x,r))\,\mathrm{d}r = -\varepsilon I_\nu(x;s_k,t_k),\end{aligned}$$

so that

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{I_\nu(x;s_k,t_k)} \geqslant 0.$$

By Theorem 4, $\mu_S \geqslant 0$. $\square$

In the above corollary, the case $n = \beta = 2$ is [12] Remark 1, which does not mention a measure ν. The choice $\nu = \lambda$ gives the result. The case $n = 2$, $\beta > 0$, is [12] Remark 2. With regard to the existence of ν, Sodin mentioned only the work of Tricot [13]. However, there are many other results in this direction. For example, if S is a q-set for some $q \in [0, n]$ (as, for example, in [8]), then the q-dimensional Hausdorff measure ν on S satisfies $\nu(B(x,r)) \sim (2r)^q$ as $r \to 0$, at every regular point of S.

5. Specific rates

The next theorem generalizes one due to Armitage [2], which we deduce as a corollary.

Theorem 5. *Let u be δ-subharmonic with associated measure μ on E. Let $\alpha > 0$, let f be a positive, increasing, absolutely continuous function on $[0, \alpha]$, and let*

$$\hat{f}(s,t) = p_n \int_s^t r^{1-n} f(r)\, \mathrm{d}r$$

whenever $0 \leqslant s < t \leqslant \alpha$. Then

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{\hat{f}(s,t)} \leqslant \limsup_{r\to 0} \frac{\mu(B(x,r))}{f(r)} \tag{16}$$

for every $x \in E$.

Proof. Given x, define a positive measure ν on $B(x,\alpha)$ by putting

$$\mathrm{d}\nu(y) = \|x-y\|^{1-n} f'(\|x-y\|)\, \mathrm{d}y + \sigma_n f(0)\, \mathrm{d}\delta_x(y),$$

where σ_n is the surface area of the unit sphere in $\mathbb{R}^n$. Then

$$\nu(B(x,r)) = \sigma_n \int_0^r f'(s)\, \mathrm{d}s + \sigma_n f(0) = \sigma_n f(r)$$

if $0 < r \leqslant \alpha$, so that

$$I_\nu(x; s, t) = p_n \int_s^t r^{1-n} \sigma_n f(r)\, \mathrm{d}r = \sigma_n \hat{f}(s,t)$$

whenever $0 < s < t \leqslant \alpha$. The result now follows from Theorem 2. $\square$

Armitage's result did not involve differences of spherical mean values, and so required an additional hypothesis on $\hat{f}$, as follows.

Corollary 1. *Let u be δ-subharmonic with associated measure μ on E. Let $\alpha > 0$, let f be a positive, increasing, absolutely continuous function on $[0,\alpha]$, and let*

$$\hat{f}(s,t) = p_n \int_s^t r^{1-n} f(r)\,\mathrm{d}r$$

whenever $0 \leqslant s < t \leqslant \alpha$. If $\hat{f}(0,\alpha) = \infty$, then

$$\limsup_{s\to 0} \frac{\mathcal{M}(u,x,s)}{\hat{f}(s,\alpha)} \leqslant \limsup_{r\to 0} \frac{\mu(B(x,r))}{f(r)}$$

for all $x \in E$.

P r o o f. Given $x \in E$, let ℓ denote the left-hand side of (16). In view of (16), it suffices to prove that

$$\limsup_{s\to 0} \frac{\mathcal{M}(u,x,s)}{\hat{f}(s,\alpha)} \leqslant \ell. \tag{17}$$

We may assume that $\ell < \infty$. Given a real number $A > \ell$, choose $\delta > 0$ such that

$$\frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{\hat{f}(s,t)} < A \qquad \text{whenever } 0 < s < t < \delta.$$

Fix $t < \delta$. Given $\varepsilon > 0$, choose $\eta < t$ such that both

$$\frac{\mathcal{M}(u,x,t)}{\hat{f}(s,\alpha)} < \varepsilon \qquad \text{and} \qquad \frac{\hat{f}(t,\alpha)}{\hat{f}(s,\alpha)} < \varepsilon$$

whenever $0 < s < \eta$. Then

$$\begin{aligned}\frac{\mathcal{M}(u,x,s)}{\hat{f}(s,\alpha)} &= \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{\hat{f}(s,t)} \cdot \frac{\hat{f}(s,t)}{\hat{f}(s,\alpha)} + \frac{\mathcal{M}(u,x,t)}{\hat{f}(s,\alpha)} \\ &< A\Big(1 - \frac{\hat{f}(t,\alpha)}{\hat{f}(s,\alpha)}\Big) + \varepsilon < \max\{A, (1-\varepsilon)A\} + \varepsilon\end{aligned}$$

if $0 < s < \eta$, and (17) follows. $\square$

The extra generality of Theorem 5 over Corollary 1 allows us to generalize the corollary of Armitage's theorem and remove its restrictions on q.

Corollary 2. *Let u be δ-subharmonic with associated measure μ on E, and let $x \in E$. Then*

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{s^{q+2-n}-t^{q+2-n}} \leqslant \Big(\frac{n-2}{n-q-2}\Big) \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^q}$$

if $0 \leqslant q < n-2$,

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\log(t/s)} \leqslant p_n \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^{n-2}},$$

and

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{t^{q+2-n}-s^{q+2-n}} \leqslant \Big(\frac{p_n}{q+2-n}\Big) \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^q}$$

if $q > n-2$.

Proof. If we take $f(r) = r^q$ $(q \geqslant 0)$ in Theorem 5, so that $\hat{f}(s,t) = p_n \int_s^t r^{q+1-n}\,\mathrm{d}r$, then $\hat{f}(s,t)$ is equal to p_n times

$$\begin{array}{ll} \dfrac{s^{q+2-n}-t^{q+2-n}}{n-q-2} & \text{if } q < n-2, \\ \log(t/s) & \text{if } q = n-2, \\ \dfrac{t^{q+2-n}-s^{q+2-n}}{q+2-n} & \text{if } q > n-2, \end{array}$$

which gives the result. □

If S is a regular q-set [8] contained in E, and μ is the q-dimensional Hausdorff measure on S, then

$$\lim_{r\to 0} \frac{\mu(B(x,r))}{r^q} = 2^q$$

for μ-almost all $x \in S$. Therefore, for such x,

$$\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{s^{q+2-n}-t^{q+2-n}} = \frac{p_n 2^q}{n-q-2}$$

if $q \neq n-2$, and

$$\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\log(t/s)} = p_n 2^q$$

if $q = n-2$, for any superharmonic function u whose associated measure is μ. These identities follow easily from Theorem 5 Corollary 2.

Theorem 2 can easily be re-written in a form that generalizes [2] Theorem 1, which we deduce as a corollary. This formulation is then used to provide conditions under which $u - Av$ is superharmonic for some real number A, as well as a new version of the domination principle.

Theorem 6. *Let u be δ-subharmonic and v superharmonic on E, with associated measures μ and ν respectively. Then*

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)} \leqslant \limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \tag{18}$$

whenever the latter exists. The reverse inequality holds for lower limits, and

$$\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)} = \lim_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))}$$

for ν-almost all $x \in E$.

P r o o f. The result follows from Theorem 2, because

$$\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t) = I_\nu(x;s,t)$$

by [2] Lemma 3. □

Corollary. *Let u be δ-subharmonic and v superharmonic on E, with associated measures μ and ν respectively. If $x \in E$ and $v(x) = \infty$, then*

$$\limsup_{s\to 0} \frac{\mathcal{M}(u,x,s)}{\mathcal{M}(v,x,s)} \leqslant \limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))}$$

and the reverse inequality holds for lower limits.

P r o o f. Given $x \in E$ such that $v(x) = \infty$, let ℓ denote the left-hand side of (18). In view of (18), it suffices to prove that

$$\limsup_{s\to 0} \frac{\mathcal{M}(u,x,s)}{\mathcal{M}(v,x,s)} \leqslant \ell. \tag{19}$$

We may assume that $\ell < \infty$. Given a real number $A > \ell$, choose $\delta > 0$ such that

$$\frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)} < A \qquad \text{whenever } 0 < s < t < \delta.$$

Since $\mathcal{M}(v,x,r) \to \infty$ as $r \to 0$, we may suppose that $\mathcal{M}(v,x,r) > 0$ for all $r < \delta$. Fix $t < \delta$. Given $\varepsilon > 0$, choose $\eta < t$ such that both

$$\frac{\mathcal{M}(v,x,t)}{\mathcal{M}(v,x,s)} < \varepsilon \qquad \text{and} \qquad \frac{\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)} < \varepsilon$$

whenever $0 < s < \eta$. Then

$$\begin{aligned}\frac{\mathcal{M}(u,x,s)}{\mathcal{M}(v,x,s)} &= \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s) - \mathcal{M}(v,x,t)}\Big(1 - \frac{\mathcal{M}(v,x,t)}{\mathcal{M}(v,x,s)}\Big) + \frac{\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)} \\ &< \max\{A, (1-\varepsilon)A\} + \varepsilon\end{aligned}$$

if $0 < s < \eta$. This proves (19). $\square$

We now use Theorem 6 to prove analogues of a domination theorem and a uniqueness theorem about Poisson integrals on half-spaces given in [1] and [16]. Conditions for the measure to be positive or null in that context translate into conditions for superharmonicity or harmonicity here.

Theorem 7. *Let u be δ-subharmonic on E, and let v be superharmonic on E with associated measure ν. If*

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s) - \mathcal{M}(v,x,t)}$$

is never $-\infty$, and is greater than or equal to A for ν-almost all x, then $u - Av$ is superharmonic on E.

Proof. Let μ be the measure associated to u. By Theorem 6, our hypotheses imply that

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))}$$

is never $-\infty$, and is greater than or equal to A for ν-almost all x. Therefore, we can use Theorem 1 Corollary 1 to show that $\mu - A\nu \geqslant 0$. Hence $u - Av$ is superharmonic on E. $\square$

Note that the case $A = 0$ of Theorem 7 gives a condition for u itself to be superharmonic. Theorem 7 is analogous to both [16] Theorem 2 and an earlier result about Poisson integrals on a disc, [5] Theorem 2. It also implies the following condition for u to be harmonic; compare [1] Theorem 4 and the comment on that result in [16] (p. 470).

Corollary. *Let u be δ-subharmonic on E, and let v be superharmonic on E with associated measure ν. If*

$$\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)}$$

is finite whenever it exists, and is 0 for ν-almost all x, then u is harmonic on E.

P r o o f. Applying Theorem 7 to both u and $-u$, we see that both functions are superharmonic on E. □

A minor variation in the proof of Theorem 7 yields a new form of the Domination Principle ([7], pp. 67, 194).

Theorem 8. *Let E be Greenian, let $v = G_E\nu$ be the Green potential of a positive measure ν on E, and let u be a positive superharmonic function on E. If*

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)}$$

is never $-\infty$, and is greater than or equal to 1 for ν-almost all x, then $u \geqslant G_E\nu$.

P r o o f. Let μ be the measure associated to u. As in the proof of Theorem 7, our hypotheses imply that $\mu \geqslant \nu$, so that $G_E\mu \geqslant G_E\nu$. Since u is positive, its greatest harmonic minorant is nonnegative, and so $u \geqslant G_E\nu$. □

7. Null sets for associated measures

Theorem 4 obviously implies a condition for a set to be null for the associated measure μ. In this section we state the result explicitly and relate it to known theorems. For example, if u is superharmonic on E and $S = \{x \in E\colon\ u(x) < \infty\}$, it is well-known that any polar subset of E is μ_S-null ([7], p. 68). That result was generalized to δ-subharmonic functions, with S replaced by $\{x \in E\colon\ \text{fine}\liminf_{y\to x} |u(y)| < \infty\}$, by Fuglede [10] Theorem 2.1. Theorem 9 Corollary 2 gives a different generalization.

Theorem 9. *Let u be δ-subharmonic and v superharmonic on E, with associated measures μ and ν respectively, and let S be a Borel subset of E. If*

$$\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)}$$

is not infinite for any $x \in S$, and is zero for ν-almost all $x \in S$, then μ_S is null.

P r o o f. Write $\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)$ as $I_\nu(x;s,t)$, and apply Theorem 4 to both u and $-u$. □

The first corollary gives a restricted version of the theorem which involves quotients of the form $\mathcal{M}(u,x,s)/\mathcal{M}(v,x,s)$, and thus parallels Theorem 6 Corollary.

Corollary 1. *Let u be δ-subharmonic with associated measure μ on E, let v be superharmonic on E, and let S be a Borel subset of E. If, for each $x \in S$, $v(x) = \infty$ and there is a null sequence $\{r_k\}$ such that*

$$\lim_{k\to\infty} \frac{\mathcal{M}(u,x,r_k)}{\mathcal{M}(v,x,r_k)} = 0, \tag{20}$$

then μ_S is null.

Proof. For any $x \in S$ we have $\mathcal{M}(v,x,r) \to \infty$ as $r \to 0$. Therefore, for any fixed t such that $B(x,t) \subseteq E$,

$$\lim_{k\to\infty} \frac{\mathcal{M}(u,x,r_k) - \mathcal{M}(u,x,t)}{\mathcal{M}(v,x,r_k) - \mathcal{M}(v,x,t)} = 0$$

in view of (20). Therefore

$$\lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s) - \mathcal{M}(v,x,t)}$$

is zero if it exists, and it exists for ν-almost all x (where ν is the measure associated to v) by Theorem 6. Now Theorem 9 shows that μ_S is null. $\square$

Corollary 2. *Let u be δ-subharmonic with associated measure μ on E. If S is a Borel subset of E such that for each $x \in S$*

$$\liminf_{r\to 0} |\mathcal{M}(u,x,r)| < \infty, \tag{21}$$

then any polar subset of E is μ_S-null.

Proof. Let N be a polar subset of E, and let v be a superharmonic function on E such that $v(x) = \infty$ for every $x \in N$. Then, for any $x \in S \cap N$, the condition (21) implies the existence of a null sequence $\{r_k\}$ such that (20) holds. By Corollary 1, $\mu_{S\cap N}$ is null. $\square$

Corollary 1 is considerably stronger than Corollary 2. To illustrate this, we consider an open ball B with a G_δ polar subset N. We construct two positive superharmonic functions u, v on B, with $u(x) = v(x) = \infty$ for all $x \in N$, such that $\mu(N) = 0$

(see [2], p. 61) and $\nu(B \setminus N) = 0$ (see [6]), where μ, ν are the measures associated to u, v respectively. Since μ and ν are mutually singular, we have

$$\lim_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} = 0$$

for ν-almost all x [3], so that

$$\lim_{r\to 0} \frac{\mathcal{M}(u,x,r))}{\mathcal{M}(v,x,r))} = 0$$

by [2] Theorem 1. So Corollary 1 confirms that there is a ν-null set M such that $\mu_{N\setminus M}$ is null, but Corollary 2 is inapplicable because (21) fails to hold for any $x \in N$.

8. More extensions of known results

We conclude with two extensions of results in [14].

Theorem 10. *Let u be δ-subharmonic and v superharmonic on E, with associated measures μ and ν respectively. Let*

$$f(x) = \lim_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s) - \mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s) - \mathcal{M}(v,x,t)}$$

whenever the limit exists, let $Z^+ = \{x \in E\colon f(x) = \infty\}$, and let $Z^- = \{x \in E\colon f(x) = -\infty\}$. Then f is defined and finite ν-a.e. on E, and there are positive ν-singular measures σ^+ and σ^-, concentrated on Z^+ and Z^- respectively, such that $\mathrm{d}\mu = f\,\mathrm{d}\nu + \mathrm{d}\sigma^+ - \mathrm{d}\sigma^-$.

Proof. By Theorem 6,

$$f(x) = \lim_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))}$$

whenever this limit exists. The result now follows from Theorem 1. □

Note that, if $f(x)$ is finite whenever it exists, then μ is absolutely continuous with respect to ν.

Theorem 10 generalizes [14] Theorem 6, which we now deduce as a corollary.

Corollary. *Let u be δ-subharmonic and v superharmonic on E, with associated measures μ and ν respectively, let $X = \{x \in E\colon v(x) = \infty\}$, let*

$$g(x) = \lim_{r\to 0} \frac{\mathcal{M}(u,x,r)}{\mathcal{M}(v,x,r)}$$

whenever the limit exists, let $Z^+ = \{x \in X\colon\ g(x) = \infty\}$, and let $Z^- = \{x \in X\colon\ g(x) = -\infty\}$. Then g is defined and finite ν-a.e. on X, and there are positive ν-singular measures σ^+ and σ^-, concentrated on Z^+ and Z^- respectively, such that $\mathrm{d}\mu_X = g\,\mathrm{d}\nu_X + \mathrm{d}\sigma^+ - \mathrm{d}\sigma^-$.

Proof. If $x \in X$, then

$$\limsup_{r\to 0} \frac{\mathcal{M}(u,x,r)}{\mathcal{M}(v,x,r)} \leqslant \limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)}$$

by (19), and the reverse inequality holds for lower limits. Therefore $g(x)$ is equal to the $f(x)$ in Theorem 10, whenever $f(x)$ exists. The result follows. □

Theorem 10 enables us to prove a corresponding generalization of [14] Theorem 8, as follows. This generalization provides conditions under which a Borel set is a positive set for the Riesz measure of a δ-subharmonic function, whereas [14] Theorem 8 applied only to a Borel *polar* set.

Theorem 11. *Let u be δ-subharmonic and v superharmonic on E, with associated measures μ and ν respectively. Let $q \in [0, n-2]$, and let S be a Borel subset of E. If*

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)} > -\infty$$

for all $x \in S \setminus Y$, where Y is an m_q-null Borel set, if

$$\limsup_{0<s<t\to 0} \frac{\mathcal{M}(u,x,s)-\mathcal{M}(u,x,t)}{\mathcal{M}(v,x,s)-\mathcal{M}(v,x,t)} \geqslant 0$$

for ν-almost all $x \in S \setminus Y$, and if

$$\liminf_{r\to 0} r^{n-q-2}\mathcal{M}(u,x,r) > -\infty \tag{22}$$

for $|\mu|$-almost all $x \in Y$, then $\mu_S \geqslant 0$. If (22) is replaced by

$$\liminf_{r\to 0} r^{n-q-2}\mathcal{M}(u,x,r) \geqslant 0,$$

then the result remains valid if $0 < m_q(Y) < \infty$.

Proof. Follow the proof of [14] Theorem 8, but use Theorem 10 above instead of [14] Theorem 6. □

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